Quiz – Introduction to String Theory

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1. **Commutator.**
   The commutator of two objects (you can assume that they are square $d \times d$-matrices) is defined as $[A, B] := AB - BA$. Show that

   (a) $[A, B] = -[B, A]$

   (b) $[A, B + C] = [A, B] + [A, C]$

   (c) $[AB, C] = A[B, C] + [A, C]B$

   We can also define an anticommutator: $\{A, B\} := AB + BA$. Decompose $\{AB, C\}$ like $[AB, C]$ is decomposed above.

2. **Euler-Lagrange equation.**
   Let $x(t)$ be the position of a point-particle moving in one spatial dimension. Its motion is parameterised by a time coordinate t. Let $\dot{x} := \frac{dx}{dt}$. Given a "Lagrangian" $L = L(x, \dot{x})$ (x and $\dot{x}$ are understood as independent variables), the equation of motion of the particle is given by the Euler-Lagrange equation

   $$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1}$$

   Given $L = \frac{1}{2}\dot{x}^2$, compute the equation of motion and find the general solution. What is the physical interpretation of the equation and of its solution?

3. **Conformal field.**
   Consider a "conformal primary field" $\phi(z)$ that has the following series expansion:

   $$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-2}, \qquad z \in \mathbb{C}, \phi_n \in \mathbb{C}. \tag{2}$$

   Compute

   $$\oint_C \phi(z) dz, \tag{3}$$

   where C is a closed contour encircling $z = 0$.

Solutions

1. We verify this by explicit computation of both sides.

   (a)

$$\begin{aligned}[A, B] &= AB - BA \\ [B, A] &= BA - AB = -(AB - BA)\end{aligned} \tag{4}$$

   (b)

$$\begin{aligned}[A, B + C] &= A(B + C) - (B + C)A = AB + AC - BA - CA \\ [A, B] + [A, C] &= AB - BA + AC - CA\end{aligned} \tag{5}$$

   (c)

$$\begin{aligned}[AB, C] &= ABC - CAB \\ A[B, C] + [A, C]B &= A(BC - CB) + (AC - CA)B \\ &= ABC - ACB + ACB - CAB = ABC - CAB.\end{aligned} \tag{6}$$

   It is easy to show that the anticommutator is

$$\{AB, C\} = A\{B, C\} - \{A, C\}B. \tag{7}$$

   Let us verify this:

$$\begin{aligned}\{AB, C\} &= ABC + CAB \\ A\{B, C\} - \{A, C\}B &= A(BC + CB) - (AC + CA)B \\ &= ABC + ACB - ACB + CAB = ABC + CAB\end{aligned} \tag{8}$$

2. Since x and $\dot{x}$ are independent we have

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial \dot{x}} = \dot{x}. \tag{9}$$

   The Euler-Lagrange equations are

$$0 - \frac{d}{dt}\dot{x} = -\ddot{x} = -\frac{d^2 x}{dt^2} = 0 \quad \to \quad \ddot{x}(t) = 0. \tag{10}$$

   This differential equation is easily solved by integration:

$$\dot{x}(t) = A, \qquad x(t) = At + B, \tag{11}$$

   where $A, B \in \mathbb{R}$ are integration constants. The physical interpretation is as follows. If $x(t)$ is the position of the particle at time t, $\dot{x}$ is its velocity and $\ddot{x}$ is the acceleration (i.e. change of velocity over time). The equation of motion describes a particle that is not accelerated and moves at a constant velocity A. Its position at $t = 0$ is $x(0) = B$.

3. The series expansion of $\phi(z)$ is a Laurent series with poles at $z = 0$. Let us write down a few terms:

$$\phi(z) = \ldots + \phi_1 z^{-3} + \phi_0 z^{-2} + \phi_{-1} z^{-1} + \phi_{-2} + \phi_{-3} z + \ldots \tag{12}$$

The integration contour encircles the poles at $z = 0$, so the residue theorem tells us that

$$\oint_C \phi(z) dz = 2\pi i \mathrm{Res}_{z=0} \phi(z). \tag{13}$$

By definition, the residue is the coefficient of z^{-1} in the Laurent series expansion. Therefore we conclude

$$\oint_C \phi(z) dz = 2\pi i \phi_{-1}. \tag{14}$$