Readiness quiz

Problems

1. Determine whether the series converges or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3+\mathrm{i}^n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2\mathrm{i}n)^n}{3n^n}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5\mathrm{i}^n}{2\mathrm{i}-n^2}$.

2. Verify or disprove "There exist nonreal complex numbers z for which $|\exp(z)| = \exp(z)$."

3. We denote by $\mathrm{Log}(z)$ the principal branch of the complex logarithm function, which we denote as $\log(z)$. Evaluate $\log(z)$ and $\mathrm{Log}(z)$, for

   $$\text{(a) } z = 1, \qquad \text{(b) } z = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}, \qquad \text{(c) } z = 4+4\mathrm{i}, \qquad \text{(d) } z = -2\mathrm{e}^{2+\mathrm{i}\frac{2\pi}{11}}.$$

4. Let γ be the path that begins with the straight line segment from 1 to -1 and returns to 1 along a circular arc in the lower half plane, centred at 0. Evaluate the path integral

   $$\int_\gamma \bar{z}\,\mathrm{d}z.$$

5. Evaluate the integral

   $$\int_C \frac{z}{3-\mathrm{i}} + \frac{3+\mathrm{i}}{z-2}\,\mathrm{d}z$$

   where C is a simple closed positively oriented path which does not pass through $z = 2$.

6. Classify the finite isolated singularities. If you find a removable singularity, then redefine the function to make it analytic there. If you find a pole, then state its order. $f_3(z) = \dfrac{\mathrm{e}^z \sin(z)}{z^5 - \pi^4 z}$.

7. Use Jordan's lemma to evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,\mathrm{d}x$.

8. Let $U(x) = x - x^2$. Find the solution of

   $$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) && (x,t) \in (0,1)\times(0,\infty), && \text{(1.PDE)}\\ u(x,0) &= U(x) && x \in [0,1], && \text{(1.IC)}\\ u(0,t) &= 0 = u(1,t) && t \in [0,\infty). && \text{(1.BC)} \end{aligned}$$

   In this problem, you should use separation of variables, solution of a Sturm-Liouville problem, and the principle of linear superposition. You should calculate the coefficients explicitly.

9. Using Cramer's rule, solve

   $$\begin{pmatrix} \lambda & \mathrm{e}^{\mathrm{i}\lambda} \\ \lambda & \mathrm{e}^{-\mathrm{i}\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

   for x and y as functions of λ. For what values of $\lambda \in \mathbb{C}$ is this possible?

Solutions

1. (a) The terms in this series form the cyclic sequence

$$\frac{1}{3+\mathrm{i}}, \frac{1}{2}, \frac{1}{3-\mathrm{i}}, \frac{1}{4}, \ldots.$$

Because this sequence does not have limit 0, the series diverges.

(b) This is the series $\frac{1}{3}\sum_{n=1}^{\infty} b_n^n$ for $b_n = (1+2\mathrm{i}n)/n$, and

$$|b_n| = \frac{\sqrt{1+4n^2}}{n} = \sqrt{\frac{1}{n^2}+4} \to 2 > 1.$$

Therefore, by the root test, the original series diverges.

(c) This is the series $\sum_{n=1}^{\infty} a_n$ for $a_n = 5\mathrm{i}^n/(2\mathrm{i}-n^2)$. But, for $n \geqslant 2$

$$|a_n| \leqslant \frac{5}{(n-1)^2},$$

and the series $\sum_{n=2}^{\infty} \frac{5}{(n-1)^2}$ converges to $5\pi^2/6$. Hence, by the comparison test, the original series converges.

Note: Evaluating the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the *Basel problem.* The solution can be found in several ways; you are not expected to know any proof.

2. This is true. Let $z = x + \mathrm{i}2k\pi$ for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then, because $\exp(\mathrm{i}2k\pi) = 1$, it follows that

$$|\exp(z)| = \exp(\mathrm{Re}(x+\mathrm{i}2k\pi)) = \exp(x) = \exp(x)\exp(\mathrm{i}2k\pi) = \exp(z).$$

3. In each answer, we provide the set of values $\log(z)$ takes, parametrized by $k \in \mathbb{Z}$. The principal value, $\mathrm{Log}(z)$, is the value corresponding to $k = 0$.

(a) $2k\pi\mathrm{i}$.

(b) $\ln(2) + \mathrm{i}\left(\frac{2\pi}{3} + 2k\pi\right)$.

(c) $5\ln(2) + \mathrm{i}\left(\frac{\pi}{4} + 2k\pi\right)$.

(d) $2 + \ln(2) + \mathrm{i}\left(\frac{-9\pi}{11} + 2k\pi\right)$.

4. For z on the unit circle, $\bar{z} = 1/z$. Hence, if $z = \exp(\mathrm{i}t)$, then $\bar{z} = \exp(-\mathrm{i}t)$. However, on the real line $\bar{z} = z$. We calculate

$$\begin{aligned}\int_\gamma \bar{z}\,\mathrm{d}z &= \int_0^2 1 - t\,\mathrm{d}t + \int_{-\pi}^0 \exp(-\mathrm{i}t)\mathrm{i}\exp(\mathrm{i}t)\,\mathrm{d}t \\ &= \left[t - \frac{t^2}{2}\right]_0^2 + \mathrm{i}\Big[t\Big]_{-\pi}^0 \\ &= 2 - 2 - 0 + 0 + \mathrm{i}(0+\pi) = \mathrm{i}\pi.\end{aligned}$$

5. The first term is entire, so the integral is equal to

$$(3+\mathrm{i})\int_C \frac{1}{z-2}\,\mathrm{d}z.$$

If C encloses $z = 2$, then, as it is a loop about a simple pole, the integral evaluates to $(3-\mathrm{i})2\pi\mathrm{i}$. If C does not enclose $z = 2$, then, by Cauchy's theorem, the integral evaluates to 0.

6. We factorize the denominator

$$f_3(z) = \frac{e^z \sin(z)}{(z - i\pi)(z + i\pi)z(z - \pi)(z + \pi)}.$$

The numerator and denominator are both entire functions, so the only points of nonanlyticity are zeros of the denominator. The denominator is a polynomial, so all its zeros correspond to isolated singularities of f_3.

At each of $z = \pm i\pi$, the numerator is nonzero and the denominator has a zero of order 1, so f_3 has a pole of order 1.

At each of $z = 0, \pi, -\pi$, both the denominator and the numerator have zeros of order 1, so these correspond to removable singularities of f_3. Indeed, we may redefine f_3 as

$$f_3(z) = \begin{cases} \frac{-1}{\pi^4} & \text{if } z = 0, \\ \frac{-e^{\pi}}{4\pi^4} & \text{if } z = \pi, \\ \frac{e^{-\pi}}{4\pi^4} & \text{if } z = -\pi, \\ \frac{e^z \sin(z)}{z^5 - \pi^4 z} & \text{otherwise,} \end{cases}$$

to remove those singularities. In doing this, we have used the Taylor series for $\sin(z)$ to obtain limits

$$\lim_{z\to 0} \frac{\sin z}{z} = 1, \qquad \lim_{z\to\pi} \frac{\sin(z - \pi)}{z - \pi} = -1, \qquad \lim_{z\to-\pi} \frac{\sin(z + \pi)}{z + \pi} = -1.$$

7. The integrand is dominated by $1/(x^2 + 1)^2$, so the improper integral converges. Hence it is equal to its Cauchy principal value, which is equal to

$$\operatorname{Re}\left(\mathrm{VP}\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)^2}\, dx\right).$$

For $R > 0$, let σ_R be the semicircular path in the upper half plane, centred at 0 with radius R, extending from R to $-R$. Let γ_R be the straight line path extending from $-R$ to R along the real axis. By Cauchy's theorem and a residue calculation, provided $R > 1$,

$$\begin{aligned}\int_{[\gamma_R,\sigma_R]} \frac{e^{iz}}{(z^2 + 1)^2}\, dz &= 2\pi i \operatorname*{Res}_{z=i}\left(\frac{e^{iz}}{(z^2 + 1)^2}\right) \\ &= 2\pi i \lim_{z\to i}\left(\frac{d}{dz}\left[\frac{e^{iz}}{(z + i)^2}\right]\right) \\ &= 2\pi i \lim_{z\to i}\left(\frac{(z + i)^2 i e^{iz} - e^{iz}(2z + 2i)}{(z + i)^4}\right) \\ &= 2\pi i \lim_{z\to i}\left(\frac{-4ie^{-1} - e^{-1}4i}{16}\right) = \frac{\pi}{e}.\end{aligned}$$

By Jordan's lemma,

$$\lim_{R\to\infty} \int_{\sigma_R} \frac{e^{iz}}{(z^2 + 1)^2}\, dz = 0.$$

Hence

$$\frac{\pi}{e} = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{(z^2 + 1)^2}\, dz = \mathrm{VP}\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + 1)^2}\, dx.$$

Because this integral is equal to its own real part, the original integral is equal to π/e.

8. Suppose initially that $u(x,t) = X(x)T(t)$, for some functions X and T to be determined. Then equation (1.PDE) implies that

$$XT' = X''T \qquad \Rightarrow \qquad \frac{T'}{T} = \frac{X''}{X}.$$

Now the left side is a function of t only, so it cannot change when x changes, and the right side depends only on x so it is independent of t. Therefore, both sides of this equation must be constant. We shall call that constant $-\lambda$. Hence we obtain ODE

$$T'(t) = -\lambda T(t) \qquad \text{and} \qquad X''(x) = -\lambda X(x).$$

Under our separation assumption, equations (1.BC) reduce to $X(0) = 0 = X(1)$. Combining these with the ODE for X, we obtain a Sturm-Liouville problem, which we shall solve next.

The X ODE has solutions

$$X(x) = \begin{cases} A + Bx & \text{if } 0 = \lambda, \\ A\cosh(kx) + B\sinh(kx) & \text{if } 0 > \lambda = -k^2;\ k > 0, \\ A\cos(kx) + B\sin(kx) & \text{if } 0 < \lambda = k^2;\ k > 0, \end{cases}$$

in each of which the constants A and B are free. In the first case, $X(0) = 0$ implies that $A = 0$, and $X(1) = 0$ then implies that $B = 0$, so $0 = \lambda$ yields no nontrivial solutions. In the second case, $X(0) = 0$ implies that $A = 0$, and, knowing that $k > 0$, $X(1) = 0$ then implies that $B = 0$, so $0 > \lambda$ yields no nontrivial solutions. In the third case, $X(0) = 0$ implies that $A = 0$, and $X(1) = 0$ then implies that k is a positive integer multiple of π. Therefore, the solutions of the Sturm-Liouville problem are

$$\text{eigenfunctions } X_n(x) = \sin(n\pi x) \text{ and eigenvalues } \lambda_n = n^2\pi^2, \text{ for } n \in \mathbb{N}.$$

Corresponding to each λ_n, the ODE for T has solution $T(t) = \mathrm{e}^{-\lambda_n t} = \mathrm{e}^{-n^2\pi^2 t}$. Therefore, the separated solutions of equation (1.PDE) are $u(x,t) = \sin(n\pi x)\mathrm{e}^{-n^2\pi^2 t}$, for positive integers n. However, none of these separated solutions evaluates to $U(x)$ at $t = 0$, so we must now abandon our original separation assumption.

Using the principle of linear superposition, any linear combination of the separated solutions also satisfies equation (1.PDE) and equations (1.BC). Therefore, we seek a solution of the full problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\mathrm{e}^{-n^2\pi^2 t},$$

for coefficients B_n to be determined. Evaluating at $t = 0$ and applying equation (1.IC), we obtain

$$x - x^2 = U(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\mathrm{e}^{-n^2\pi^2 0} = \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

For each positive integer m, we take the inner product of both sides of this equation against the function $\sin(m\pi x)$, and use the orthogonality property of these sine functions to determine

$$\begin{aligned} \int_0^1 (x - x^2)\sin(m\pi x)\,\mathrm{d}x &= \int_0^1 \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} B_n \frac{\delta_{m,n}}{2} = \frac{B_m}{2}. \end{aligned}$$

Hence, integrating by parts thrice,

$$\begin{aligned}
B_m &= 2\int_0^1 (x - x^2)\sin(m\pi x)\,\mathrm{d}x \\
&= \left[-(x-x^2)\frac{\cos(m\pi x)}{m\pi} + (1-2x)\frac{\sin(m\pi x)}{m^2\pi^2} - 2\frac{\cos(m\pi x)}{m^3\pi^3}\right]_{x=0}^{x=1} \\
&= \qquad (0-0) \qquad + \qquad (0-0) \qquad - \frac{2}{m^3\pi^3}\left(\cos(m\pi) - 1\right) \\
&= \begin{cases} \frac{4}{m^3\pi^3} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

Therefore,

$$u(x,t) = \sum_{j=1}^{\infty} \frac{4}{[2j-1]^3\pi^3}\sin([2j-1]\pi x)\mathrm{e}^{-[2j-1]^2\pi^2 t}.$$

9. By Cramer's rule, the solution is given by

$$x(\lambda) = \frac{\det\begin{pmatrix} 3 & \mathrm{e}^{\mathrm{i}\lambda} \\ -1 & \mathrm{e}^{-\mathrm{i}\lambda}\end{pmatrix}}{\det\begin{pmatrix} \lambda & \mathrm{e}^{\mathrm{i}\lambda} \\ \lambda & \mathrm{e}^{-\mathrm{i}\lambda}\end{pmatrix}} = \frac{3\mathrm{e}^{-\mathrm{i}\lambda} + \mathrm{e}^{\mathrm{i}\lambda}}{\lambda\left(\mathrm{e}^{-\mathrm{i}\lambda} - \mathrm{e}^{\mathrm{i}\lambda}\right)} = \frac{\mathrm{e}^{-\mathrm{i}\lambda} + \cos(\lambda)}{-\mathrm{i}\lambda\sin(\lambda)},$$

$$y(\lambda) = \frac{\det\begin{pmatrix} \lambda & 3 \\ \lambda & -1\end{pmatrix}}{\det\begin{pmatrix} \lambda & \mathrm{e}^{\mathrm{i}\lambda} \\ \lambda & \mathrm{e}^{-\mathrm{i}\lambda}\end{pmatrix}} = \frac{-\lambda - 3\lambda}{\lambda\left(\mathrm{e}^{-\mathrm{i}\lambda} - \mathrm{e}^{\mathrm{i}\lambda}\right)} = \frac{2}{\mathrm{i}\sin(\lambda)},$$

wherever it exists. The solution exists if and only if the system is nondegenerate, that is if and only if the determinant of the linear system is nonzero, i.e. when

$$0 \neq \det\begin{pmatrix} \lambda & \mathrm{e}^{\mathrm{i}\lambda} \\ \lambda & \mathrm{e}^{-\mathrm{i}\lambda}\end{pmatrix} = -2\mathrm{i}\lambda\sin(\lambda),$$

so when λ is not an integer multiple of π.