AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

The Improving Mathematics Education in Schools (TIMES) Project

SCALE DRAWINGS AND SIMILARITY

A guide for teachers - Years 8–10

MEASUREMENT AND GEOMETRY : Module 22

June 2011

YEARS 8 10

Scale drawings and Similarity

(Measurement and Geometry: Module 22)

For teachers of Primary and Secondary Mathematics

510

Cover design, Layout design and Typesetting by Claire Ho

The Improving Mathematics Education in Schools (TIMES) Project 2009-2011 was funded by the Australian Government Department of Education, Employment and Workplace Relations.

The Improving Mathematics Education in Schools (TIMES) Project

SCALE DRAWINGS AND SIMILARITY

A guide for teachers - Years 8–10

MEASUREMENT AND GEOMETRY : Module 22

June 2011

Peter Brown
Michael Evans
David Hunt
Janine McIntosh
Bill Pender
Jacqui Ramagge

YEARS 8 10

SCALE DRAWINGS AND SIMILARITY

ASSUMED KNOWLEDGE

- Translations, reflections and rotations.
- Elementary Euclidean geometry, including constructions and transversals to parallel lines.
- Ratio, fractions and the unitary method.
- Sections 3–6 on similarity assume that congruence has already been studied.

MOTIVATION

Scale drawings are used when we increase or reduce the size of an object so that it fits nicely on a page or computer screen. For example, we would want to reduce the size when drawing:

- the plan of the facade of a building
- a map of a suburb or a country
- a photograph of a distant galaxy,

and we would want to increase the size when drawing:

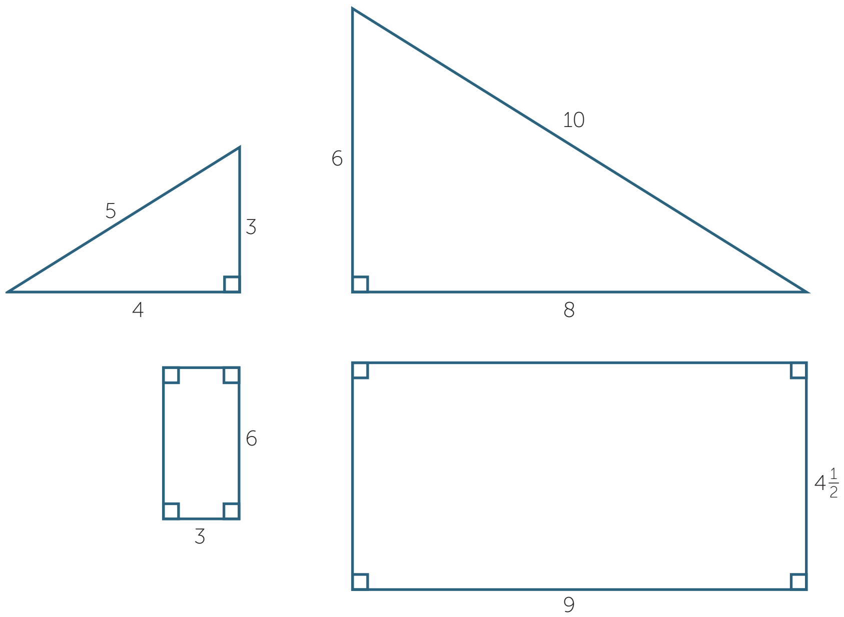
- a diagram of a printed circuit
- a magnified photograph of an insect
- a drawing of a human cell.

The proportional increase or decrease in lengths is called the scale of the drawing. It is usually expressed in terms of a ratio, so the topic of scale drawings is closely related to ratios and fractions.

The transformation that produces a scale drawing is an enlargement. An enlargement transformation preserves the shape of the figure, but increases or decreases all distances by a constant ratio.

This is different from the three transformations that we have already introduced – translations, rotations and reflections all produce an image that is the same size and shape as the original figure.

The module, *Congruence* studied congruent figures, which are figures that can be mapped one to the other by a sequence of translations, rotations and reflections. Figures that can be mapped one to the other by these transformations and enlargements are called similar. Thus two figures are similar if an enlargement of one is congruent to the other.



Any two figures that have the same shape are similar. Matching angles in similar figures are equal, but matching lengths in two similar figures are all in the same ratio. This constant ratio is the same ratio that appears in scale drawings and enlargements.

The theory of similarity develops in the same way as congruence. First, most situations involving similarity can be reduced to similar triangles, and we shall establish four similarity tests for triangles, corresponding to the four congruence tests for triangles. Secondly, just as congruence was used to prove many basic theorems about triangles and special quadrilaterals, so similarity will allow us to establish further important theorems in geometry.

The treatment of similarity and enlargements in this module has been guided by well-established classroom practice. Scale drawings and enlargements are usually discussed a year or so earlier than similarity, and these topics therefore receive a self-contained treatment in Sections 1–2.

Section 3 can then introduce similarity in terms of enlargement transformations. In Section 4–6, the discussion of similar triangles begins with the AAA similarity test, which is usually considered the most straightforward test to use.

CONTENT

SCALE DRAWINGS

A scale drawing has exactly the same shape as the original object, but usually has a different size.

This means that:

- Matching angles in the original and the drawing are equal, and
- All the lengths of the original object are reduced or magnified in the drawing in exactly the same ratio. This ratio is called the scale of the drawing.

  Scale = length on the drawing : length on the actual object

The scale of a drawing is thus given as the ratio of two numbers. For example, in the photograph below, showing the side of a train engine,

Scale = 1 : 200.

This means that a length of 1 cm on the photograph above corresponds to a length of 200 cm , or 2 metres, on the actual engine. The scale can also be written as the ratio of two lengths,

Scale = 1 cm : 2 m

This second notation, using the ratio of two lengths with different units, is often more convenient for maps, where, for example, a scale of 1 cm : 100 km is easier to interpret than 1 : 10 000 000.

EXAMPLE

a Measure the overall length of the engine in the photograph above, including the couplings. Then use the scale to find the approximate length of the actual engine.

b Find the approximate width and height of the doors, and the area of each door.

SOLUTION

a Engine length in photograph = 11 cm

$\boxed{\times 200}$ Actual engine length = 2200 cm

= 22 metres

b Width of door in photograph = 0.35 cm

$\boxed{\times 200}$ Actual width of door = 70 cm

= 0.7 metres

Height of door in photograph = 1.1 cm

$\boxed{\times 200}$ Actual height of door = 220 cm

= 2.2 metres

Area of door = 0.7×2.2

= 1.54 m^2

EXAMPLE

a Convert each scale to a ratio of two numbers.

i 6 m : 1 cm

ii 6 cm : 10 km

b Convert each scale to a ratio of lengths in the units indicated.

i 400 : 1 = 1 cm : ... mm

ii 1 : 150 000 = 1 cm : ...km

SOLUTION

a i 6 m : 1 cm = 600 cm : 1 cm

= 600 : 1.

ii 6 cm : 10 km = 6 cm : 1 000 000 cm

= 3 : 500 000.

When the scale is written as a ratio of two numbers, the ratio is normally cancelled down to simplest form.

b i 400 : 1 = 400 cm : 1 cm

= 4 cm : 0.1 mm

= 1 cm : 0.025 mm

ii 1 : 150 000 = 1 cm : 150 000 cm

= 1 cm : 1500 m

= 1 cm : 1.5 km

SCALE DRAWINGS:

- A **scale drawing** of an object has the same shape as the object, but a different size.
- The **scale** of the drawing is the ratio length on the drawing : length on the actual object.
- A scale can be written as the ratio of two lengths, or as the ratio of two numbers.
  For example: scale = 1 cm : 5 m or scale = 1 : 500
- Matching angles are equal, and the ratio of matching lengths equals the scale.

EXERCISE 1

The photograph below shows the facade of the neo-Gothic St Petrus and Paulus Church in Ostend, Belgium. Assume that the person dressed in a black suit in the bottom right-hand corner of the photograph is 2 metres tall.

a What is the approximate scale of the photograph?

b What is the approximate height of the top of each spire from the ground?

Scales involving very large and very small distances – Scientific notation

Science routinely uses scale drawings and photographs of astronomical and microscopic objects. When the scale of such drawings is expressed in terms of pure numbers, the numbers are so large, or so small, that scientific notation becomes appropriate.

EXAMPLE

a A distant galaxy with diameter 100 000 light years appears on a photograph as an image with diameter 3 cm. Given that a light year is very roughly 10^{13} km, express this scale as the ratio of two numbers.

b A molecule with diameter 10 Angstrom units across appears on an electron microscope photograph as an image with diameter 5 cm. Given that 1 Angstrom unit is 10^{-10} metres, express this scale as the ratio of two numbers.

SOLUTION

a $3\text{ cm} : 10^5$ light years $= 3\text{ cm} : 10^5 \times 10^{13} \times 10^5\text{ cm} = 3 : 10^{23}$.

b $5\text{ cm} : 10$ Angstrom units $= 5\text{ cm} : 10 \times 10^{-10} \times 100\text{ cm} = 5 : 10^{-7} = 5 \times 10^7 : 1$.

The best form for the answer to this question is $1 : 2 \times 10^{-8}$ or $5 \times 10^7 : 1$.

ENLARGEMENTS

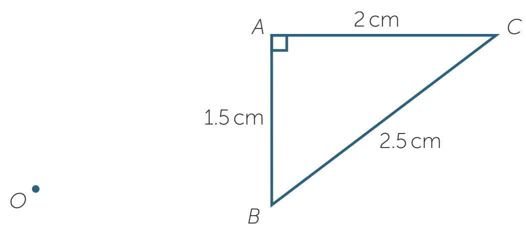
We have already dealt with three transformations of the plane – translations, rotations and reflections. These three transformations are examples of congruence transformations, because the image of a figure under one of these transformations is congruent to the original. Indeed, we defined two figures to be congruent if one could be mapped to the other by a sequence of these transformations.

This section introduces a fourth type of transformation of the plane called an **enlargement**, in which all lengths are increased or decreased in the same ratio.

To specify an enlargement, we need to specify two things:

- The **centre** O of the enlargement. The centre of enlargement stays fixed in the one place, while the enlargement expands or shrinks everything else around it.
- An **enlargement factor** k (or **enlargement ratio** $1 : k$). The distances of all points from the enlargement centre increase or decrease by this factor.

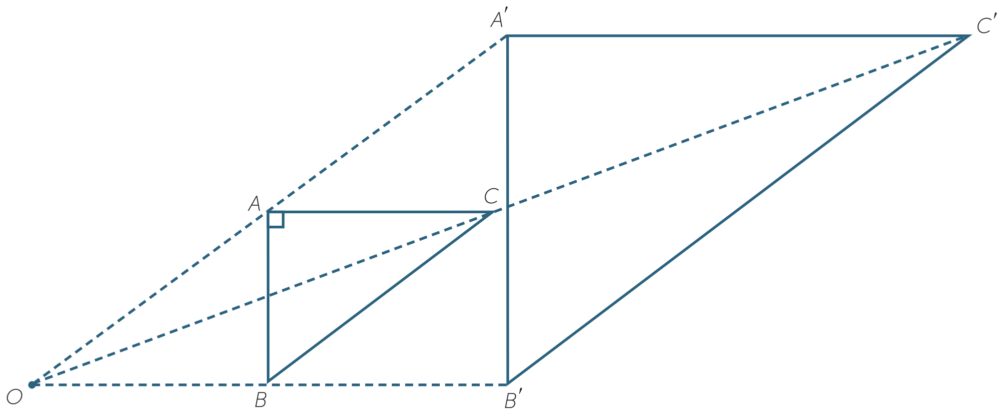
In this module, we will only deal with positive enlargement factors. For example, the diagram below shows a point O and a triangle ABC.



The next figure below shows how to construct the **image** $\triangle A'B'C'$ of $\triangle ABC$ under the enlargement with centre O and enlargement factor 2. Because the construction only involves doubling, it can be done with straight edge and compasses.

- Join OA, then extend OA to OA' so that $OA' = 2 \times OA$.
- Join OB, then extend OB to OB' so that $OB' = 2 \times OB$.

- Join OC, then extend OC to OC' so that $OC' = 2 \times OC$.
- Join up the triangle $A'B'C'$, which is called **the image** of $\triangle ABC$.



We can use this diagram to verify three important properties of enlargements.

First, it is easy to verify with compasses that each side of the image triangle $A'B'C'$ is twice the length of the matching sides of the original triangle ABC. That is,

$$\frac{A'B'}{AB} = 2, \quad \frac{B'C'}{BC} = 2 \quad \text{and} \quad \frac{C'A'}{CA} = 2.$$

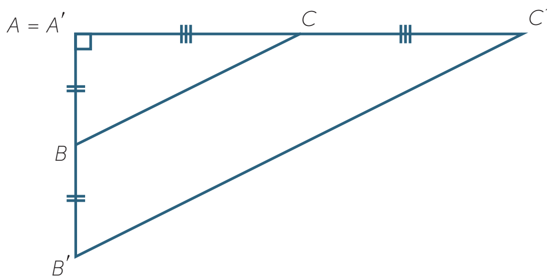
Secondly, we can verify that each angle of $\triangle A'B'C'$ is equal to the matching angle of $\triangle ABC$.

$$\angle B'A'C' = \angle BAC, \quad \angle A'C'B' = \angle ACB \quad \text{and} \quad \angle C'B'A' = \angle CBA.$$

Thirdly, since the corresponding angles are equal, each side of the image triangle is parallel to the matching side of the original triangle:

$$A'B' \parallel AB, \quad B'C' \parallel BC \quad \text{and} \quad C'A' \parallel CA.$$

It follows from the first and second points above that the image $\triangle A'B'C'$ is a scale drawing of the original figure $\triangle ABC$, with ratio 1 : 2. If we had used a different centre of enlargement, but the same enlargement factor 2, then the image $\triangle A'B'C'$ would end up in a different place, but it would still be a scale drawing with ratio 1 : 2. For example, the diagram below uses the vertex A as the centre of enlargement.

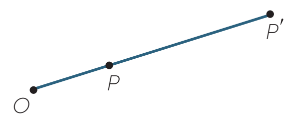


Enlargements are reversible. In our examples above, $\triangle ABC$ is the image of $\triangle A'B'C'$ under an enlargement with the same centre O, but enlargement factor $\frac{1}{2}$, and $\triangle ABC$ is a scale drawing of $\triangle A'B'C'$ with ratio 2 : 1. To perform the construction given the point O and the triangle $A'B'C'$, we would obtain the points A, B and C by bisecting each interval OA', OB' and OC'.

An enlargement can have any positive real number as its enlargement factor. The definition and the properties of enlargements are summarised in the box below.

ENLARGEMENT TRANSFORMATIONS:

- An enlargement is specified by a **centre of enlargement** and an **enlargement factor** $k > 0$.
- The enlargement moves each point P to a point P' on the ray OP.
- The distance OP' is k times the distance OP. That is, $OP' = k \times OP$.



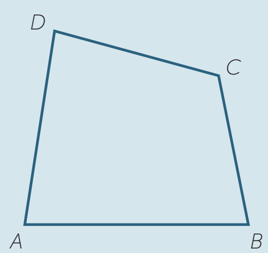
- If $k > 1$, then the image is larger than the original.
- If $k < 1$, then the image is smaller than the original.
- If $k = 1$, then no point moves, and the image is the same as the original.

PROPERTIES OF AN ENLARGEMENT:

- The image of an interval is k times the length of the original interval.
- The image of an angle has the same size as the original angle.
- The image of an interval is parallel to the original interval.

The following exercise is intended to present two further examples of enlargements, and to confirm that in these two cases, each distance is increased by the enlargement factor.

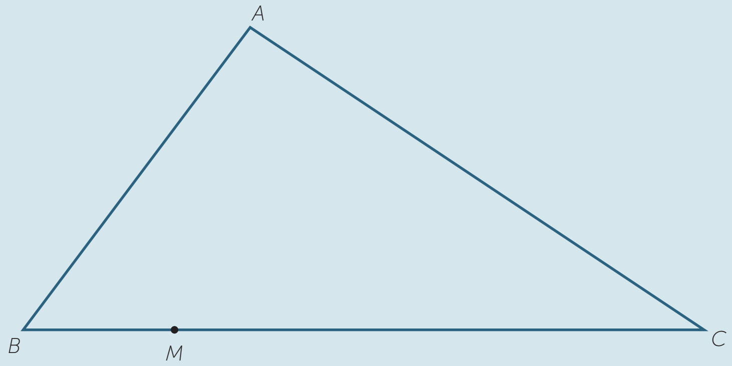
EXERCISE 2



a i Using A as the centre of enlargement and enlargement factor 3, construct the image $AB'C'D'$ of the quadrilateral $ABCD$.

ii Confirm that:

- $B'C' = 3 \times BC$ and $D'C' = 3 \times DC$, and
- the images $\angle B'$, $\angle C'$ and $\angle D'$ are each equal to the original angle.



b **i** Using M as the centre of enlargement and enlargement factor $\frac{1}{2}$, construct the image $\triangle A'B'C'$ of the isosceles triangle ABC.

ii Confirm that:

- each interval of the image has half the length of the original interval, and
- each angle of the image is equal to the original angle.

SIMILARITY

Two plane figures are called **similar** if an enlargement of one figure is congruent to the other.

That is, if one can be mapped to the other by a sequence of translations, rotations, reflections and enlargements. Similar figures thus have the same shape, but not necessarily the same size.

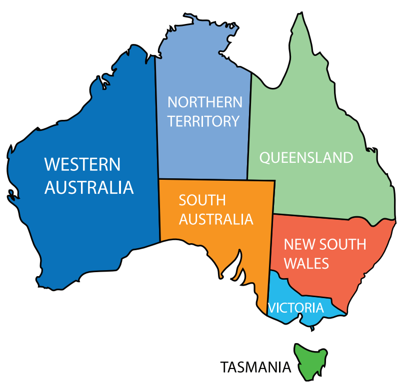
Thus a scale drawing of a two-dimensional object, and an enlargement of a plane figure, bith produce figures similar to the original.

It follows from the properties of enlargements discussed in the previous section that when two figures are similar:

- matching angles are equal, and
- matching distances are in a constant ratio.

Conversely, if two figures can be matched up so that these two conditions apply, we can enlarge the first figure by the constant ratio. The enlarged figure is thus congruent to the second figure, so the first and second figures are similar.

The constant ratio is called the **similarity ratio** or **similarity factor**. If two figures are similar with similarity ratio 1 : 1, then the two figures are congruent, otherwise they will have different sizes.



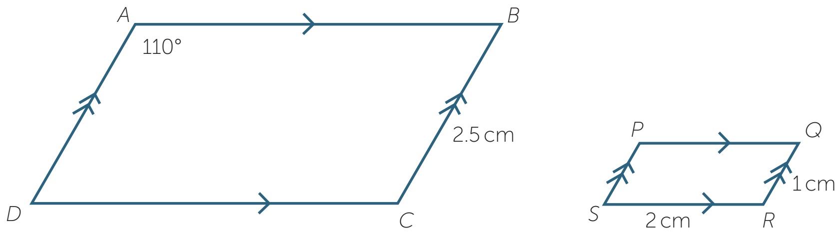
The two maps above are similar to each other. Each distance on the second map is twice the matching distance on the first map, so we say that the similarity ratio is 1 : 2, or that the similarity factor is 2.

In the language of the first section, each map is a scale drawing of Australia (ignoring the curvature of the Earth), and each map is a scale drawing of the other map. We can continue to solve problems in similarity using the unitary method approach adopted with scale drawings, but it is more usual to use algebra and fractions, as in the following example.

As with congruence, vertices must be kept in matching order.

EXAMPLE

The two parallelograms below are known to be similar.



a What is the size of $\angle P$?

b Find the base DC of the large parallelogram.

c Find the similarity ratio of:

i $PQRS$ to $ABCD$, ii $ABCD$ to $PQRS$.

SOLUTION

a $\angle P = 110°$ (matching angles of similar figures)

b $\frac{DC}{SR} = \frac{BC}{QR}$ (matching sides of similar figures) or $\frac{DC}{BC} = \frac{SR}{QR}$

$\frac{DC}{2} = \frac{2.5}{1}$ $\frac{DC}{2.5} = \frac{2}{1}$

$DC = 5\,\text{cm}$ $DC = 5\,\text{cm}$

c i 1 : 2.5 or 2 : 5 ii 2.5 : 1 or 5 : 2

Both methods of writing the algebra of part **b** are equally good.

- In the first method, one pair of matching intervals is on the left of the equation, the other pair is on the right.
- In the second method, two intervals of one figure are on the left of the equation, and two matching intervals of the other figure are on the right.

One should use whichever method seems more natural at the time – this will often depend on the particular problem. The subsequent algebra will be easier, however, if one always starts by placing the unknown length on the top of the left-hand side.

It is worthwhile writing down the algebra that proves that the two identities are equivalent:

$$\frac{DC}{SR} = \frac{BC}{QR}$$

$DC \times QR = BC \times SR$ (multiplying both sides by SR and by QR)

$\frac{DC}{BC} = \frac{SR}{QR}$ (dividing both sides by BC and by QR).

The middle identity says that two products of lengths are equal. In later work, particularly with circle geometry, this middle identity will be a common way of expressing the consequences of similarity.

Some may prefer to find the length DC by continuing with the arithmetic approach used in scale drawings. Such an approach is not recommended, because it is unsuitable when similarity is applied to more general situations where the similarity ratio is not readily apparent.

Comparing the sides QR and BC, the similarity ratio is 1 : 2.5.

Hence $DC = 2.5 \times SR$

$= 2.5 \times 2$

$= 5\,\text{cm}$

EXERCISE 3

Is each statement true or false? If it is false, can you qualify the statement to make it true?

a Any two squares are similar.

b Any two rectangles are similar.

c Any two rhombuses are similar.

d Any two circles are similar.

e Any two sectors of circles are similar.

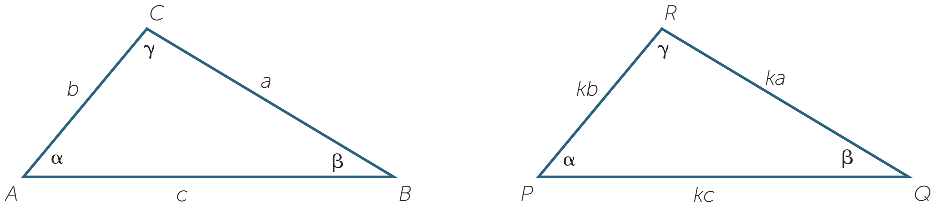
f Any two equilateral triangles are similar.

g Any two isosceles triangles are similar.

FOUR SIMILARITY TESTS FOR TRIANGLES

Two figures are congruent when they are similar with similarity factor 1. Similarity is thus a generalisation of congruence, and we would expect the theory of similarity to proceed along similar lines to the theory of congruence that we have already developed. As with congruence, discussions involving the similarity of straight-sided figures can be reduced to discussions of similar triangles.

Triangles have three vertex angles and three side lengths. If the vertices of two triangles can be matched up so that matching angles are equal and matching sides are in a constant ratio, then the two triangles are similar.



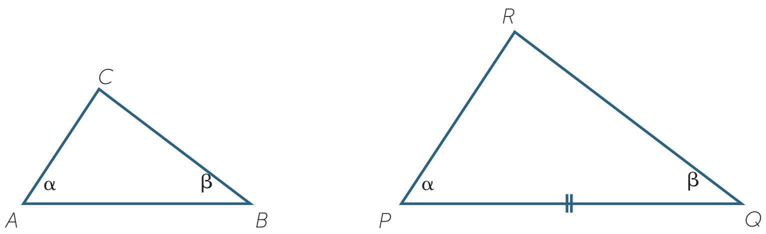
This can be demonstrated with the two triangles above, where matching angles are equal and matching sides are in the ratio 1 : k. We can enlarge $\triangle ABC$ on the left by an enlargement with centre A and enlargement factor k to produce the triangle $AB'C'$, which is congruent to $\triangle PQR$ on the right. This shows that $\triangle ABC$ is similar to $\triangle PQR$.

As with congruent triangles, however, we do not need to check all six measurements to be sure that the two triangles are similar. There are four standard tests for two triangles to be similar, corresponding to the four standard congruence tests.

We can develop each similarity test from the corresponding congruence test. First, the AAS congruence test corresponds to the AAA similarity test, which states:

- **The AAA similarity test:** If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

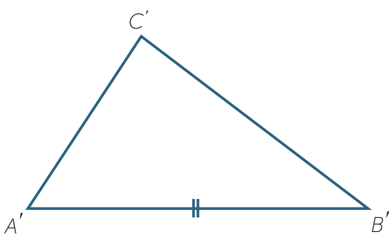
If two triangles have two pairs of equal angles, then their third pair is also equal because the angle sum of a triangle is 180°. Thus two such triangles are called equi-angular, and the test is often referred to as the AAA similarity test.



To prove this test, let $\triangle ABC$ and $\triangle PQR$ be triangles with

$$\angle A = \angle P = \alpha \text{ and } \angle B = \angle Q = \beta.$$

Enlarge $\triangle ABC$ to $\triangle A'B'C'$ by a suitable enlargement factor so that $A'B' = PQ$.

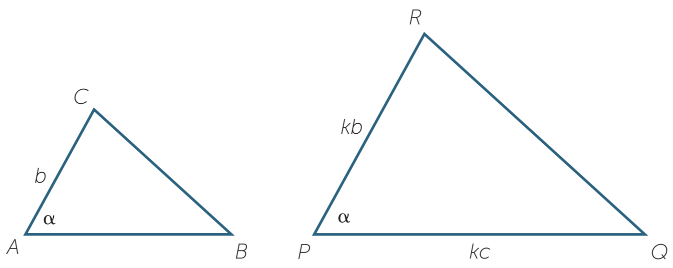


Then $\angle A' = \alpha$ and $\angle B' = \beta$ because angles are preserved by enlargements.

Hence $\triangle A'B'C'$ is congruent to $\triangle PQR$ by the AAS congruence test, so $\triangle ABC$ is similar to $\triangle PQR$, because an enlargement of $\triangle ABC$ is congruent to $\triangle PQR$.

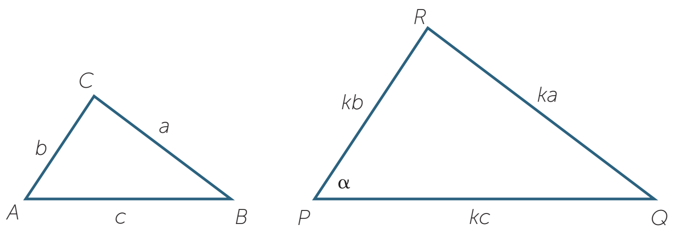
This same argument can be applied to develop a similarity test from each of the other three congruence tests. From the SAS congruence test we obtain the SAS similarity test:

- **The SAS similarity test:** If the ratio of the lengths of two sides of one triangle is equal to the ratio of the lengths of two sides of another triangle, and the included angles are equal, then the two triangles are similar.



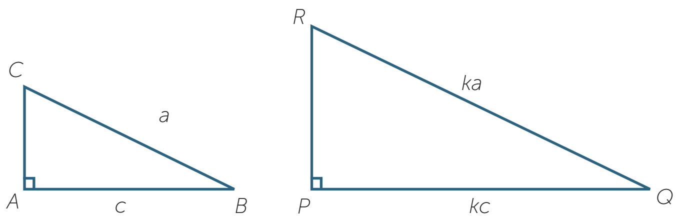
From the SSS congruence test we obtain the SSS similarity test:

- **The SSS similarity test:** If we can match up the sides of one triangle with the sides of another so that the ratios of matching sides are equal, then the two triangles are similar.



From the RHS congruence test we obtain the RHS similarity test:

- **The RHS similarity test:** If the ratio of the hypotenuse and one side of a right-angled triangle is equal to the ratio of the hypotenuse and one side of another right-angled triangle, then the two triangles are similar.



EXERCISE 4

Develop the SSS similarity test from the SSS congruence test. Use the same argument as was used above to develop the AAA similarity test from the AAS congruence test.

Refer to the triangles *ABC* and *PQR* that illustrate the statement of the SSS similarity test above.

Using the AAA similarity test

Similarity problems are set out in much the same way as congruence problems.
In particular, vertices must always be written in matching order in every statement.

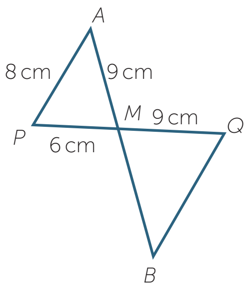
The following example shows how to use the AAA similarity test to find lengths. There are two ways of writing the ratio condition on the sides, as discussed in the previous section, and the first example has been solved both ways.

EXAMPLE

In the diagram to the right:

a Show that ΔAMP is similar to ΔBMQ.

b Hence find: i BQ, ii BM.



SOLUTION

a In the triangles AMP and BMQ:

$\angle AMP = \angle BMQ$ (vertically opposite angles)

$\angle A = \angle B$ (alternate angles, $AP \parallel QB$)

so ΔAMP is similar to ΔBMQ (AAA similarity test).

b i Hence $\frac{BQ}{AP} = \frac{QM}{PM}$ (matching sides of similar triangles) or $\frac{BQ}{QM} = \frac{AP}{PM}$

$\frac{BQ}{8} = \frac{9}{6}$ $\qquad$ $\frac{BQ}{9} = \frac{8}{6}$

$BQ = 8 \times \frac{3}{2}$ $\qquad$ $BQ = 9 \times \frac{4}{3}$

$= 12\,\text{cm}$ $\qquad$ $= 12\,\text{cm}$

ii Also $\frac{MB}{MA} = \frac{MQ}{MP}$ (matching sides of similar triangles)

$\frac{MB}{9} = \frac{9}{6}$

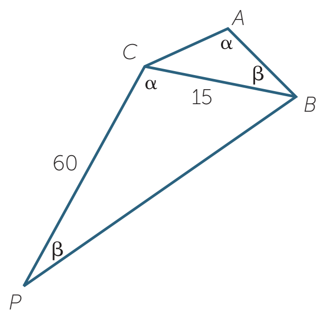
$MB = 9 \times \frac{3}{2}$

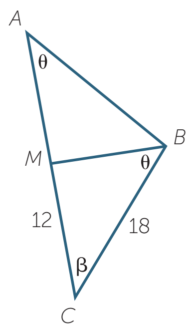
$= 13\frac{1}{2}\text{cm}$

EXAMPLE

a In each diagram below, complete the similarity statement 'ABC is similar to…', stating the similarity test used.

b Hence find the length of AC in each case.





SOLUTION

a $\triangle ABC$ is similar to $\triangle CPB$ (AAA similarity test)

$\triangle ABC$ is similar to $\triangle BMC$ (AAA similarity test)

b We apply matching sides of similar triangles in both diagrams.

In the first diagram,

$$\frac{AC}{BC} = \frac{CB}{PB}$$

$$\frac{AC}{15} = \frac{15}{60}$$

$$AC = 15 \times 14$$

$$= 334.$$

In the second diagram,

$$\frac{AC}{BC} = \frac{BC}{MC}$$

$$\frac{AC}{18} = \frac{18}{12}$$

$$AC = 18 \times 32$$

$$= 27.$$

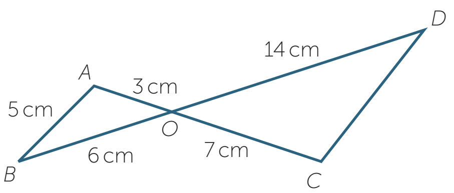
Using the SAS similarity test

The first example below displays both ways of writing the ratio condition on the sides, first in the proof of similarity in part **a**, then in the subsequence calculation of a side length.

Part **c** shows how the equality of two angles can be used to prove that two lines are parallel.

EXAMPLE

In the diagram to the right:



a Prove that $\triangle AOB$ is similar to $\triangle COD$.

b Find the length of CD.

c Prove that $AB \parallel DC$.

SOLUTION

a In the triangles AOB and COD:

$\angle AOB = \angle COD$ (vertically opposite angles)

$\frac{AO}{BO} = \frac{3}{7}$ (given) or $\frac{AO}{BO} = \frac{1}{2}$

$\frac{BO}{DO} = \frac{3}{7}$ (given) $\quad \frac{CO}{DO} = \frac{1}{2}$

so $\triangle AOB$ is similar to $\triangle COD$ (SAS similarity test).

b $\frac{CD}{AB} = \frac{CO}{AO}$ (matching sides of similar triangles) or $\frac{CD}{CO} = \frac{AB}{AO}$

$\frac{CD}{5} = \frac{3}{7}$ $\quad \frac{CD}{7} = \frac{5}{3}$

$CD = 11\frac{2}{3}$cm. $\quad CD = 11\frac{2}{3}$cm.

c $\angle A = \angle C$ (matching angles of similar triangles), so $AB \parallel DC$ (alternate angles are equal).

The second method in parts **a** and **b** above compares lengths within each triangle.

Using the SSS similarity test

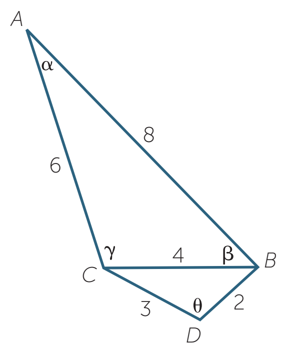
The following example shows how to use the SSS similarity test to prove that angles are equal.

EXAMPLE

a Prove that the two triangles in the diagram are similar.

b Which of the marked angles are equal?

SOLUTION



a In the triangles ABC and CBD:

$AB = 2 \times CB$ (given)

$BC = 2 \times BD$ (given)

$CA = 2 \times DC$ (given)

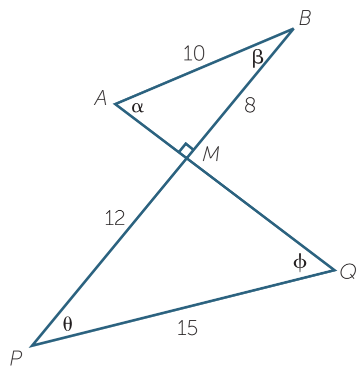
so $\triangle ABC$ is similar to $\triangle CBD$ (SSS).

b Hence $\gamma = \theta$ (matching angles of similar triangles).

Using the RHS similarity test

As with congruence, the SAS similarity test requires that the pair of equal angles be included between the pairs of sides that are in ratio. When the equal angles are right angles, however, we can use the RHS similarity test, in which the pair of equal angles are not included between the pairs of sides that are in ratio.

EXAMPLE



a Prove that the two triangles in the diagram to the right are similar.

b Identify the equal angles in the two triangles.

c Prove that $AB \parallel PQ$.

SOLUTION

a In the triangles ABM and QPM:

$\angle AMB = \angle QMP = 90°$ (given)

$\frac{AB}{QP} = \frac{2}{3} = \frac{BM}{PM}$ (given) or $\frac{AB}{BM} = \frac{5}{4} = \frac{QP}{PM}$

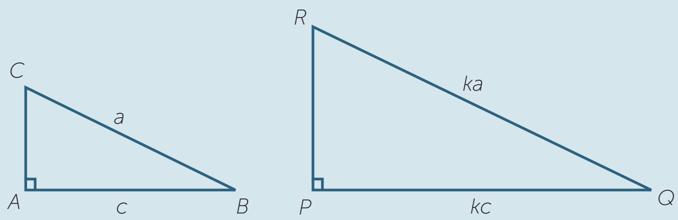
so $\triangle ABM$ is similar to $\triangle QPM$ (RHS similarity test).

b	Hence $\theta = \beta$ and $\phi = \alpha$	(matching angles of similar triangles).
c	Hence $AB \parallel PQ$	(alternate angles are equal).

We indicated above how the RHS similarity test can be developed from the RHS congruence test. It is also possible to use Pythagoras' theorem to develop the RHS congruence test from the SSS congruence test. The following exercise gives the details.

EXERCISE 5

In the two right-angled triangles in the diagram,



- the two hypotenuses are in the ratio $1 : k$, and
- the sides AB and PQ are in the ratio $1 : k$.

a Show that $PR = k\sqrt{a^2 - c^2}$.

b Hence show that $\triangle ABC$ is similar to $\triangle PQR$.

The similarity tests as generalisations of the congruence tests

Two figures are congruent when they are similar with similarity ratio 1 : 1. The four congruence tests can be regarded as special cases of the four similarity tests when the similarity ratio is 1 : 1.

For example, the SSS similarity test specifies that the matching sides of the two triangles are in a constant ratio, and the SSS congruence test specifies that the matching sides have equal length, that is, that the constant ratio is 1 : 1.

Similarly, the SAS and RHS similarity tests each specify that the ratios of the two pairs of matching sides are equal, and the SAS and RHS congruence tests each specify that the two pairs of matching sides have equal length, that is, that the constant ratio is 1 : 1.

The AAS congruence test requires that matching angles are equal, and that one pair of matching sides are equal. This is generalised by the AA congruence test, which only specifies that matching angles are equal – the ratio of any pair of matching sides is then the similarity ratio of the pair.

SIMILARITY AND INTERCEPT THEOREMS

A point F on an interval AB divides the interval into two subintervals AF and FB called **intercepts**.

A point on a side of a triangle divides that side into two intercepts. Several important theorems about intercepts on the sides of triangles can be proven quickly using similarity.

The intercept theorem – A special case involving midpoints

We shall deal first with the special case involving the midpoints of two sides.

Theorem

The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Proof

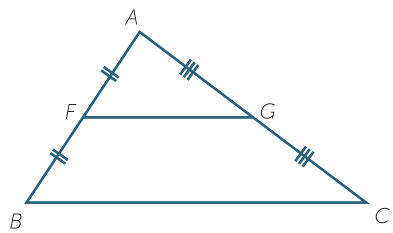
Let F and G be the midpoints of the sides AB and AC of $\triangle ABC$.

In the triangles AFG and ABC:

$\angle FAG = \angle BAC$ (common)

$\frac{AF}{AB} = \frac{AG}{AC} = \frac{1}{2}$ (given)

so $\triangle AFG$ is similar to $\triangle ABC$ (SAS similarity test).

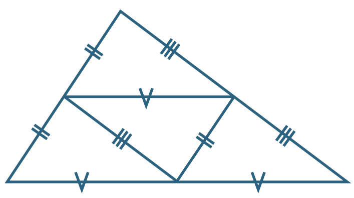


Hence $BC = 2 \times FG$ (matching sides of similar triangles).

Also $\angle AFG = \angle ABC$ (matching angles of similar triangles),

so $FG \parallel BC$ (corresponding angles are equal).

As a consequence of this theorem, if we join up the three midpoints of the sides of a triangle, as in the diagram on the right, we obtain four triangles that are congruent to each other by the SSS congruence test, each similar to the original triangle by the SAS similarity test.

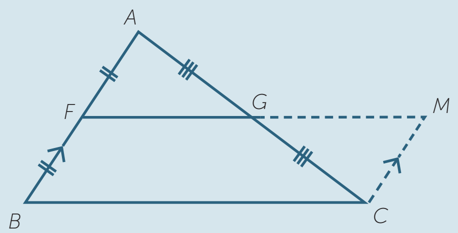


This dissection also demonstrates that each small triangle has one quarter the area of the large triangle.

The special case can also be proven using congruence alone, as in the following exercise, although the construction is a little more elaborate.

EXERCISE 6

In the triangle ABC, F and G are the midpoints of AB and AC.



Construct the line through C parallel to BA, and let it meet GF produced at M.

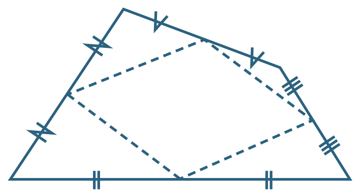
a Prove that $AFG \times CMG$.

b Prove that $FG \parallel BC$.

c Prove that $FG = \frac{1}{2}BC$.

The midpoints of the sides of a quadrilateral

The intercept theorem for triangles has an interesting application to quadrilaterals. The midpoints of the sides of any quadrilateral form a parallelogram, whose area is half the area of the original quadrilateral.

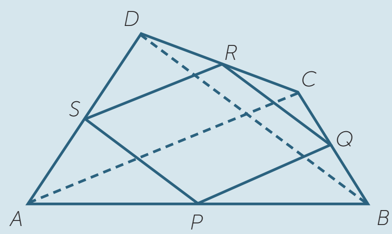


This theorem is proven in the following exercise.

EXERCISE 7

In the diagram to the right, the points P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively of a quadrilateral $ABCD$.

Join the diagonals AC and BD of the quadrilateral.



a Show that $PS \parallel QR$, and that $PS = QR$.

b Show that $PQ \parallel SR$, and that $PQ = SR$.

c Show that area $ASP = \frac{1}{4}$area ADB.

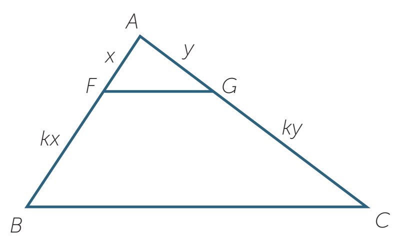
d Show that area $PQRS = \frac{1}{2}$area $ABCD$.

e What conclusion can you draw about the diagonals of $ABCD$ if:

i $PQRS$ is a rectangle? **ii** $PQRS$ is a rhombus?

The intercept theorem – The general case

The intercept theorem as stated above is a special case – the points F and G can divide the sides AB and AC in any given ratio.



Theorem

Let the points F and G divide the sides AB and AC of a triangle ABC in the same ratio $1 : k$.

Then $FG \parallel BC$ and $\frac{FG}{BC} = \frac{1}{k+1}$.

Proof

Let $AF = x$ and $AG = y$.

Then $FB = kx$ and $GC = ky$.

In the triangles AFG and ABC.

$\angle FAG = \angle BAC$ (common)

$\frac{AF}{AB} = \frac{AG}{AC} = \frac{1}{k+1}$ (given)

so $\triangle AFG$ is similar to $\triangle ABC$ (SAS similarity test) with similarity ratio $1 : k + 1$.

Hence $\frac{FG}{BC} = \frac{1}{k+1}$ (matching sides of similar triangles).

Also $\angle AFG = \angle ABC$ (matching angles of similar triangles),

so $FG \parallel BC$ (corresponding angles are equal).

The converse of the intercept theorem for triangles – A special case involving midpoints

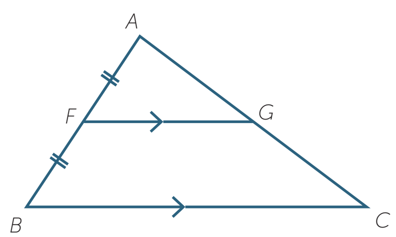
The intercept theorem has an important converse. Again we shall deal first with the special case where F is the midpoint of the side AB.

Theorem

The interval from the midpoint of one side of a triangle parallel to a second side is half the length of the second side and bisects the third side.

Proof

Let F be the midpoint of the side AB of ABC.
Let the line through F parallel to BC meet AC at G.



In the triangles AFG and ABC:

$\angle FAG = \angle BAC$ (common)

$\angle AFG = \angle ABC$ (corresponding angles, $FG \parallel BC$)

so AFG is similar to ABC (AAA similarity test).

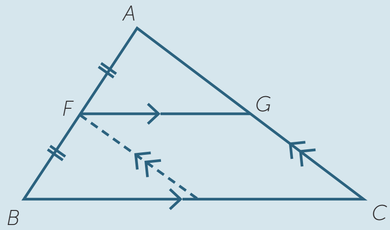
Hence, using matching sides of similar triangles,

$\frac{AG}{AC} = \frac{AF}{AB} = \frac{1}{2}$ and $\frac{FG}{BC} = \frac{AF}{AB} = \frac{1}{2}$.

As before, this special case of the theorem does not actually require similarity. It can be proven using congruence alone, as in the following exercise.

EXERCISE 8

Construct *E* on *BC* so that *FE* || *AC*.



a Prove that *AFG* ≡ *FBE*.

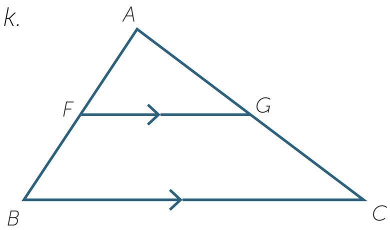
b Prove that *BE* = *FG* = *EC* and that *AG* = *FE* = *GC*.

The converse of the intercept theorem – The general case

This special case can be generalised to apply to any line parallel to a side of a triangle.

Theorem

Let the point *F* divide the side *AB* of △*ABC* in the ratio *1 : k*. Let the line through *F* parallel to *BC* meet *AC* at *G*.



Then

$\frac{AG}{GC} = \frac{1}{k}$ and $\frac{FG}{BC} = \frac{1}{k+1}$.

Proof

In the triangles *AFG* and *ABC*:

$\angle FAG = \angle BAC$ (common)

$\angle AFG = \angle ABC$ (corresponding angles, FG || BC)

so △*AFG* is similar to △*ABC* (AAA similarity test).

We know that $\frac{AF}{FB} = \frac{1}{k}$,

that is, $\frac{AF}{AB} = \frac{1}{k+1}$

so $\frac{FG}{BC} = \frac{1}{k+1}$ (matching sides of similar triangles).

Also $\frac{AG}{AC} = \frac{1}{k+1}$ (matching sides of similar triangles),

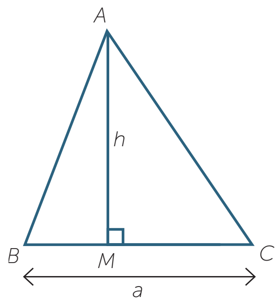
so $\frac{AG}{GC} = \frac{1}{k}$.

The ratio of areas of similar triangles

We saw above that when two triangles are similar with similarity factor 2, then their areas are in the ratio 1 : 4. This is a special case of a more general result.

Theorem

If two triangles are similar with similarity factor k, then their areas are in the ratio $1 : k^2$.

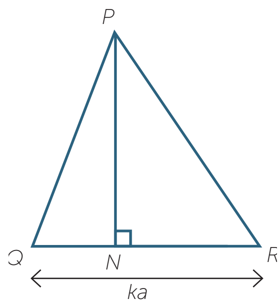


Proof

Let ABC be similar to PQR, and let $BC = a$ and $QR = k_a$.

Let AM and PN be altitudes of the two triangles, and let $AM = h$.

Then $PN = kh$ (matching lengths within similar triangles).



$$\begin{aligned}\text{Hence area of } PQR &= \tfrac{1}{2} \times (ka) \times (kh) \\ &= k^2 \times \tfrac{1}{2}ah \\ &= k^2 \times (\text{area of } \triangle ABC).\end{aligned}$$

FURTHER APPLICATIONS OF SIMILARITY

This section contains some further applications of similarity in geometry.

The altitude to the hypotenuse of a right-angled triangle

The altitude to the hypotenuse of a right-angled triangle divides the triangle into two triangles each similar to the original triangle. Several interesting results follow from this observation. The exercise below uses the construction to provide another proof of Pythagoras' theorem, and then to prove these two further results:

Theorem

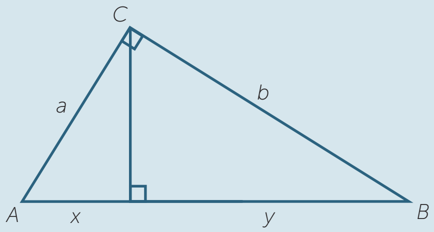
Let the altitude to the hypotenuse of a right-angled triangle divide the hypotenuse into two intercepts.

a The square of the altitude equals the product of the intercepts.

b The ratio of the squares of the other two sides equal the ratio of the intercepts.

EXERCISE 9

In the diagram to the right, *CM* is an altitude of the triangle *ABC*, which is right-angled at *C*.



a Prove that all three triangles in the figure are similar, and write down the similarity statement.

b Use the similarity of the left-hand triangle and the large triangle to prove that $a^2 = x^2 + xy$.

c Use a similar method to prove that $b^2 = y^2 + xy$.

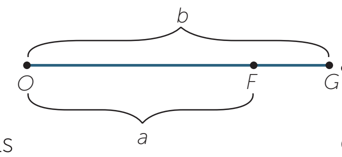
d Hence prove Pythagoras' theorem by showing that $a^2 + b^2 = (x + y)^2$.

e Use the identities of parts **b** and **c** to prove that $\frac{a^2}{b^2} = \frac{x}{y}$

f Use the similarity of the two smaller triangles to prove that $h^2 = xy$.

An enlargement construction

Suppose that we are given a centre of enlargement *O* and asked to construct, using straight edge compasses, an enlargement of some figure in the ratio *a* : *b*, where *a* and *b* are the lengths of two given intervals and *OG* on a common ray *OFG*.



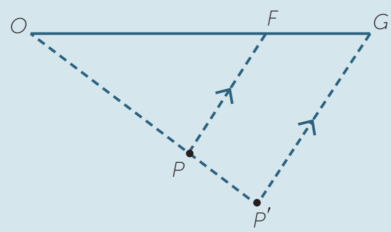
and

OF

The following exercise shows how this can done.

EXERCISE 10

Let *P* be any point on the figure.



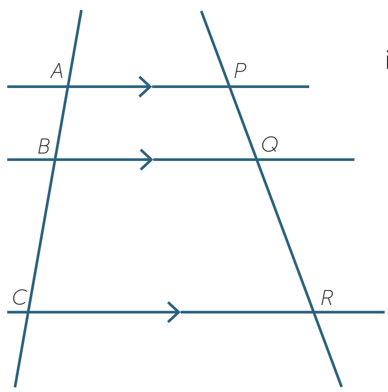
Join *FP*, then construct the line through *G* parallel to *FP*.

Let this line meet *OP*, produced if necessary, at *P*′.

Use similarity to prove that *OP* : *OP*′ = *a* : *b*.

The intercept theorem for three parallel lines

The intercept theorems have an interesting application to transversals of three parallel lines. The proof presented in the following exercise.



is

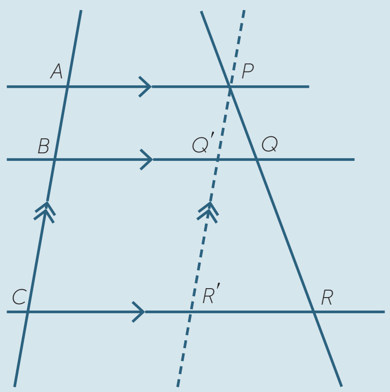
Theorem

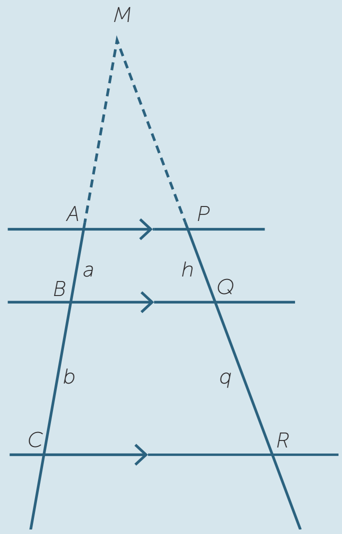
When two transversals cross three parallel lines, the intercepts cut of one transversal are in the same ratio as the intercepts cut off the other transversal.

This means that in the diagram to the right, *AB* : *BC* = *PQ* : *QR*.

EXERCISE 11

a Prove the result when $AC \parallel PR$.



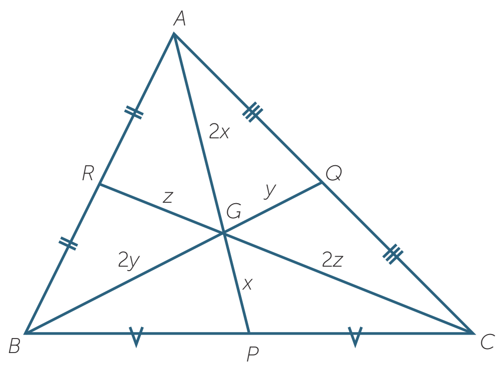


b Assume that AC is not parallel to PR, construct the line through P parallel to AC, and complete the proof.

c Alternatively, assume again that AC is not parallel to PR, let AC meet PR at M, and complete the proof.

Extension – The centroid of a triangle

A median of a triangle is the interval joining the midpoint of a side to the opposite vertex. There are three medians in a triangle, and we can use the intercept theorem above to prove that they are concurrent in a point called the centroid of the triangle. Moreover, the centroid trisects each median, or more precisely, divides it in the ratio 2 : 1.



Theorem

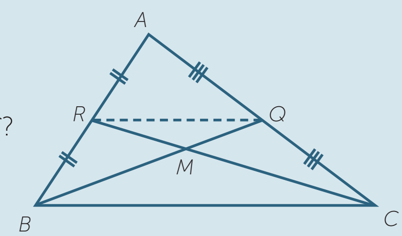
The medians AP, BQ and CR of a triangle ABC are concurrent at a point G called the centroid. The centroid trisects each median, in the sense that

$$AG : GP = BG : GQ = CG : GR = 2 : 1.$$

EXERCISE 12

a Let the medians BQ and CR intersect at a point M.

i What does the intercept theorem for triangles tell us about the relationship between RQ and BC?



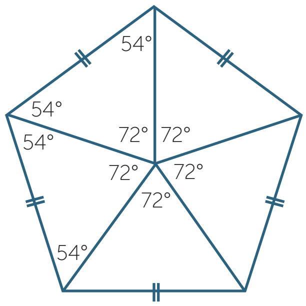
ii Prove that $\triangle BCM$ is similar to $\triangle QRM$.

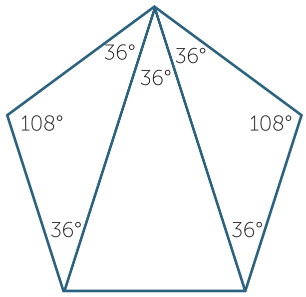
iii Hence prove that M divides BQ and CR in the ratio 2 : 1.

b Complete the proof of the theorem.

Extension – The golden mean and the diagonals of the regular pentagon

The five angles at the centre of a regular pentagon are each $360° \div 5 = 72°$, so each interior angle of the pentagon is $54° + 54° = 108°$.



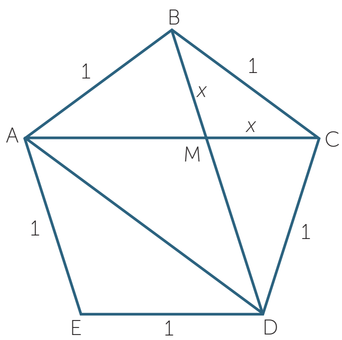


When we draw a diagonal, it forms an isosceles triangle with base angles 36° and 36°.

Thus at each vertex, the two diagonal trisect the vertex angle into three angles, each 36°.

Let $ABCDE$ be a regular pentagon with side length 1, and join the diagonals AC, AD and BD.

Let AC meet BD at M, and let $CM = x$. Then by isosceles triangles, the lengths are as shown.



EXERCISE 13

a Find the angles of $\triangle ACD$ and $\triangle DCM$. Hence show that $\triangle ACD$ is similar to $\triangle DCM$.

b Hence show that $\frac{x+1}{1} = \frac{1}{x}$.

c By solving the quadratic equation, show that $x = \frac{1}{2}(\sqrt{5} - 1)$.

d Show that each diagonal has length $\frac{1}{2}(\sqrt{5} - 1)$.

e The number $f = \frac{1}{2}(\sqrt{5} - 1)$ is called the *golden mean*. Use the identity in part **b** to prove that

$\frac{1}{\phi} = \phi - 1$.

Use the calculator to find an approximation for ϕ correct to three decimal places. Then verify this identity on your calculator.

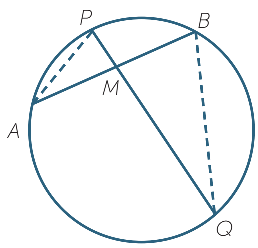
LINKS FORWARD

SIMILARITY IN CIRCLE GEOMETRY

Similarity is a useful tool in circle geometry, where equal angles turn up in what seem at first surprising positions. It is also more natural in circle geometry to express the theorems in terms of products of lengths than ratios of lengths. One example is worth giving in detail to illustrate the usefulness of similarity.

Theorem

When two chords intersect within a circle, the products of the intercepts are equal.



Proof

Let AB and PQ be chords intersecting at M.

Join AP and BQ.

In the triangles APM and QBM:

1. $\angle PAM = \angle BQM$ (angles on the same arc PB)
2. $\angle APM = \angle QBM$ (angles on the same arc AQ)

so $\triangle APM$ is similar to $\triangle QBM$ (AAA similarity test).

Hence $\frac{AM}{PM} = \frac{QM}{BM}$ (matching sides of similar triangles)

so $AM \times BM = PM \times QM$.

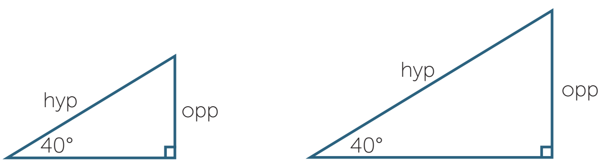
Similarity is also useful in the study of the other conic sections – parabolas, ellipses and hyperbolas.

SIMILARITY IN TRIGONOMETRY

The most important use of similarity in Years 9–10 mathematics, however, is in trigonometry, where similarity is required in the definitions of the trigonometric ratios. For example, we define

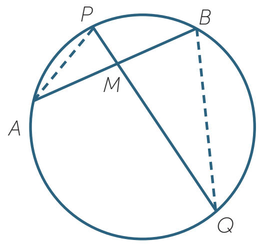
$$\sin 40^\circ = \frac{\text{opposite side}}{\text{hypotenuse}}$$

in a right-angled triangle with an angle of 40°. Such a definition requires the AAA similarity test to ensure that in any two such triangles, the ratio of the two sides is the same, no matter what the size of the triangle:



Because similarity is built into trigonometry, many geometric proofs using similarity also have an alternative trigonometric proof, where trigonometry can function as a sort of 'automated similarity', transferring ratios around the diagram. For example, here is an alternative proof of the intersecting chord theorem above. It uses trigonometry, in particular the sine rule, to transfer the ratio of the lengths.

Proof



In the triangles APM and QBM:

$\angle A = \angle Q$ (angles on the same arc PB)

and $\angle P = \angle B$ (angles on the same arc AQ),

so $\frac{AM}{PM} = \frac{\sin P}{\sin A}$ (sine rule in AMP)

$= \frac{\sin B}{\sin Q}$ (proven above)

$= \frac{QM}{BM}$ (sine rule in BMQ)

so $AM \times BM = PM \times QM$.

SIMILARITY OF OTHER CONIC SECTIONS

We have already noted that any two circles are similar. The definition of the other conic sections in terms of focus and directrix and eccentricity means that

- any two parabolas are similar, and
- any two rectangular hyperbolas are similar.

To prove this, apply an enlargement to one conic so that the distance from focus to directrix is the same in its enlargement and in the other conic. The enlarged conic is then congruent to the other conic. More generally,

- any two ellipses with the same eccentricity are similar, and
- any two hyperbolas with the same eccentricity are similar.

ENLARGEMENTS AND OTHER TRANSFORMATIONS IN COORDINATE GEOMETRY

Suppose that a figure in the coordinate plane is defined by an equation, and that an enlargement with centre the origin and similarity factor k is applied to the figure. The equation of the transformed figure can be found by replacing x by $\frac{x}{k}$ and y by $\frac{y}{k}$. For example, when the unit circle $x^2 + y^2 = 1$ is enlarged by a factor of 5, the equation of the enlarged figure is

$$\frac{x}{5}^2 + \frac{y}{5}^2 = 1$$

that is, $x^2 + y^2 = 25$.

There are many other important transformations in mathematics besides the four introduced so far. The study of functions uses dilations or stretchings, which enlarge a figure in one direction only, so that a circle becomes an ellipse. For example, when we stretch the unit circle $x^2 + y^2 = 1$ by a factor of 5 in the x-direction only, we replace x by $\frac{x}{5}$ and obtain

$$\left(\frac{x}{5}\right)^2 + y^2 = 1$$

or equivalently, $x^2 + 25y^2 = 25$.

Another interesting transformation is a shear. For example, if we move every point $P(a, b)$ in the plane to the point $P(a + kb, b)$ with the same height as P, but with x-coordinate increased by k times its y-coordinate, then a square with vertical and horizontal sides becomes a parallelogram.

In these last two situations, the resulting ellipse and parallelogram no longer have the same shape as the original figures. Nevertheless, they share important properties of the original – the diagonals of the parallelogram, for example, still bisect each other, and many theorems about tangents to circles have analogies in ellipses. A great deal of geometry investigates properties of figures that are preserved under various sets of transformations of them.

HISTORY AND APPLICATIONS

SCALE DRAWINGS

One can scarcely imagine a more useful piece of geometry than scale drawings. Plans for buildings and other public works were prepared routinely in Roman times, and had already been used in slightly less systematic form at least by ancient Egyptian and Mesopotamian architects.

The notes in this module carefully talked about scale drawings of 'the side of a train engine' and 'the facade of a cathedral', so that all scale drawings were scale drawings of two-dimensional objects. The normal situation in architecture or engineering or biology, however, is scale drawings of three-dimensional objects. This can be done in various ways.

In architecture, the most obvious way to draw a plan for a building is to project the building onto a plane such as the ground, or onto a side wall. More complicated drawings project the build onto a slant plane, making it easier for the viewer to imagine a three-dimensional picture of the building.

Such projections are extremely useful for drawing up building plans, but they do not mimic the view of someone looking at the building. Whenever we see parallel lines with our eyes, they appear to meet at a point way off in the distance, and these considerations lead to the projective projections of buildings. Projective projections were made famous by painters in the Renaissance, but they are now used routinely in the computer programmes used by architects, who can invite viewers to take a virtual walk through a proposed building with the projection constantly changing as they go. Projective transformations also allow the various conic sections – circles, ellipses, parabolas and hyperbolas – to be transformed into each other.

Maps of anything larger than a small city need to take account of the curvature of the Earth when they represent the landscape on a two-dimensional map. A typical atlas will use various projections for different purposes, because each projection has its characteristic advantages and disadvantages.

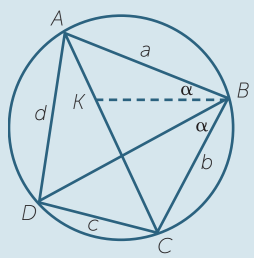
Similarity

Similarity is an essential part of Greek geometry, and was used to great effect. An important theorem about cyclic quadrilaterals proven by the Egyptian mathematician Ptolemy (cAD90 – cAD 168) theorem shows how similarity can be used to prove surprising results about lengths and products of lengths. Its proof is given below in a structured exercise that is suitable as extension material.

Ptolemy's theorem: *The product of the diagonals of a cyclic quadrilateral equals the sum of the products of opposite sides.*

EXERCISE 14

In the diagram to the right, $ABCD$ is a cyclic quadrilateral with sides $AB = a$, $BC = b$, $CD = c$ and $DA = d$, with diagonals $AC = s$ and $BD = t$.



Construct K on AC with $\angle ABK = \angle CBD$.

a Explain why $\angle BAC = \angle BDC$, $\angle ACB = \angle ADB$ and $\angle ABD = \angle CBK$.

b Show that $\triangle ABK$ is similar to $\triangle DBC$, and hence show that $AK \times t = ac$.

c Show that $\triangle CBK$ is similar to $\triangle DBA$, and hence show that $CK \times t = bd$.

d Hence show that $st = ac + bd$.

The converse of this theorem is also true, 'If the product of the diagonals of a quadrilateral equals the sum of the products of opposite sides, then the quadrilateral is cyclic.'

Because the Greeks had no coherent theory of irrational numbers, there is always an uneasy relationship between similarity and the rest of Greek geometry. For example, one can construct with straight edge and compasses enlargements whose enlargement factor is any given rational number or square root of a rational number, but it is now known that one cannot construct an enlargement with enlargement factor 32 (the proof of this is not straightforward). This is the basis of one the three most famous unsolved problems passed on to the modern world by Greek mathematicians, 'Given a cube, construct another cube of twice the volume.' The problems of the relationships between the arithmetic of the real numbers and the geometry of the plane were only sorted out in the late 19th century.

APPENDIX

The study of similarity is quite a challenge to students. The discussion in this module, using enlargement transformations to introduce similarity, was chosen because it makes good sense to students, and because it is well established in schools. It would be unwise to make the topic more complicated by introducing logical difficulties. Nevertheless, the preceding discussion has logical problems that some students begin asking questions about. For example:

- The four similarity tests can be proven from the congruence tests without enlargements when the similarity factor is a rational number.
- We never actually proved that enlargement transformations had the properties that we claimed they had, that is, that matching angles are equal and that matching lengths are in the same ratio. We only verified these properties in examples.

This appendix gives one of several possible approaches to dealing with these issues.

The first section, which essentially proves the AAA similarity test for positive rational values of k, may be suitable as Extension material for some students.

A. EQUI-ANGULAR TRIANGLES IN WHICH TWO MATCHING SIDES HAVE POSITIVE RATIONAL RATIO k : 1

Let us then begin the whole discussion again, with neither similarity nor enlargements defined.

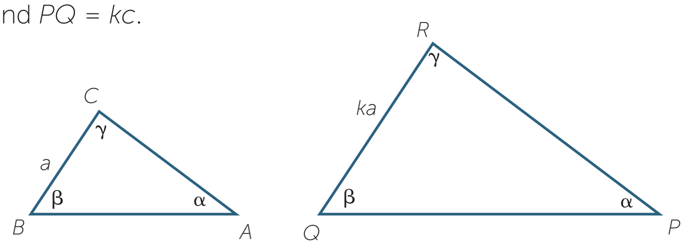
The first step is a theorem which essentially shows that the AAA similarity test holds when the similarity ratio k is a positive rational number. Because similarity has not yet been defined, the theorem is expressed not in terms of similarity, but as a test for the sides being in ratio.

Theorem

Let ABC and PQR be equi-angular triangles, with $\angle A = \angle P = \alpha$, $\angle B = \angle Q = \beta$, $\angle C = \angle R = \gamma$.

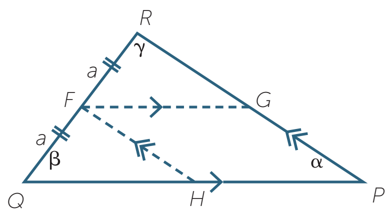
Let $BC = \alpha$, $CA = \beta$ and $AB = c$, and let $QR = ka$, where k is a positive rational number.

Then $RP = kb$ and $PQ = kc$.



Proof

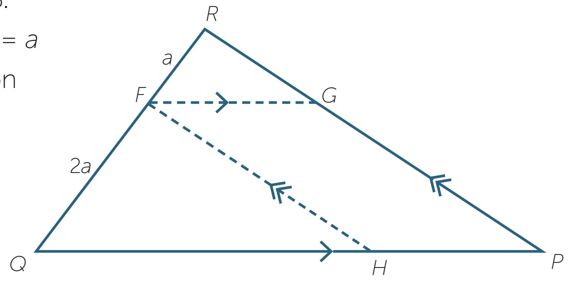
First, we prove the result when $k = 2$.
Let F be the midpoint of QR. Then $QF = FR = a$.
Construct G on RP and H on QP so that $FG \parallel QP$ and $FH \parallel RP$. Omitting the details of the angle-chasing, $\Delta RFG \equiv \Delta FQH$ (AAS congruence test).



Using opposite sides of parallelograms and matching sides of congruent triangles, $HP = FG = QH$ and $RG = FH = GP$, as required.

Secondly, we prove the result when $k = 3$. Let F divide RQ in the ratio 1 : 2. Then $FR = a$ and $FQ = 2a$. Construct G on RP and H on QP so that $FG \parallel QP$ and $FH \parallel RP$.

Omitting the details of the angle-chasing, RFG is similar to FQH (AAA similarity test with ratio 1 : 2).



Using opposite sides of parallelograms and matching sides of congruent triangles,

$HP = FG = \frac{1}{2}QH$ and $RG = \frac{1}{2}FH = \frac{1}{2}GP$, as required.

Thirdly, to prove the result for all whole numbers k, each successive step uses the previous step and adapts the same argument. For example, the argument for $k = 4$ uses exactly the same argument as used for $k = 3$, except that the length FQ is now $3a$.

Eventually the result can be proven for any whole number k. This method of successive argument for one whole number after the other will be formalised in Years 11–12 as *mathematical induction*.

Lastly, we have now outlined congruence proofs of the result for $k = 2, 3, 4$.... The test is therefore also proven for the reciprocal similarity factors $k = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$,.... Then applying two tests in sequence, the test also follows for all fractions formed from the whole numbers, that is, for all rational numbers.

B. A NEW AXIOM

Since every irrational number is 'as close as we like' to a rational number, it is reasonable to take as a new axiom of our geometry that the previous result holds for all positive real numbers.

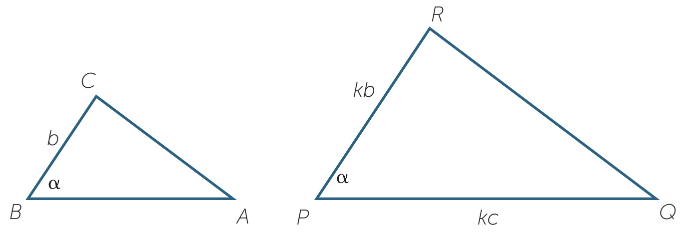
- AAA: *(A new axiom of geometry) If two triangles are equi-angular, then matching sides are in a constant ratio.*

Using this axiom, we can now establish the other three tests in a form that still avoids the word 'similarity'.

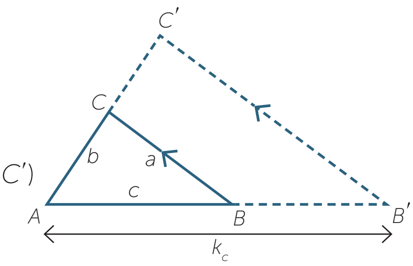
- SAS: *If two triangles have a pair of equal angles, and the including sides are in a constant ratio, then the triangles are equi-angular and the third pair of sides are in the same ratio.*
- SSS: *If we can match up the sides of one triangle with the sides of another so that all the ratio of matching sides are the same, then the two triangles are equi-angular.*
- RHS: *If the ratio of the hypotenuse and one side of a right-angled triangle is equal to the ratio of the hypotenuse and one side of another right-angled triangle, then the two triangles are equi-angular and the third pair of sides are in the same ratio.*

Here is a proof of the SAS test as stated above. In the two triangles below,

$AB : PQ = AC : PR = 1 : k$ and $\angle A = \angle P$.



Construct B' on AB, produced if necessary, so that $AB' = kc$. Let the line through B' parallel to BC meet AC, produced if necessary, at C'.



Then $\angle ABC = \angle AB'C'$ (corresponding angles, $BC \parallel B'C'$)

so $AB'C' \equiv PQR$ by the AAS congruence test.

Hence ABC and PQR are equi-angular, so using the new axiom introduced above,

$$\frac{PR}{AC} = \frac{RQ}{CB} = k.$$

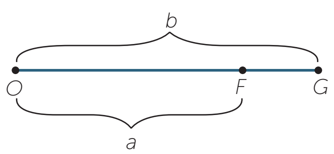
The other two tests can be proven with exactly analogous constructions.

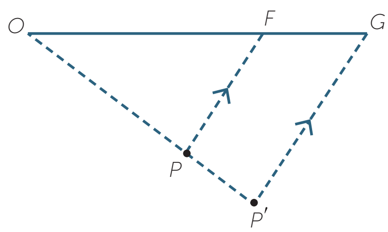
C. ENLARGEMENTS

The definition of enlargement given in this module is not really satisfactory as a geometric definition. It was based on constructing an interval that is k times a given interval, where k is any positive real number – such a construction cannot be done in general.

What we need is a definition of enlargement that is based not on a given real number k, but on the ratio of two given lengths. Such a construction was the subject of Exercise 10, and we now redefine enlargement to be the result of this construction.

This new definition of enlargement, together with the new axiom, will enable us to prove the properties of enlargements that could only be established by demonstration in the module.





The enlargement transformation: Suppose that we are given a centre of enlargement O, and two intervals OF and OG of lengths a and b, lying on the same ray from O.

Define the enlargement with centre O and ratio $a : b$ to be the transformation whose image of a point P in the plane is the point P' as in the diagram to the right. That is, P′ is the intersection of OP, produced if necessary, with the line through G parallel to FP.

Using corresponding angles on parallel lines, the triangles OFP and OGP' are equi-angular, so by the new axiom introduced in part B, it follows that

$$OP : OP' = a : b.$$

The three properties of enlargements can now be proven.

Theorem

When an enlargement with centre O and ratio $OF : OG = a : b$ is applied.

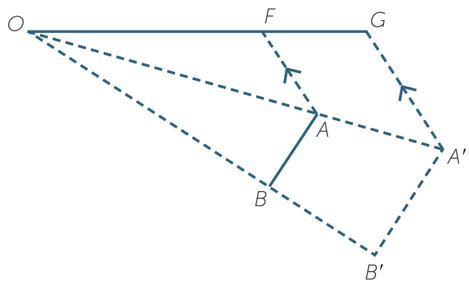
a $AB : A'B' = a : b$, for each interval AB in the plane.

b $AB \parallel A'B'$, for each interval AB in the plane.

c $\angle ABC = \angle A'B'C'$, for each angle $\angle ABC$ in the plane.

Proof

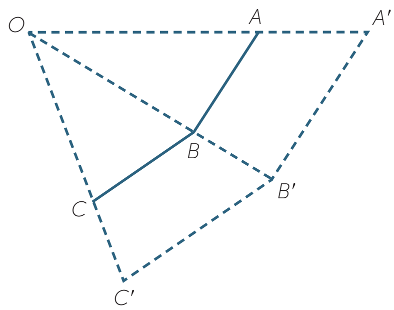
a Using the upper diagram to the right, we have shown above that $OA : OA' = OB : OB' = a : b$. Hence using the SAS situation discussed in part B above, applied to OAB and $OA'B'$, $AB : A'B' = a : b$.



b Also $\quad \angle OAB = \angle OA'B'$

so $\quad AB \parallel A'B'$.

c Using the lower diagram to the right, we know from part **b** that $AB \parallel A'B'$ and $BC \parallel B'C'$.Hence using corresponding angles, $\angle ABC = \angle A'B'C'$.



D. SIMILARITY

We can now define similarity exactly as before.

- *Two figures are called similar if an enlargement of one is congruent to the other.*

The theorem of part C now shows that two figures are similar when the points of one can be paired up with the points of the other so that matching angles are equal and matching sides are in the same ratio.

ANSWERS TO EXERCISES

EXERCISE 1

a The height of the man on the photograph is about 5 mm. so the scale is about 5 mm : 2 m = 5 mm = 2000 mm = 1 : 400.

b The height of each spire on the photograph is about 16.3 cm, so actual height ≈ 400 × 16.3 cm ≈ 65 metres. (The height of the actual church in Belgium is about 72 metres.)

EXERCISE 3

a True.

b False. Any two rectangles whose sides are in the same ratio are similar.

c False. Any two rhombuses whose angles are equal are similar (or whose diagonals are in the same ratio).

d True.

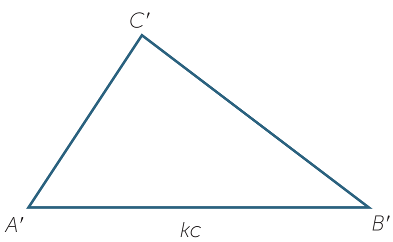
e False. A sector of one circle is similar to a sector of another circle if the two sectors have the same angle at the centre.

f True.

g False. Any two isosceles triangles whose base angles are equal are similar (or whose sides are in the same ratio, or whose height and base are in the same ratio).

EXERCISE 4

Enlarge $\triangle ABC$ to $\triangle A'B'C'$ by a suitable enlargement factor so that $A'B' = PQ = kc$.



Then $B'C' = ka$ and $C'A' = kb$ because the ratios of sides is preserved by enlargements.

Hence $\triangle A'B'C'$ is congruent to $\triangle PQR$ by the SSS congruence test, so ABC is similar to PQR, because an enlargement of $\triangle ABC$ is congruence to $\triangle PQR$.

EXERCISE 5

a Using Pythagoras' theorem in each triangle,

$$AC = k\sqrt{a^2 - c^2}$$

and $$PR = \sqrt{k^2a^2 - k^2c^2}$$

$$= k\sqrt{a^2 - c^2}$$

b Hence the sides AC and PR are in the same ratio $1 : k$, so the two triangles are similar by the SSS similarity test.

EXERCISE 6

a In the triangles AFG and CMG:

$AG = CG$	(given)
$\angle AGF = \angle CGM$	(vertically opposite angles)
$\angle AFG = \angle CMG$	(alternate angles, $BA \parallel CM$)
so $AFG = CMG$	(AAS congruence test)

b Hence $CM = AF$ (matching sides of congruent triangles)

$= BF$ (given);

so $BCMF$ is a parallelogram, because a pair of opposite sides are equal and parallel.

c Hence $FM = BC$ (opposite sides of parallelogram $BCMF$)

and $FG = MG$ (matching sides of congruent triangles),

so $FG = \frac{1}{2}BC$.

EXERCISE 7

a Apply the intercept theorem to $\triangle ABD$ and $\triangle CBD$, and combine the results.

b Apply the intercept theorem to $\triangle BAC$ and $\triangle DAC$, and combine the results.

c Use the dissection immediately above this exercise.

d Using part **c** applied to all four outer triangles,

area $\triangle ASP$ + area $\triangle CQR$ $= \frac{1}{4}$area $ABCD$.

and area $\triangle BPQ$ + area $\triangle DRS$ $= \frac{1}{4}$area $ABCD$.

Adding, sum of areas of four outer triangles $= \frac{1}{2}$area $ABCD$ and subtracting, area $PQRS = \frac{1}{2}$area $ABCD$.

e **i** The diagonals of $ABCD$ are perpendicular.

ii The diagonals of $ABCD$ are equal.

EXERCISE 8

a Use the AAS congruence test.

b Use matching sides of the congruent triangles AFG and FBE, and opposite sides of the parallelogram $EFGC$.

EXERCISE 9

a Let $\angle A = \alpha$. Then using adjacent angles and the angle sums of the triangles, $\angle ACM = 90° - \alpha$, $\angle BCM = \alpha$, and $\angle B = 90° - \alpha$.

Hence $\triangle ABC$ is similar to $\triangle CBM$ is similar to ACM (AAA similarity test).

b $\frac{a}{x} = \frac{x+y}{a}$ (matching sides of similar triangles ACM and ABC)

$$a^2 = x(x+y)$$
$$= x^2 + xy.$$

c $\frac{b}{y} = \frac{x+y}{b}$ (matching sides of similar triangles CBM and ABC)

$$b^2 = y(x+y)$$
$$= xy + y^2.$$

d Adding the identities of parts **b** and **c**, $a^2 + b^2 = x^2 + xy + xy + y^2 = (x+y)^2$.

e Dividing the identities of parts **b** and **c**,

$$\frac{a^2}{b^2} = \frac{x^2 + xy}{y^2 + xy}$$
$$= \frac{x(x+y)}{y(x+y)}$$
$$= \frac{x}{y}$$

f $\frac{x}{h} = \frac{y}{h}$ (matching sides of similar triangles CBM and ACM)

$$h^2 = xy.$$

EXERCISE 10

In the triangles OFP and OGP':

$\angle FOP = \angle GOP'$ (common)

$\angle OFP = \angle OGP'$ (corresponding angles, $FP \parallel GP'$)

so $\triangle OFP$ is similar to $\triangle OGP'$ (AAA similarity test):

Hence $\frac{OP}{OP'} = \frac{OF}{OG}$ (matching sides of similar triangles)

$$= \frac{a}{b}.$$

EXERCISE 11

a The opposite sides of both parallelograms are equal.

b Let the line through P parallel to AC meet BQ at Q', and CR at R'.

Then the opposites sides of the parallelograms $ABQ'P$ and $BCR'Q'$ are equal, that is,

$PQ' = AB$ and $Q'R' = BC$.

Applying the intercept theorem for triangles to $\triangle QR'R$,

$PQ' : Q'R' = PQ : QR$

so $AB : BC = PQ : QR$.

c Let $AB = a$, $BC = b$, $PQ = p$ and $QR = q$.

Using the intercept theorem for triangles on $\triangle MBQ$ and on $\triangle MCR$,

$$\frac{a}{p} = \frac{MA}{MP} = \frac{a+b}{p+q}$$

so $\quad ap + aq = ap + bp$

$\quad aq = bp.$

Note: The working in part **c** assumed that the transversals meet outside the three parallel lines on the same side as A. There are, of course, three other cases to consider. What other cases should be considered in part **b**?

EXERCISE 12

a **i** $RQ \parallel BC$ and $RQ = \frac{1}{2}BC$.

ii In the triangles BCM and QRM:

$\angle QMR = \angle BMC$ (vertically opposite angles)

$\angle QRM = \angle BCM$ (alternate angles, $RQ \parallel BC$)

so $\triangle BCM$ is similar to $\triangle QRM$ (AAA similarity test)

$BC = 2 \times RQ$

iii Hence using matching sides of similar triangles,

$MQ : MB = MR : MC = QR : BC = 1 : 2.$

b We have now proven that any two medians of a triangle intersect in point that divides each median in the ratio 2 : 1.

Let G be the point that divides the median AP in the ratio 2 : 1. Then the other two medians BQ and CR pass through G, and G also divides these medians in the ratio 2 : 1.

EXERCISE 13

a Each triangle is isosceles with apex angle 72° and base angles 36°.

b $\frac{AC}{CD} = \frac{DC}{CM}$ (matching sides of similar triangles)

$\frac{x+1}{1} = \frac{1}{x}$

c Hence $x^2 + x - 1 = 0$

$x = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$.

Since x is positive, $x = \frac{1}{2}(\sqrt{5} - 1)$

d Hence $AD = x + 1$

$= \frac{1}{2}(\sqrt{5} - 1)$.

e Substituting $\phi = x + 1$ into the second line of part **b** gives $\phi - 1 = \frac{1}{\phi}$.

From the calculator, $\phi \approx 1{:}628$, and $\frac{1}{\phi} \approx 0.618 \approx \phi - 1$

EXERCISE 14

a Use angles on the same arc, and adjacent angles at a point.

b $\triangle ABK$ is similar to $\triangle DBC$ (AAA)

so $\frac{AK}{a} = \frac{c}{t}$ (matching sides of similar triangles)

$AK \times t = ac$.

c $\triangle CBK$ is similar to $\triangle DBC$ (AAA)

so $\frac{CK}{b} = \frac{d}{t}$ (matching sides of similar triangles)

$CK \times t = bd$.

d Adding the results of parts **b** and **c**,

$(AK + CK) \times t = ac + bd$

$st = ac + bd$.

INTERNATIONAL CENTRE
OF EXCELLENCE FOR
EDUCATION IN
MATHEMATICS

The aim of the International Centre of Excellence for Education in Mathematics (ICE-EM) is to strengthen education in the mathematical sciences at all levels-from school to advanced research and contemporary applications in industry and commerce.

ICE-EM is the education division of the Australian Mathematical Sciences Institute, a consortium of 27 university mathematics departments, CSIRO Mathematical and Information Sciences, the Australian Bureau of Statistics, the Australian Mathematical Society and the Australian Mathematics Trust.

AUSTRALIAN MATHEMATICS TRUST

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

The ICE-EM modules are part of *The Improving Mathematics Education in Schools* (TIMES) *Project.*

The modules are organised under the strand titles of the Australian Curriculum:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

www.amsi.org.au