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The Improving Mathematics Education in Schools (TIMES) Project

## FURTHER TRIGONOMETRY

A guide for teachers - Year 10

MEASUREMENT AND  
GEOMETRY • Module 24

June 2011

YEAR  
**10**

## Further Trigonometry

### (Measurement and Geometry: Module 24)

For teachers of Primary and Secondary Mathematics

510

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## FURTHER TRIGONOMETRY

A guide for teachers - Year 10

MEASUREMENT AND  
GEOMETRY • Module 15

June 2011

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YEAR  
**10**

# FURTHER TRIGONOMETRY

## ASSUMED KNOWLEDGE

- Familiarity with the content of the module *Introductory Trigonometry*.
- Familiarity with basic coordinate geometry.
- Facility with simple algebra, formulas and equations.
- Familiarity with surds.

## MOTIVATION

In the module, *Introductory Trigonometry*, we showed that if we know the angles and one side in a right-angled triangle we can find the other sides using the trigonometric ratios sine, cosine and tangent. Similarly, knowing any two of the sides in a right-angled triangle enables us to find all the angles.

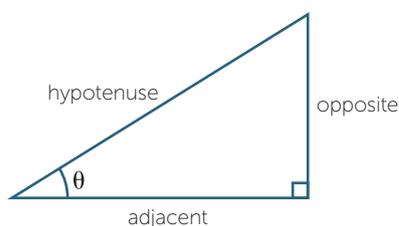
Not all triangles contain a right-angle. We can relate sides and angles in an arbitrary triangle using two basic formulas known as the **sine rule** and the **cosine rule**.

Armed with these we can solve a greater range of problems in two dimensions and extend these ideas to three-dimensional problems as well. This is an essential tool for surveyors and civil engineers.

It soon becomes apparent that in some cases we need to be able to define the trigonometric ratio of an obtuse angle. This allows us to deal with a broader range of problems and applications. It will also provide the model for extending the definition of the trigonometric ratios to any angle. This idea will be picked up in the module, *The Trigonometric Functions*.

## CONTENT

In the module, *Introductory Trigonometry*, we defined the three standard trigonometric ratios **sine**, **cosine** and **tangent** of an angle  $\theta$ , called the **reference angle**, in a right-angled triangle.

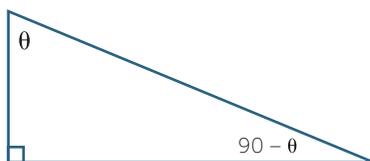


These are defined by:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}, \quad \text{where } 0^\circ < \theta < 90^\circ.$$

Students should learn these ratios thoroughly. One simple mnemonic that might assist them is SOH CAH TOA, consisting of the first letter of each ratio and the first letter of the sides making up that ratio.

In a right-angled triangle, the other two angles are complements of each other. As the diagram below shows, the side opposite one of these angles is adjacent to the other.



Thus, it can be seen that,

$$\sin \theta = \cos (90^\circ - \theta) \quad \text{and} \quad \cos \theta = \sin (90^\circ - \theta) \quad \text{if } 0^\circ < \theta < 90^\circ$$

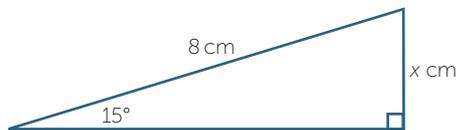
The cosine (co-sine) is so named since the cosine of an angle is the sine of its complement.

These ratios can be used to find sides and angles in right-angled triangles.

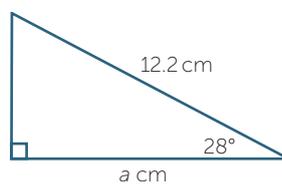
### EXAMPLE

Find, correct to two decimal places, the value of the pronumeral in each triangle.

a



b



Solution

a  $\sin 15^\circ = \frac{\text{opposite}}{\text{hypotenuse}}$

$$= \frac{x}{8}$$

$$x = 8 \times \sin 15^\circ$$

$$\approx 2.07 \quad (\text{correct to two decimal places})$$

b  $\cos 28^\circ = \frac{\text{adjacent}}{\text{hypotenuse}}$

$$= \frac{a}{12.2}$$

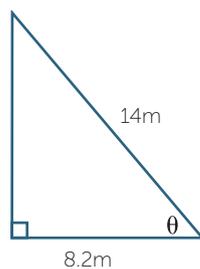
$$a = 12.2 \times \cos 28^\circ$$

$$\approx 10.77 \quad (\text{correct to two decimal places})$$

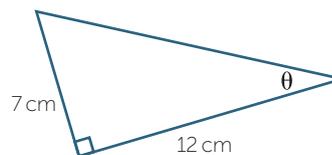
### EXAMPLE

Calculate the value of  $\theta$ , correct to one decimal place.

a



b



### SOLUTION

a  $\cos \theta = \frac{8.2}{14}$

$$\text{So, } \theta = \cos^{-1}\left(\frac{8.2}{14}\right)$$

$$\approx 54.1^\circ \quad (\text{correct to one decimal place})$$

b  $\tan \theta = \frac{7}{12}$

$$\text{So, } \theta = \tan^{-1}\left(\frac{7}{12}\right)$$

$$\approx 30.3^\circ \quad (\text{correct to one decimal place})$$

Note that for  $0 < x < 1$  the statement  $\sin^{-1} x = \theta$  means that  $\sin \theta = x$ . This notation is standard but it is essential that students do not confuse the inverse notation with the usual meaning of the index -1 used in algebra. To help avoid this confusion, it is best to always read  $\sin^{-1} x$  as *inverse sine of x* and  $\tan^{-1} x$  as *inverse tangent of x*.

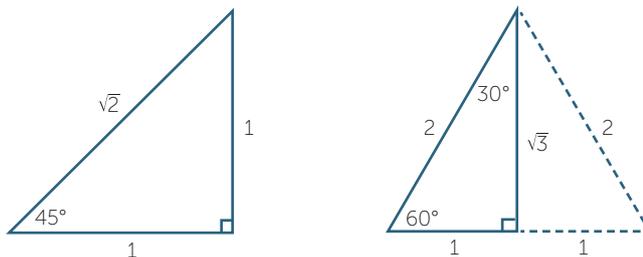
## EXACT VALUES

The trigonometric ratios for the angles  $30^\circ$ ,  $45^\circ$  and  $60^\circ$  can be expressed using surds and occur very frequently in introductory trigonometry, in senior mathematics and in calculus. It is thus important for students to become familiar with them.

One way to find them quickly is to sketch the following triangles and then simply write down the ratios.

A right-angled triangle containing a  $45^\circ$  angle will be isosceles, so we choose the two shorter sides to be 1 unit in length and use Pythagoras' theorem to find the hypotenuse.

For the angles  $30^\circ$  and  $60^\circ$ , we start with an equilateral triangle of side 2 units in length and drop a perpendicular as shown. Simple geometry and Pythagoras' theorem gives the remaining information as shown in the diagram.



The table of values can now be completed from these diagrams.

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$
$30^\circ$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45^\circ$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60^\circ$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

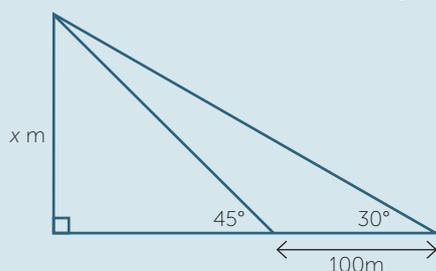
## INDEX NOTATION

There are several deviations from the usual index notation that arise in trigonometry. Students may initially find them confusing.

We write, for example  $(\tan \theta)^2$  as  $\tan^2 \theta$ ,  $(\sin \theta)^3$  as  $\sin^3 \theta$  and so on. This must not be confused with the inverse notation discussed above. We do not write, for example,  $\sin^{-2} \theta$  for  $(\sin \theta)^{-2}$ , since this would confuse the usual meaning of indices with inverses.

## EXERCISE 1

- a Simplify  $\tan^2 30 - \frac{1}{\cos^2 30^\circ}$ .
- b The top of a tower has an angle of elevation of  $45^\circ$  from a point on the ground. From a point 100m further on, the angle of elevation is  $30^\circ$ , as shown in the diagram. Find the exact value,  $x$ , of the height of the tower.



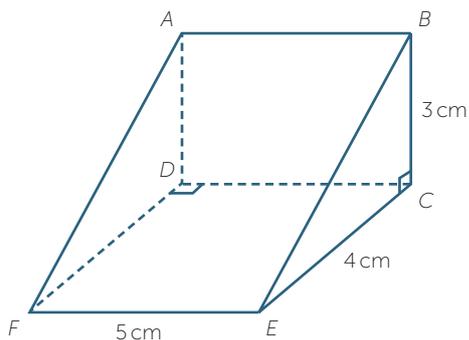
## THREE-DIMENSIONAL PROBLEMS

We can use our knowledge of trigonometry to solve problems in three dimensions.

### EXAMPLE

In the triangular prism opposite, find:

- a the length  $CF$
- b the length  $BF$
- c the angle  $BFC$ , correct to one decimal place.

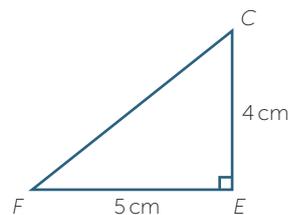


### SOLUTION

- a Applying Pythagoras' theorem to  $\triangle CEF$ ,

$$\begin{aligned} CF^2 &= 4^2 + 5^2 \\ &= 41 \end{aligned}$$

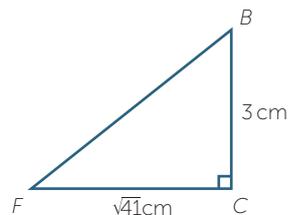
Hence  $CF = \sqrt{41}$  cm



- b Applying Pythagoras' theorem to  $\triangle BCF$ ,

$$\begin{aligned} BF^2 &= 3^2 + (\sqrt{41})^2 \\ &= 50 \end{aligned}$$

So  $BF = 5\sqrt{2}$  cm

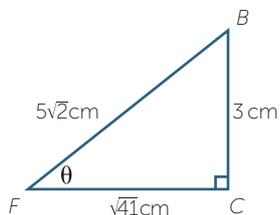


- c To find the angle  $BFC$ , draw  $\triangle BCF$  and let  $\angle BFC = \theta$ .

Now  $\tan \theta = \frac{3}{\sqrt{41}}$

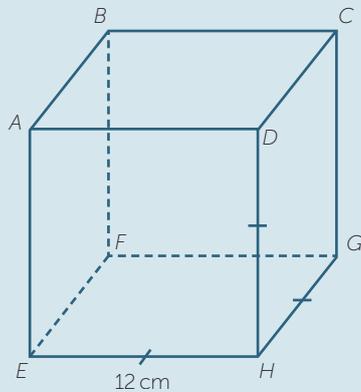
giving  $\theta = \tan^{-1}\left(\frac{3}{\sqrt{41}}\right)$

So  $\angle BFC \approx 25.1^\circ$  (to one decimal place)



## EXERCISE 2

Find  $\angle CEG$  in the cube shown below.

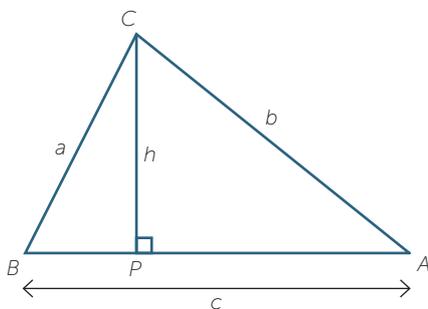


## THE SINE RULE

In many applications we encounter triangles that are not right-angled. We can extend our knowledge of trigonometry to deal with these triangles. This is done using two basic formulas, the first of which is called the **sine rule**.

We will assume, for the moment, that we are dealing with an acute-angled triangle  $ABC$ .

As shown in the diagram, we drop a perpendicular  $CP$  of length  $h$  from  $C$  to  $AB$ .



Then in  $\triangle APC$  we have  $\sin A = \frac{h}{b}$ , so  $h = b \sin A$ .

Similarly, in  $\triangle CPB$  we have  $\sin B = \frac{h}{a}$ , so  $h = a \sin B$ .

Equating these two expressions for  $h$  we have  $b \sin A = a \sin B$  which we can write as

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The same result holds for the side and the angle  $C$ , so we can write

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is known as the **sine rule**. In words it says: *any side of a triangle over the sine of the opposite angle equals any other side of the triangle over the sine of its opposite angle.*

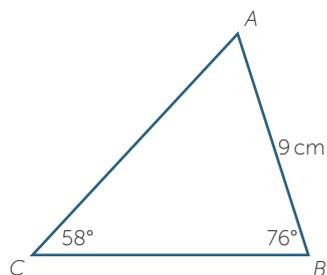
We will soon see how to extend this result to obtuse-angled triangles.

### EXAMPLE

In  $\triangle ABC$ ,  $AB = 9$  cm.  $\angle ABC = 76^\circ$  and  $\angle ACB = 58^\circ$ .

Find, correct to two decimal places:

- a**  $AC$                       **b**  $BC$



### SOLUTION

- a** Apply the sine rule:

$$\frac{AC}{\sin 76^\circ} = \frac{9}{\sin 58^\circ}$$

and so 
$$AC = \frac{9 \sin 76^\circ}{\sin 58^\circ}$$

$$\approx 10.30 \text{ cm} \quad (\text{to two decimal places})$$

- b** To find  $BC$ , we need the angle  $\angle CAB$  opposite it.

$$\begin{aligned} \angle CAB &= 180^\circ - 58^\circ - 76^\circ \\ &= 46^\circ \end{aligned}$$

Thus, by the sine rule:

$$\frac{BC}{\sin 46^\circ} = \frac{9}{\sin 58^\circ}$$

$$BC = \frac{9 \sin 46^\circ}{\sin 58^\circ}$$

$$\approx 7.63 \text{ cm} \quad (\text{to two decimal places})$$

### BEARINGS

True bearings were covered in the module, *Introductory Trigonometry Years*.

We can now use the sine rule to solve simple surveying problems involving non-right-angled triangles.

**EXAMPLE**

From the points A and B, 800 metres apart, on a straight North-South road, the bearing of a house is  $125^\circ\text{T}$  and  $050^\circ\text{T}$  respectively. Find how far each point is from the house, correct to the nearest metre.

**SOLUTION**

We draw a diagram to represent the information.

We can find the angles in  $\triangle AHB$ .

$$\begin{aligned}\angle HAB &= 180^\circ - 125^\circ \\ &= 55^\circ\end{aligned}$$

$$\begin{aligned}\text{and } \angle AHB &= 180^\circ - 50^\circ - 55^\circ \\ &= 75^\circ\end{aligned}$$

Apply the sine rule to  $\triangle ABH$ :

$$\frac{BH}{\sin 55^\circ} = \frac{800}{\sin 75^\circ}$$

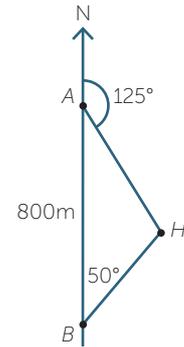
$$\begin{aligned}\text{and so } BH &= \frac{800 \sin 55^\circ}{\sin 75^\circ} \\ &\approx 678.44 \text{ m} \quad (\text{to two decimal places})\end{aligned}$$

Thus B is approximately 678 metres from the house.

$$\text{Similarly, } \frac{AH}{\sin 50^\circ} = \frac{800}{\sin 75^\circ}$$

$$\begin{aligned}\text{and so } AH &= \frac{800 \sin 50^\circ}{\sin 75^\circ} \\ &\approx 634.45 \text{ m} \quad (\text{to two decimal places})\end{aligned}$$

Thus A is approximately 634 metres from the house.

**EXERCISE 3**

From a point at  $P$ , west of a building  $OA$ , the angle of elevation of the top  $A$  of the building  $OA$  is  $28^\circ$ . From a point  $Q$  10m further west from  $P$  the angle of elevation is  $20^\circ$ . Draw a diagram and then use the sine rule to find the distance  $AP$  and hence the exact height of the building. Finally, evaluate the height  $OA$  to the nearest centimetre.

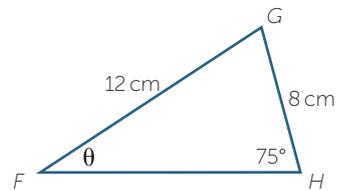
**FINDING ANGLES**

The sine rule can be used to find angles as well as sides in a triangle. One of the known sides, however, must be opposite one of the known angles.

### EXAMPLE

Assuming that all the angles are acute.

Find the angle  $\theta$  in the triangle  $FGH$ , to the nearest degree.



### SOLUTION

Apply the sine rule to  $\triangle FGH$ ,

$$\frac{8}{\sin \theta} = \frac{12}{\sin 75^\circ}$$

To make the algebra easier, take the reciprocal of both sides.

$$\frac{\sin \theta}{8} = \frac{\sin 75^\circ}{12}$$

Hence

$$\sin \theta = \frac{8 \sin 75^\circ}{12}$$

$$= 0.644\dots$$

Hence

$$\theta = 40^\circ \quad (\text{to the nearest degree})$$

As seen in the example above, it is easier when finding angles, to write the sine rule as  $\frac{\sin A}{a} = \frac{\sin B}{b}$  before substituting in the given information.

### DEALING WITH OBTUSE ANGLES

Both the sine rule and the cosine rule are used to find angles and sides in triangles. What happens when one of the angles is obtuse? To deal with this we need to extend the definition of the basic trigonometric ratios from acute to obtuse angles. We use coordinate geometry to motivate the extended definitions as follows.

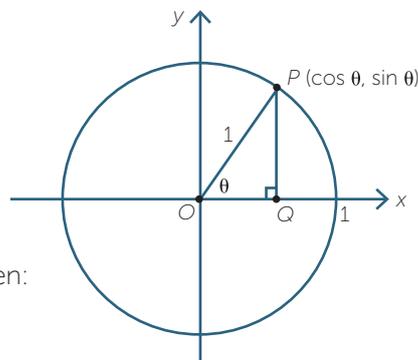
We draw the unit circle centre the origin in the Cartesian plane and mark the point on the circle in the first quadrant.

In the diagram shown, since  $\frac{OQ}{1} = \cos \theta$ , we can see that the  $x$ -coordinate of  $P$  is  $\cos \theta$ . Similarly, the  $y$ -coordinate of  $P$  is  $\sin \theta$ .

Hence the coordinates of  $P$  are  $(\cos \theta, \sin \theta)$ .

We can now turn this idea around and say that if  $\theta$  is the angle between  $OP$  and the positive  $x$ -axis, then:

- the cosine of  $\theta$  is defined to be the  $x$ -coordinate of the point  $P$  on the unit circle and
- the sine of  $\theta$  is defined to be the  $y$ -coordinate of the point  $P$  on the unit circle.

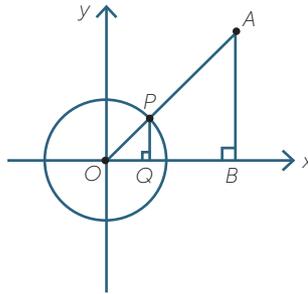


This definition can be applied to all angles, both positive and negative, but in this module we will restrict the angle to be between  $0^\circ$  and  $180^\circ$ .

## CONSISTENCY OF THE DEFINITIONS

In the module, *Introduction to Trigonometry*, we defined  $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$  and  $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ , where  $0^\circ < \theta < 90^\circ$ . In the previous section we defined  $\cos \theta = OQ$  and  $\sin \theta = PQ$ . We must show that the two definitions agree.

The diagram below shows the right-angled triangle  $OAB$ , and the triangle  $OPQ$  both containing the angle  $\theta$ . Triangle  $OPQ$  has its vertex  $P$  on the unit circle. These triangles are similar and so the ratio  $\frac{AB}{OA} = \frac{PQ}{OP} = PQ$ , which is the  $y$ -coordinate of the point  $P$ . Similarly  $\frac{OB}{OA} = \frac{OQ}{OP} = OQ$ , which is the  $x$ -coordinate of  $P$ .



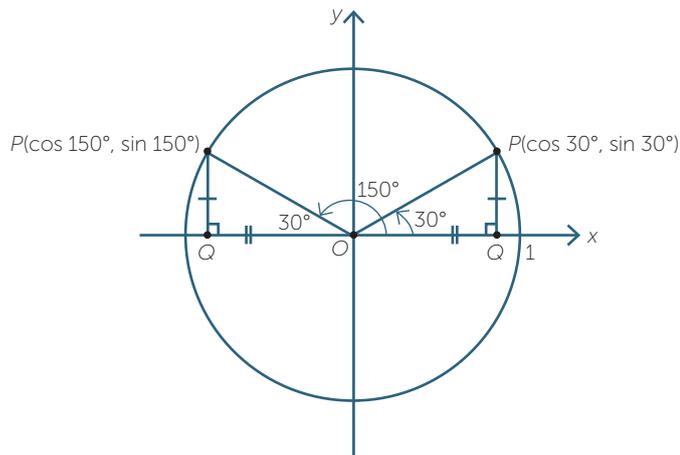
So we have shown the two definitions agree.

## ANGLES IN THE SECOND QUADRANT

As an example, let us take  $\theta$  to be  $30^\circ$ , so  $P$  has coordinates  $(\cos 30^\circ, \sin 30^\circ)$ .

Now move the point  $P$  around the circle to  $P'$  so that  $OP'$  makes an angle of  $150^\circ$  with the positive  $x$ -axis. Note that  $30^\circ$  and  $150^\circ$  are supplementary angles.

The coordinates of  $P'$  are  $(\cos 150^\circ, \sin 150^\circ)$ .



But we can see that the triangles  $OPQ$  and  $OP'Q'$  are congruent, so the  $y$ -coordinates of  $P$  and  $P'$  are the same. Thus,  $\sin 150^\circ = \sin 30^\circ$ .

Also, the  $x$ -coordinates of  $P$  and  $P'$  have the same magnitude but opposite sign, so  $\cos 150^\circ = -\cos 30^\circ$ .

From this typical example, we see that if  $\theta$  is any obtuse angle, then its supplement,  $180^\circ - \theta$  is acute, and the sine of  $\theta$  is given by

$$\sin \theta = \sin (180^\circ - \theta), \text{ where } 90^\circ < \theta < 180^\circ.$$

Similarly, if  $\theta$  is any obtuse angle then the cosine of  $\theta$  is given by

$$\cos \theta = -\cos (180^\circ - \theta), \text{ where } 90^\circ < \theta < 180^\circ.$$

In words this says:

- the sine of an obtuse angle equals the sine of its supplement,
- the cosine of an obtuse angle equals **minus** the cosine of its supplement.

### EXAMPLE

Find the exact value of:

**a**  $\sin 150^\circ$

**b**  $\cos 150^\circ$

**c**  $\sin 120^\circ$

**d**  $\cos 120^\circ$

### SOLUTION

$$\begin{aligned} \mathbf{a} \quad \sin 150^\circ &= \sin (180 - 150)^\circ \\ &= \sin 30^\circ \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos 150^\circ &= -\cos (180 - 150)^\circ \\ &= -\cos 30^\circ \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \sin 120^\circ &= \sin (180 - 120)^\circ \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

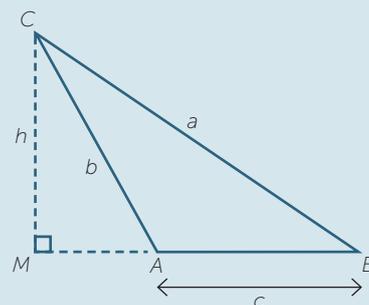
$$\begin{aligned} \mathbf{d} \quad \cos 120^\circ &= -\cos (180 - 120)^\circ \\ &= -\cos 60^\circ \\ &= -\frac{1}{2} \end{aligned}$$

Note: You can verify these results using your calculator.

The sine rule is also valid for obtuse-angled triangles.

### EXERCISE 4

Reprove the sine rule  $\frac{a}{\sin A} = \frac{b}{\sin B}$  for a triangle in which angle  $A$  is obtus.



## THE ANGLES $0^\circ$ , $90^\circ$ , $180^\circ$

We can use the extended definition of the trigonometric functions to find the sine and cosine of the angles  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ .

### EXERCISE 5

Draw a diagram showing the point on the unit circle at each of the above angles. Use the coordinates of to complete the entries in the table below.

$\theta$	$0^\circ$	$90^\circ$	$180^\circ$
$\sin \theta$			
$\cos \theta$			

## THE TANGENT OF AN OBTUSE ANGLE

For  $\theta$  in the range  $0^\circ < \theta < 90^\circ$  or  $90^\circ < \theta < 180^\circ$  we define the tangent of an angle  $\theta$  by

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \text{ for } \cos \theta \neq 0.$$

In the case when  $\cos \theta = 0$ , the tangent ratio is undefined. This will happen, when  $\theta = 90^\circ$ .

If  $\theta$  is in the range  $0^\circ < \theta < 90^\circ$ , this definition agrees with the usual definition of

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

Hence, if  $\theta$  is an obtuse angle, then

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} && \text{(from the definition)} \\ &= \frac{\sin (180^\circ - \theta)}{-\cos (180^\circ - \theta)} && \text{(since } \theta \text{ is obtuse)} \\ &= -\tan (180^\circ - \theta) && \text{(from the definition).} \end{aligned}$$

Hence the tangent of an obtuse angle is the negative of the tangent of its supplement.

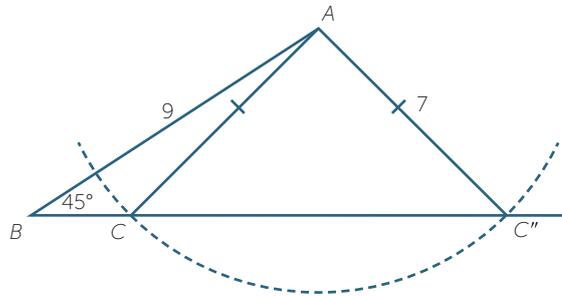
Note that  $\tan 0^\circ = 0$  and  $\tan 180^\circ = 0$  since the sine of these angles is 0 and that  $\tan 90^\circ$  is undefined since  $\cos 90^\circ = 0$ .

### EXERCISE 6

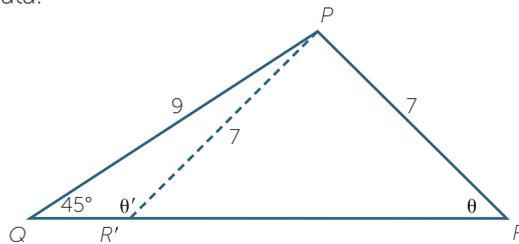
Find the exact values of  $\tan 150^\circ$  and  $\tan 120^\circ$ .

## THE AMBIGUOUS CASE

In our work on congruence, it was emphasized that when applying the SAS congruence test, the angle in question had to be the angle **included** between the two sides. Thus, the following diagram shows two non-congruent triangles  $ABC$  and  $ABC'$  with two pairs of matching sides sharing a common (non-included) angle.



Suppose we are told that a triangle  $PQR$  has  $PQ = 9$ ,  $\angle PQR = 45^\circ$ , and  $PR = 7$ . Then the angle opposite  $PQ$  is not uniquely determined. There are two non-congruent triangles that satisfy the given data.



Applying the sine rule to triangle we have

$$\frac{\sin \theta}{9} = \frac{\sin 45^\circ}{7}$$

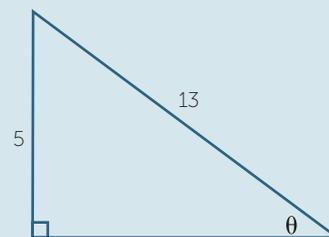
and so  $\sin \theta = \frac{9 \sin 45^\circ}{7} \approx 0.9091$ .

Thus  $\theta \approx 65^\circ$ , assuming that  $\theta$  is acute. But the supplementary angle  $\theta' = 115^\circ$ . The angle  $PR'Q$  also satisfies the given data. This situation is sometimes referred to as **the ambiguous case**.

Since the angle sum of a triangle is  $180^\circ$ , in some circumstances only one of the two angles calculated is geometrically valid.

## EXERCISE 7

Find the value of  $\theta$  in the following diagram, explaining why the answer is unique.

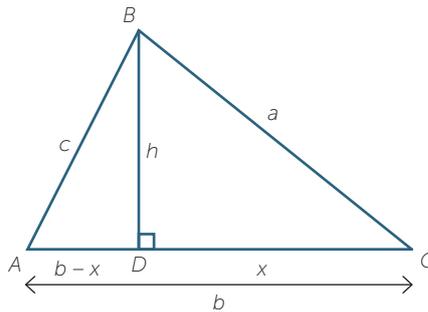


## THE COSINE RULE

We know from the SAS congruence test, that a triangle is completely determined if we are given two sides and the included angle. However, if we know two sides and the included angle in a triangle, the sine rule does not help us determine the remaining side.

The second important formula for general triangles is the **cosine rule**.

Suppose  $ABC$  is a triangle and that the angles  $A$  and  $C$  are acute. Drop a perpendicular from  $B$  to  $AC$  and mark the lengths as shown in the diagram.



In  $\triangle BDA$ , Pythagoras' theorem gives

$$c^2 = h^2 + (b - x)^2.$$

Also in  $\triangle CBD$ , another application of Pythagoras' theorem gives

$$h^2 = a^2 - x^2.$$

Substituting this expression for  $h^2$  into the first equation and expanding,

$$\begin{aligned} c^2 &= a^2 - x^2 + (b - x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx. \end{aligned}$$

Finally, from  $\triangle CBD$ , we have  $x = a \cos C$  and so

$$c^2 = a^2 + b^2 - 2ab \cos C$$

This last formula is known as the cosine rule. By relabeling the sides and angle, we can also write  $a^2 = b^2 + c^2 - 2bc \cos A$ , and  $b^2 = a^2 + c^2 - 2ac \cos B$ .

Notice that if  $C = 90^\circ$  then, since  $\cos C = 0$ , we obtain Pythagoras' theorem, and so we can regard the cosine rule as Pythagoras' theorem with a *correction term*.

The cosine rule is also true when  $C$  is obtuse, but note that in this case the final term in the formula will produce a positive number, because the cosine of an obtuse angle is negative. Some care must be taken in this instance.

### EXAMPLE

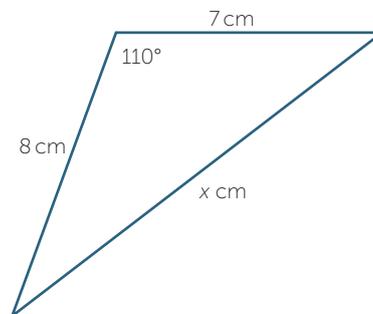
Find the value of  $x$  to one decimal place.

### SOLUTION

Applying the cosine rule:

$$\begin{aligned} x^2 &= 7^2 + 8^2 - 2 \times 7 \times 8 \cos 110^\circ \\ &= 151.30\dots \end{aligned}$$

so  $x = 12.3$  (to one decimal place)



## EXERCISE 8

Prove that the cosine rule also holds in the case when  $C$  is obtuse.

### FINDING ANGLES

We know from the SSS congruence test that if the three sides of a triangle are known then the three angles are uniquely determined. Again, the sine rule is of no help in finding them since it requires the knowledge of (at least) one angle, but we can use the cosine rule instead.

We can substitute the three side lengths  $a$ ,  $b$  and  $c$  into the formula  $c^2 = a^2 + b^2 - 2ab \cos C$  where  $C$  is the angle opposite the side  $c$ , and then re-arrange to find  $\cos C$  and hence  $C$ .

Alternatively, we can re-arrange the formula to obtain

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

and then substitute. Students may care to rearrange the cosine rule or learn a further formula. Using this form of the cosine rule often reduces arithmetical errors.

Recall that in any triangle  $ABC$ , if  $a > b$  then  $\angle A > \angle B$ .

**EXAMPLE**

A triangle has side lengths 6 cm, 8 cm and 11 cm. Find the smallest angle in the triangle.

**SOLUTION**

The smallest angle in the triangle is opposite the smallest side.

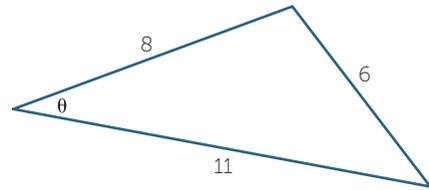
Applying the cosine rule:

$$6^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos \theta$$

$$\cos \theta = \frac{8^2 + 11^2 - 6^2}{2 \times 8 \times 11}$$

$$= \frac{149}{176}$$

and so  $\theta \approx 32.2$  (correct to one decimal place)

**EXTENSION — THE LONGEST SIDE AND THE LARGEST ANGLE OF A TRIANGLE**

In the module *Congruence*, we proved an important relationship between the relative sizes of the angles of a triangle and the relative lengths of its sides: *The angle of a triangle opposite a longer side is larger than the angle opposite a shorter side.*

For scalene triangles, this can be restated in terms of inequalities of all three sides as follows:

If  $\triangle ABC$  is a triangle in which  $a > b > c$ , then  $A > B > C$ .

This result can be proved in an interesting way using either the sine rule or the cosine rule.

**THE LONGEST SIDE AND THE SINE RULE**

The following exercise uses the fact that  $\sin \theta$  increases from 0 to 1 as  $\theta$  increases from  $0^\circ$  to  $90^\circ$ .

**EXERCISE 9**

Let  $\triangle ABC$  be a triangle in which  $a > b > c$ .

- What can you conclude about the relative sizes of  $\sin A$ ,  $\sin B$  and  $\sin C$  using the sine rule?
- If no angle is obtuse, what can you conclude about the relative sizes of  $A$ ,  $B$  and  $C$ ?
- If a triangle  $PQR$  has an obtuse angle  $\angle P = 180^\circ - \theta$ , where  $\theta$  is acute, use the identity  $\sin(180^\circ - \theta) = \sin \theta$  to explain why  $\sin P$  is larger than  $\sin Q$  and  $\sin R$ .
- Hence prove that if the triangle  $ABC$  has an obtuse angle, then  $A > B > C$ .

## THE LONGEST SIDE AND THE COSINE RULE

This exercise uses the fact that  $\cos \theta$  decreases from 1 to  $-1$  as  $\theta$  increases from  $0^\circ$  to  $180^\circ$ .

### EXERCISE 10

Let  $\triangle ABC$  be a triangle in which  $a > b > c$ .

- Write down  $\cos A$  and  $\cos B$  in terms of  $a$ ,  $b$  and  $c$ , and express each in terms of their common denominator  $2abc$ .
- Show that  $\cos B - \cos A = \frac{(a^2b - ab^2) - (ac^2 - bc)^2 - (a^3 - b^3)}{2abc}$   
 $= \frac{(a - b)(ab - c^2) + (a^3 - b^3)}{2abc}$ .
- Hence explain why  $\cos B > \cos A$ .
- Similarly explain why  $\cos C > \cos B$ , and hence show that  $A > B > C$ .

### EXERCISE 11

- Use Pythagoras' theorem to show that the hypotenuse is the longest side of a right-angled triangle
- Why is the result of part a special case of the theorem above?

## AREA OF A TRIANGLE

We saw in the module, *Introductory Trigonometry* that if we take any triangle with two given sides and about a given (acute) angle  $\theta$ , then the area of the triangle is given by

$$\text{Area} = \frac{1}{2}ab \sin \theta.$$

### EXERCISE 12

Derive this formula in the case when  $\theta$  is obtuse.

### EXERCISE 13

A triangle has two sides of length 5 cm and 4 cm containing an angle  $\theta$ . Its area is  $5 \text{ cm}^2$ . Find the two possible (exact) values of  $\theta$  and draw the two triangles that satisfy the given information.

### EXERCISE 14

Write down two expressions for the area of triangle and derive the sine rule from them.

## LINKS FORWARD

The sine and cosine rules can be used to solve a range of practical problems in surveying and navigation.

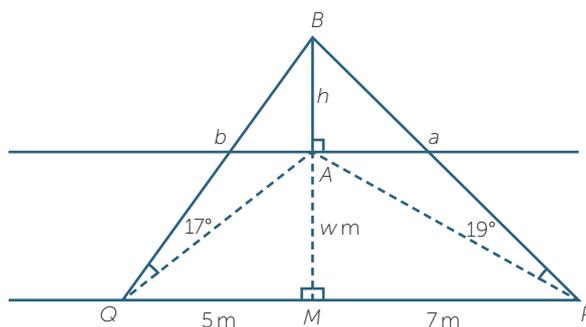
## THREE-DIMENSIONAL PROBLEMS

### EXAMPLE

The point  $M$  is directly across the river from the base  $B$  of a tree  $AB$ . From a point 7 metres upstream from  $M$ , the angle of elevation of the top  $A$  of the tree is  $17^\circ$ . From a point  $Q$ , 5 metres downstream from  $M$  the angle of elevation of the top of the tree is  $19^\circ$ . Assuming that  $PMQ$  is a straight line and that the tree is on the edge of the river, we wish to find the width  $w$  metres of the river.

### SOLUTION

In problems such as these, it is imperative to draw a careful diagram.



Let  $BP = a$ ,  $BQ = b$ , and  $AB = h$ , and then applying Pythagoras' theorem to triangles  $BMP$  and  $BMQ$ , we have  $a = \sqrt{49 + w^2}$ ,  $b = \sqrt{25 + w^2}$ .

From triangles  $ABP$  and  $ABQ$  we have  $h = a \sin 17^\circ$ ,  $h = b \sin 19^\circ$ .

Equating these, substituting in the values of  $a$  and  $b$  and squaring we arrive at

$$(49 + w^2)\sin^2 17^\circ = (25 + w^2)\sin^2 19^\circ$$

We can now make  $w^2$  the subject and obtain  $w^2 = \frac{49\sin^2 17^\circ - 25\sin^2 19^\circ}{\sin^2 19^\circ - \sin^2 17^\circ}$ , and so  $w \approx 8.66$  m correct to 2 decimal places.

Note: Do not evaluate until the last step to retain full calculator accuracy.

Civil engineers analyzing forces and stresses in buildings and other structures often use vectors to represent the direction and magnitude of these forces. A vector is an arrow which has both direction and magnitude. The sine and cosine rules are used in vector diagrams to find resultant forces and stresses. This is an important application.

## TRIGONOMETRIC IDENTITIES

As well as having practical uses, the sine and cosine rules can be used to derive theoretical results, known as **trigonometric identities** that have important implications and applications in later work. Among these are the **double angle** results which we will describe below.

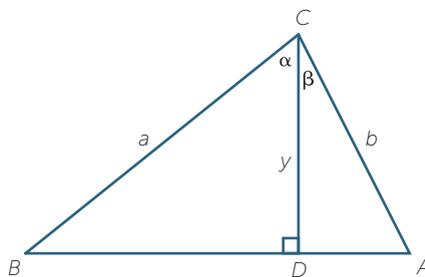
Using the area formula,  $A = \frac{1}{2}ab \sin C$ , for a triangle with two sides  $a$  and  $b$ , containing an angle  $C$ , we can do the following:

Fix acute angles  $\alpha$  and  $\beta$  and let angle  $C = \alpha + \beta$ . From the point  $C$  draw  $CD$  of length  $y$  and construct triangle  $ABC$  as shown in the diagram, where  $BA$  is perpendicular to  $CD$ .

From the diagram, we have

$$\frac{y}{a} = \cos \alpha \Rightarrow y = a \cos \alpha \quad (1)$$

and  $\frac{y}{b} = \cos \beta \Rightarrow y = b \cos \beta \quad (2).$



Comparing areas,

$$\frac{1}{2}ab \sin (\alpha + \beta) = \frac{1}{2}ay \sin \alpha + \frac{1}{2}by \sin \beta.$$

Substituting in the value of  $y$  from (2) into the first term and that from (1) into the second, we have, after some simplification,

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

In the discussion above we assumed that  $\alpha, \beta$  were acute angles. This identity holds for all  $\alpha$  and  $\beta$  but to show this requires a different approach.

Note that  $\sin (\alpha + \beta) \neq \sin \alpha + \sin \beta$ . For example,  $\sin (60^\circ + 30^\circ) = \sin 90^\circ = 1$ , whereas  $\sin 60^\circ + \sin 30^\circ = \frac{(1 + \sqrt{3})}{2} \neq 1$ .

## EXERCISE 15

Use the above formula to show that the exact value of  $\sin 75^\circ$  is  $\frac{\sqrt{2} + \sqrt{6}}{4}$ .

Putting  $\alpha = \beta = \theta$  in the above formula, we obtain the double angle formula for sine, namely

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

There is a similar double angle formula for cosine,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

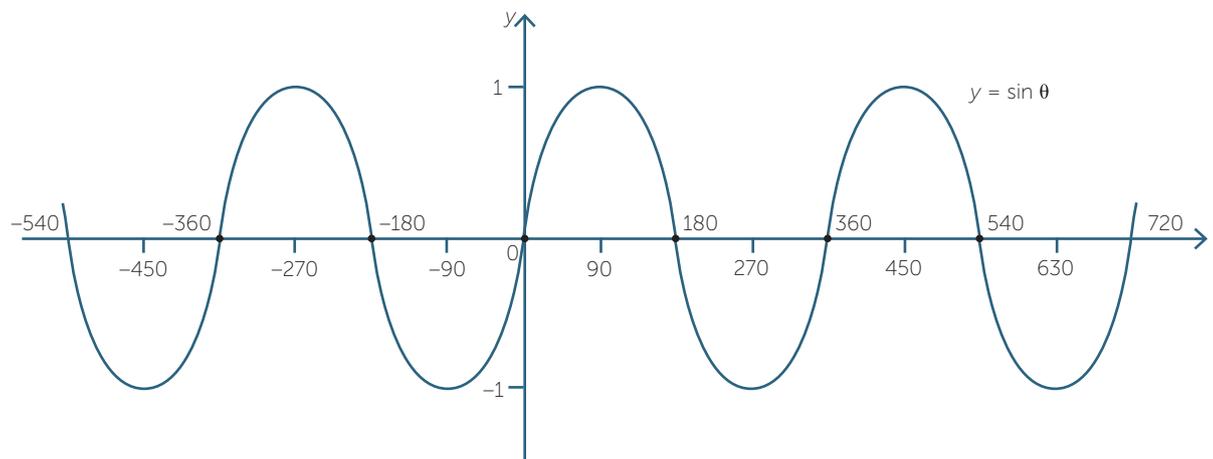
Both formulas are extremely useful when calculus is applied to the trigonometric functions.

## ANGLES OF ANY MAGNITUDE AND THE TRIGONOMETRIC FUNCTIONS

We saw in this module how to use the unit circle to give meaning to the sine and cosine of an obtuse angle. This definition can be extended to include angles greater than  $180^\circ$  and also to negative angles.

Thus, for example, if  $\theta$  is between  $180^\circ$  and  $270^\circ$ , then  $\sin \theta = -\sin(\theta - 180^\circ)$  and  $\cos \theta = -\cos(\theta - 180^\circ)$ .

Once we can find the values of  $\sin \theta$  and  $\cos \theta$  for values of  $\theta$ , we can plot graphs of the functions  $y = \sin \theta$ ,  $y = \cos \theta$ .

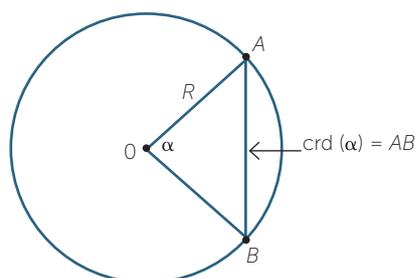


These ideas will be developed in the module, *The Trigonometric Functions*.

The graphs of the sine and cosine functions are used to model wave motion and electrical signals. They are an essential part of modern signal processing and telecommunications. This provides a breathtaking example of how a simple idea involving geometry and ratio was abstracted and developed into a remarkably powerful tool that has changed the world.

## HISTORY

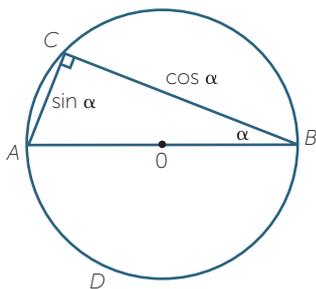
In the module, *Introductory Trigonometry*, we mentioned that the Greeks had a version of trigonometry involving chords. This is shown in the diagram below.



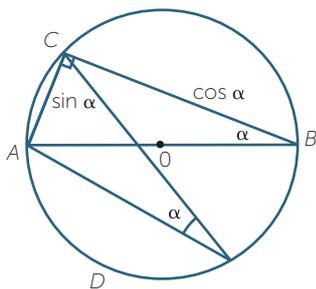
In the diagram, the **chord** of the angle is the length of the chord that subtends an angle  $\alpha$  at the centre in the circle of radius  $R$ .

Ptolemy (85-185AD), who lived and worked in Alexandria, wrote an extremely influential book called the *Mathematical Syntaxis*. It was translated into Arabic and given the Arabic title of *Almagest*.

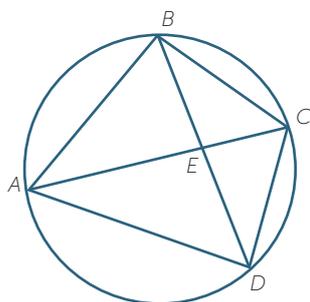
Ptolemy considered chords subtending an angle  $\alpha$  on the circumference. Using modern notation, if we take a diameter of  $AB$  length 1 unit as shown and a chord  $AC$  subtending an angle  $\alpha$  at the circumference, then the lengths  $AC$  and  $BC$  are respectively  $\sin \alpha$  and  $\cos \alpha$ .



Since angles on the circumference subtended by the same arc are equal, if  $\alpha$  is the angle subtended by any chord in this circle, then the length of that chord is always  $\sin \alpha$ .



Ptolemy also showed that if  $ABCD$  is a cyclic quadrilateral, then

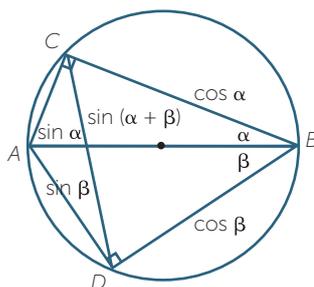


$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

That is, the sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals.

This result is known as *Ptolemy's theorem*.

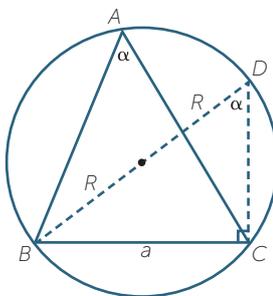
Applying Ptolemy's theorem in the diagram below, where the circle has diameter 1, we obtain the result  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ , that we derived above.



Using this, and other formulas, Ptolemy was able to construct a detailed table of chords of angles. Since chords are closely related to the sine ratio, he essentially had a table of sines.

### THE SINE RULE IN THE CIRCLE

Given a triangle  $ABC$ , we can draw its circumcircle with diameter  $BD$  as shown. Let  $2R$  be the diameter of the circumcircle.



Then for  $\alpha$  acute, the sine of angle  $BDC$  is given by

$$\sin \angle BDC = \sin \alpha = \frac{a}{2R}.$$

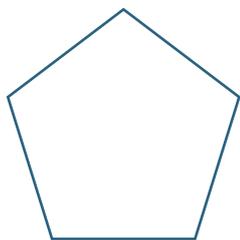
Re-arranging gives  $\frac{a}{\sin \alpha} = 2R$ .

Thus the quantities  $\frac{a}{\sin A}$ ,  $\frac{b}{\sin B}$  and  $\frac{c}{\sin C}$  in the sine rule are all equal to the diameter of the circumcircle of the triangle  $ABC$ .

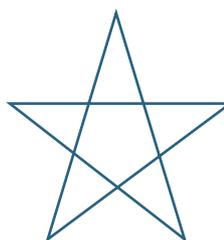
Regiomontanus (1436-1476), who wrote the first *modern* European book on trigonometry, included the sine rule and its derivation in his work.

### THE REGULAR PENTAGRAM AND REGULAR PENTAGON

The regular pentagram and the regular pentagon have always been a source of fascination and is often used in astrology. It is based on a triangle whose properties are investigated in the following exercise.

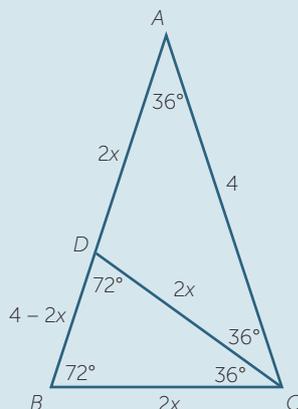


Regular pentagon



Regular pentagram

## EXERCISE 16



We begin with an isosceles triangle with angles  $72^\circ$ ,  $72^\circ$ ,  $36^\circ$ , as shown. This triangle occurs naturally inside both the regular pentagon and the regular pentagram. For ease of calculation we take the two equal sides to be 4 units in length.

We take the point  $D$  on  $AB$  such that  $\angle BDC = 72^\circ$ . Finally, we let  $BC = 2x$ .

- a Show that the information marked on the diagram is correct.
- b Prove that the triangles  $ABC$  and  $CDB$  are similar.
- c Deduce that  $\frac{4-2x}{2x} = \frac{2x}{4}$  and solve this equation to obtain  $x = \sqrt{5} - 1$ .
- d By dropping a perpendicular from  $A$  to  $BC$  show that  $\cos 72^\circ = \frac{\sqrt{5}-1}{4}$ .
- e Use the identity  $\cos^2 \theta + \sin^2 \theta = 1$  to show that  $\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$ .

## ANSWERS TO EXERCISES

### EXERCISE 1

**a**  $-1$                       **b**  $50(\sqrt{3} + 1)$

### EXERCISE 2

$\tan^{-1}\left(\frac{\sqrt{2}}{2}\right) \approx 35.26^\circ$ , correct to two decimal places.

### EXERCISE 3

$AP = 24.57\dots$  m and  $OA = 11.54$  m to the nearest cm.

### EXERCISE 4

Let  $\angle CAM = \theta$

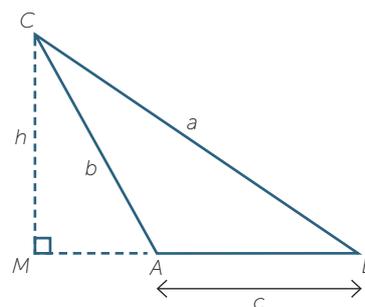
Therefore  $\angle CAB = 180^\circ - \theta$

$h = b \sin \theta$  (triangle  $CAM$ ) and  $h = a \sin B$  (triangle  $CAB$ )

Therefore  $a \sin B = b \sin \theta = b \sin (180^\circ - A)$   
 $= b \sin A$

$$\text{so } \frac{a}{\sin A} = \frac{b}{\sin B}$$

hence the result.



### EXERCISE 5

$\sin 0^\circ = 0$ ,  $\sin 90^\circ = 1$ ,  $\sin 180^\circ = 0$

$\cos 0^\circ = 1$ ,  $\cos 90^\circ = 0$ ,  $\cos 180^\circ = -1$

### EXERCISE 6

$-\frac{\sqrt{3}}{3}$ ,  $-\sqrt{3}$

### EXERCISE 7

$\sin^{-1}\left(\frac{5}{13}\right) \approx 22.62^\circ$  (correct to two decimal places). The two sides and the right angle define a unique triangle (RHS congruence).

### EXERCISE 8

In triangle  $BCM$ ,  $h^2 = a^2 - (c + MA)^2$

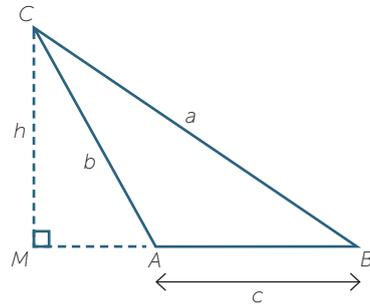
In triangle  $CMA$ ,  $h^2 = b^2 - MA^2$

Therefore,  $a^2 - (c + MA)^2 = b^2 - MA^2$

$$a^2 = b^2 + c^2 + 2c \times MA$$

But  $MA = b \cos(180^\circ - A) = -b \cos A$

$$\text{Hence } a^2 = b^2 + c^2 - 2bc \cos A$$



### EXERCISE 9

- $\sin A > \sin B > \sin C$  (if  $a > b$  and  $a \sin B = b \sin A$  then  $\sin A > \sin B$ )
- By the remark at the beginning of this paragraph,  $A > B > C$ .
- The angles  $P$  and  $Q$  add to  $\theta$ , because the angle sum of the triangle is  $180^\circ$ . Hence  $P$  and  $Q$  are smaller than  $\theta$ , so  $\sin P$  and  $\sin Q$  are less than  $\sin \theta = \sin P$ .
- From part **c**, the obtuse angle is  $A$ , which is therefore the largest of the three angles. Hence  $\sin B > \sin C$ , where  $B$  and  $C$  are acute, so  $A > B > C$ .

### EXERCISE 10

$$\begin{aligned} \text{a } \cos A &= \frac{b^2 + c^2 - a^2}{2abc} \quad \text{and} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \\ &= \frac{ab^2 + ac^2 - a^3}{2abc} & &= \frac{bc^2 + a^2b - b^3}{2abc} \end{aligned}$$

**b** Grouping terms,

$$\begin{aligned} \cos B - \cos A &= \frac{(a^2b - ab^2) - (ac^2 - bc^2) + (a^3 - b^3)}{2abc} \\ &= \frac{ab(a - b) - c^2(a - b) + (a^3 - b^3)}{2abc} \\ &= \frac{(ab - c^2)(a - b) + (a^3 - b^3)}{2abc} \end{aligned}$$

- From  $a > b > c > 0$ , it follows that  $ab > c^2$  and  $a^3 > b^3$ . Hence  $\cos B > \cos A$ .
- A similar argument proves that  $\cos C > \cos B$ , so  $\cos C > \cos B > \cos A$ .  
By the remark at the start of this paragraph,  $A > B > C$ .

### EXERCISE 11

- Let  $\triangle ABC$  be right-angled at  $C$ . Then  $AB^2 = AC^2 + BC^2$ . Hence  $AB^2$  is larger than both  $AC^2$  and  $BC^2$ , so  $AB$  is longer than both  $AC$  and  $BC$ .

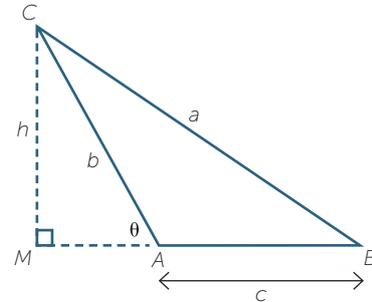
- b In a right-angled triangle, the other two angles are acute, because the angle sum of the triangle is  $180^\circ$ . Hence the right angle is the largest angle.

### EXERCISE 12

We use the same diagram as that used for the obtuse angle case proof of the sine and cosine rule.

$$\begin{aligned} \text{Area} &= \frac{1}{2}c \times h = \frac{1}{2}cb \sin \theta \\ &= \frac{1}{2}cb \sin A \end{aligned}$$

The stated result is obtained by symmetry of argument.



### EXERCISE 13

$$\theta = 30^\circ \text{ or } 150^\circ$$

### EXERCISE 14

For a given triangle  $ABC$ ,  $\frac{1}{2}cb \sin A = \frac{1}{2}ca \sin B$ . Therefore  $b \sin A = a \sin B$  and  $\frac{a}{\sin A} = \frac{b}{\sin B}$ .

### EXERCISE 15

$$\sin 75^\circ = \sin (45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \sin 30^\circ \cos 45^\circ$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4} \end{aligned}$$

### EXERCISE 16

- a  $\angle DCB = 36^\circ$  (angle sum of triangle),  $\angle ACD = 36^\circ$  ( $\angle DCA = 36^\circ$  and  $\angle ACD = 72^\circ$ )

$CD = 2x$  (triangle  $CBD$  is isosceles),  $DA = 2x$  (triangle  $CDA$  is isosceles),

$$DB = AB - AD = 4 - 2x.$$

- b Triangle  $ABC$  is similar to triangle  $CDB$  (AA)

c  $\frac{DB}{BC} = \frac{BC}{CB}$

d  $\cos B = \frac{x}{4}$

e  $\sin^2 B = 1 - \frac{x^2}{16} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$

hence  $\frac{4 - 2x}{2x} = \frac{2x}{4}$

$$\frac{2 - x}{x} = \frac{x}{2}$$

$$x^2 - 2x = 4$$

$$x^2 + 2x + 1 = 5$$

$$(x + 1)^2 = 5$$

$$x = -1 + \sqrt{5} \text{ as } x > 0$$



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The aim of the International Centre of Excellence for Education in Mathematics (ICE-EM) is to strengthen education in the mathematical sciences at all levels—from school to advanced research and contemporary applications in industry and commerce.

ICE-EM is the education division of the Australian Mathematical Sciences Institute, a consortium of 27 university mathematics departments, CSIRO Mathematical and Information Sciences, the Australian Bureau of Statistics, the Australian Mathematical Society and the Australian Mathematics Trust.



AUSTRALIAN MATHEMATICS TRUST



The ICE-EM modules are part of *The Improving Mathematics Education in Schools (TIMES) Project*.

The modules are organised under the strand titles of the Australian Curriculum:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

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