AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

The Improving Mathematics Education in Schools (TIMES) Project

CONES, PYRAMIDS AND SPHERES

A guide for teachers - Years 9–10

MEASUREMENT AND GEOMETRY : Module 12

June 2011

YEARS 9 10

Cones, Pyramids and Spheres

(Measurement and Geometry : Module 12)

For teachers of Primary and Secondary Mathematics

510

Cover design, Layout design and Typesetting by Claire Ho

The Improving Mathematics Education in Schools (TIMES) Project 2009-2011 was funded by the Australian Government Department of Education, Employment and Workplace Relations.

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MEASUREMENT AND GEOMETRY : Module 12

June 2011

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YEARS 9 10

CONES, PYRAMIDS AND SPHERES

ASSUMED KNOWLEDGE

- Familiarity with calculating the areas of the standard plane figures including circles.
- Familiarity with calculating the volume of a prism and a cylinder.
- Familiarity with calculating the surface area of a prism.
- Facility with visualizing and sketching simple three-dimensional shapes.
- Facility with using Pythagoras' theorem.
- Facility with rounding numbers to a given number of decimal places or significant figures.
- Facility with manipulation of formulas and equations

MOTIVATION

In the earlier module, *Area Volume and Surface Area* we developed formulas and principles for finding the volume and surface areas for prisms. The volume of a prism, whose base is a polygon of area A and whose height is h, is given by

Volume of a prism = Ah.

This formula is also valid for cylinders. Hence, if the radius of the base circle of the cylinder is r and its height is h, then:

Volume of a cylinder = $\pi r^2 h$

Also in that module, we defined the surface area of a prism to be the sum of the areas of all its faces. For a rectangular prism, this is the sum of the areas of the six rectangular faces. For other prisms, the base and top have the same area and all the other faces are rectangles.

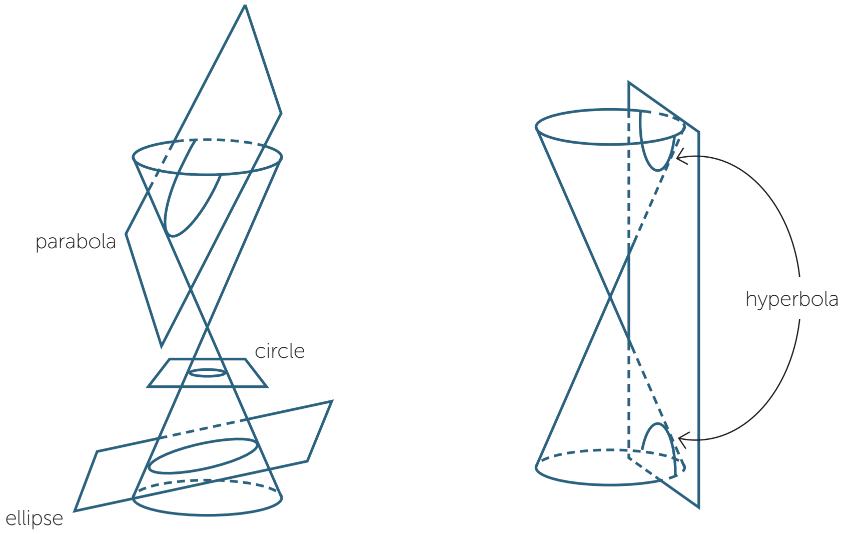
In this module, we will examine how to find the surface area of a cylinder and develop the formulae for the volume and surface area of a pyramid, a cone and a sphere. These solids differ from prisms in that they do not have uniform cross sections.

This will complete the discussion for all the standard solids.

Pyramids have been of interest from antiquity, most notably because the ancient Egyptians constructed funereal monuments in the shape of square based pyramids several thousand years ago. Conical drinking cups and storage vessels have also been found in several early civilisations, confirming the fact that the cone is also a shape of great antiquity, interest and application. The word *sphere* is simply an English form of the Greek *sphaira* meaning a *ball*.

Conical and pyramidal shapes are often used, generally in a truncated form, to store grain and other commodities. Similarly a silo in the form of a cylinder, sometimes with a cone on the bottom, is often used as a place of storage. It is important to be able to calculate the volume and surface area of these solids.

The ancient Greeks discovered the various so-called *quadratic curves*, the *parabola*, the *ellipse*, the *circle* and the *hyperbola,* by slicing a double cone by various planes. Of these, the parabola, obtained by slicing a cone by a plane as shown in the diagram below, is studied in some detail in junior secondary school. These *conic sections*, as they are also called, all occur in the study of planetary motion.

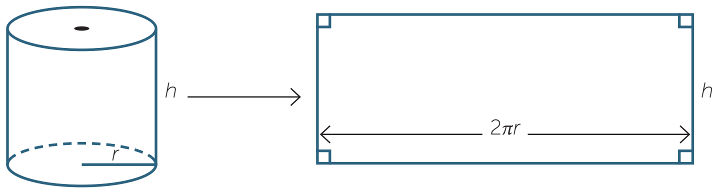


The sphere is an example of what mathematicians call a *minimal surface*. The sphere is a smooth surface that bounds a given volume using the smallest surface area, just as the circle bounds the given area using the smallest perimeter. The sphere is a three-dimensional analogue of the circle. As we saw in the module, The *Circle*, we use the word *sphere* to refer to either the closed boundary surface of the sphere, or the solid sphere itself.

CONTENT

SURFACE AREA OF A CYLINDER

Suppose we have a cylinder with base radius r and height h. If we roll the cylinder along a flat surface through one revolution, as shown in the diagram, the curved surface traces out a rectangle.



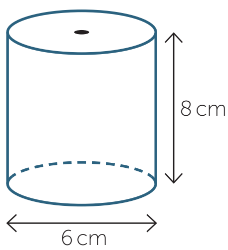
The width of the rectangle is equal to the height of the cylinder. The length of the rectangle is equal to the circumference of the circle, which is $2\pi r$. Hence the area of the curved portion of the cylinder is $2\pi rh$. Adding in the area of the circles at each end of the cylinder, we obtain,

Surface Area of a cylinder $= 2\pi rh + 2\pi r^2$.

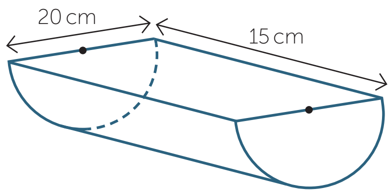
EXAMPLE

Calculate the surface area of each solid, correct to 2 decimal places.

a



b



SOLUTION

a Here, $r = 3$ and $h = 8$.

$$\begin{aligned}\text{Surface area} &= 2\pi rh + 2\pi r^2\\ &= 2 \times \pi \times 3 \times 8 + 2 \times \pi \times 3^2\\ &= 66\pi \text{ cm}^2\\ &\approx 207.35 \text{ cm}^2 \quad \text{(to 2 decImal places).}\end{aligned}$$

b Here, $r = 10$ and $h = 15$.

Area of curved section $= \frac{1}{2} \times 2\pi rh = 150\pi \text{ cm}^2$

Area of top rectangle $= 300 \text{ cm}^2$

Area of two semicircles $= \pi r^2$

$= 100\pi \text{ cm}^2$

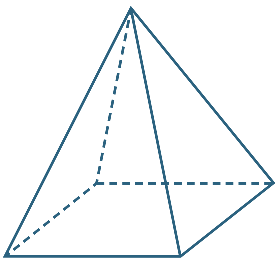
Surface area $= (250\pi + 300) \text{ cm}^2$

$\approx 1085.40 \text{ cm}^2$ (to 2 decimal places).

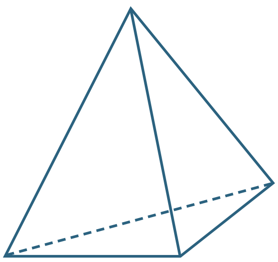
If the base of a pyramid is a regular polygon, then it has a well-defined centre. If the vertex of the pyramid lies vertically above the centre, then the pyramid is called a **right pyramid**. In most of what follows, we assume the pyramid is a right pyramid.

PYRAMIDS

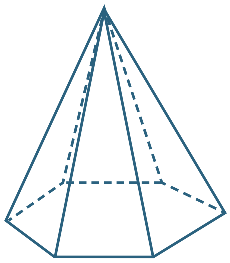
A pyramid is a polyhedron with a polygonal base and triangular faces that meet at a point called the **vertex**. The pyramid is named according to the shape of the base.



square-based pyramid



triangular-based pyramid

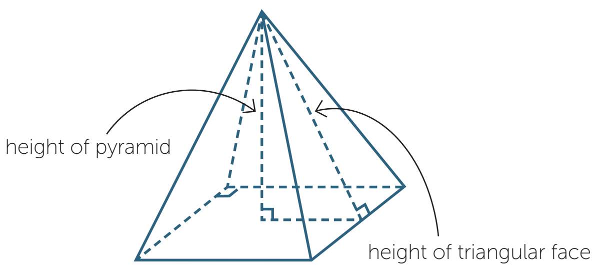


hexagonal-based pyramid

If we drop a perpendicular from the vertex of the pyramid to the base, then the length of the perpendicular is called the **height** of the pyramid.

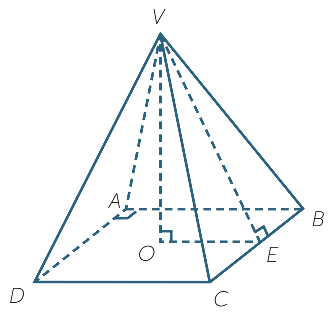
Surface area of a right pyramid

The faces bounding a right pyramid consist of a number of triangles together with the base. To find the surface area, we find the area of each face and add them together. Depending on the information given, it may be necessary to use Pythagoras' Theorem to calculate the height of each triangular face. If the base of the pyramid is a regular polygon, then the triangular faces will be congruent to each other.



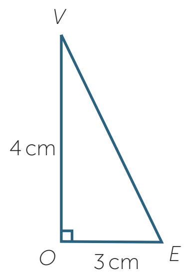
EXAMPLE

VABCD is a square-based pyramid with vertex *V* and base *ABCD*, with *V* vertically above the centre of the square base. The height of the pyramid is 4 cm and the side length of the base is 6 cm, find the surface area of the pyramid.



SOLUTION

We need to find the height *VE* of triangle *VBC*, using Pythagoras' Theorem.



$$\begin{aligned} VE^2 &= VO^2 + OE^2 \\ &= 4^2 + 3^2 \\ &= 25 \end{aligned}$$

Hence $VE = 5$ cm.

$$\begin{aligned} \text{Area of } VCB &= \tfrac{1}{2} \times CB \times VE \\ &= \tfrac{1}{2} \times 6 \times 5 \\ &= 15\text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of base} &= 6 \times 6 \\ &= 36\text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= 4 \times 15 + 36 \\ &= 96\text{ cm}^2 \end{aligned}$$

The surface area of the pyramid is 96 cm^2.

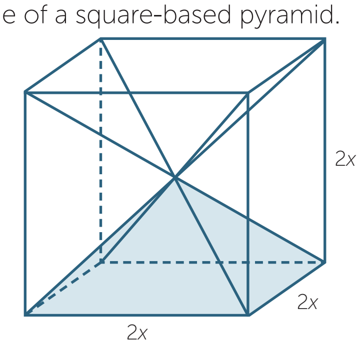
EXERCISE 1

When it was built, the Great Pyramid of Cheops in Egypt had a height of $145.\frac{3}{4}$ m and its base was a square of side length 229 m. Find its surface area in square metres, correct to three significant figures.

Volume of a pyramid

Here is a method for determining the formula for the volume of a square-based pyramid.

Consider a cube of side length 2*x*. If we draw the four long diagonals as shown, then we obtain six square-based pyramids, one of which is shaded in the diagram.



Each of these pyramids has base area $2x \times 2x$ and height x. Now the volume of the cube is $8x^3$. So the volume of each pyramid is $\frac{1}{6} \times 8x^3 = \frac{4}{3}x^3$. Since the base area of each pyramid is $4x^2$ it makes sense to write the volume as

$$\text{Volume} = \frac{1}{3} \times 4x^2 \times x = \frac{1}{3} \times \text{area of the base} \times \text{height}.$$

We can extend this result to any pyramid by using a geometric argument, giving the following important result.

$$\text{Volume of a pyramid} = \frac{1}{3} \times \text{area of the base} \times \text{height}.$$

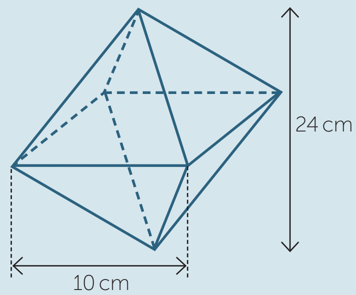
See the appendix on the pyramid for details.

EXERCISE 2

Find the volume of the Great Pyramid of Cheops whose height is 145.75 m and whose base is a square of side length 229 m. Give answer in cubic metres correct to two significant figures.

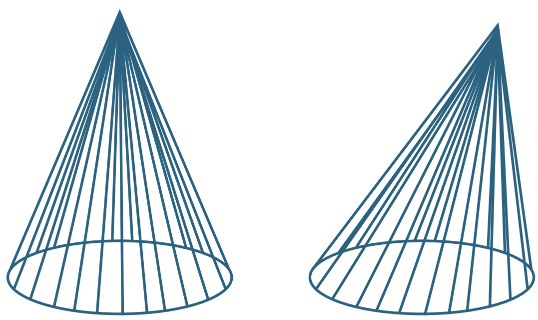
EXERCISE 3

Find the volume of the 'diamond', with height 24 cm and side length 10 cm as shown.



CONES

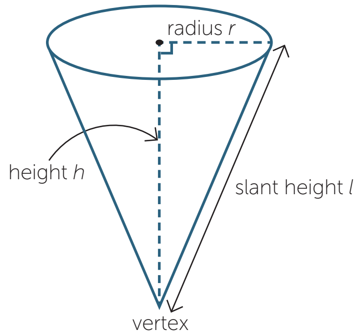
To create a **cone** we take a circle and a point, called the **vertex**, which lies above or below the circle. We then join the vertex to each point on the circle to form a solid.



If the vertex is directly above or below the centre of the circular base, we call the cone a **right cone**. In this section only right cones are considered.

If we drop a perpendicular from the vertex of the cone to the circular base, then the length of this perpendicular is called the **height** h of the cone.

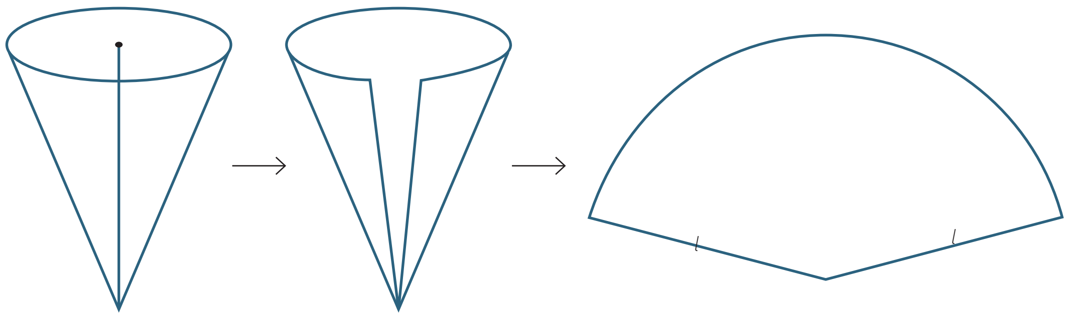
The length of any of the straight lines joining the vertex to the circle is called the **slant height** of the cone. Clearly $l^2 = r^2 + h^2$, where r is the radius of the base.



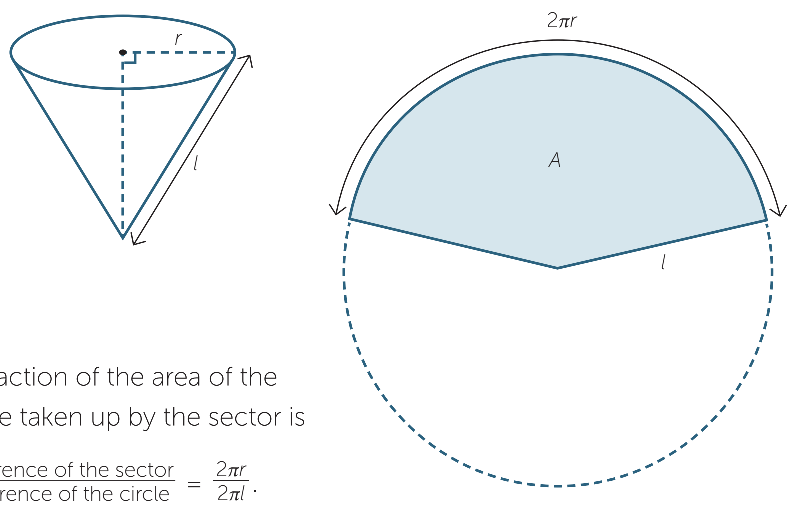
Surface area of a cone

Suppose the cone has radius r, and slant height l, then the circumference of the base of the cone is $2\pi r$.

To find the area of the curved surface of a cone, we cut and open up the curved surface to form a sector with radius l, as shown below.



In the figure to the right below the ratio of the area of the shaded sector to the area of the circle is the same as the ratio of the length of the arc of the sector to the circumference of the circle.



Thus the fraction of the area of the whole circle taken up by the sector is

$$\frac{\text{circumference of the sector}}{\text{circumference of the circle}} = \frac{2\pi r}{2\pi l}.$$

Hence, the area of the sector is $\frac{2\pi r}{2\pi l} \times \pi l^2 = \pi r l$.

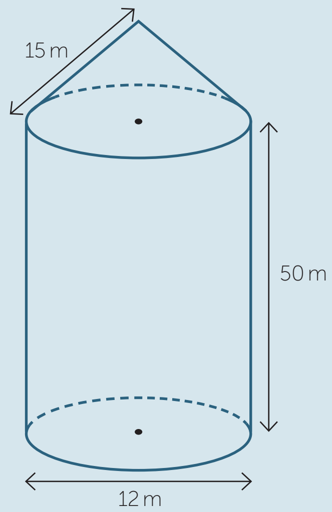
In conclusion, the area of the curved surface of the cone is πrl

Adding this to the base, we have

Surface area of a cone = $\pi rl + \pi r^2$.

EXERCISE 4

Find the surface area of the solid with dimensions shown.



Volume of a cone

When developing the formula for the volume of a cylinder in the module *Area Volume and Surface Area,* we approximated the cylinder using inscribed polygonal prisms. By taking more and more sides in the polygon, we obtained closer and closer approximations to the volume of the cylinder. From this, we deduced that the volume of the cylinder was equal to the area of the base multiplied by the height.

We can use a similar approach to develop the formula for the volume of a cone.

Given a cone with base radius r and height h, we construct a polygon inside the circular base of the cone and join the vertex of the cone to each of the vertices of the polygon, producing a polygonal pyramid. By increasing the number of sides of the polygon, we obtain closer and closer approximations to the cone. Hence,

$$\begin{aligned}\text{Volume of a cone} &= \tfrac{1}{3} \times \text{area of the base} \times \text{height}\\ &= \tfrac{1}{3}\pi r^2 h\end{aligned}$$

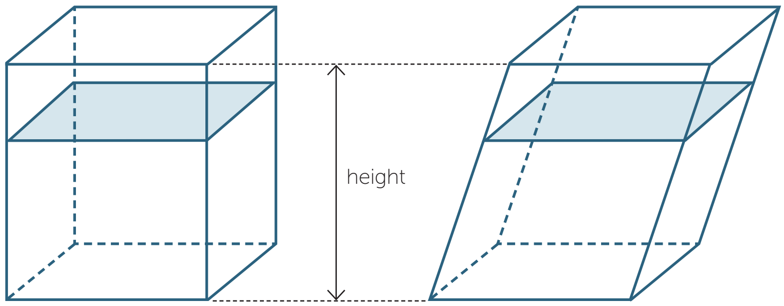
EXERCISE 5

Find the volume of the solid described in the previous exercise.

OBLIQUE PRISMS, CYLINDERS AND CONES

We have seen that the volume of a right rectangular prism is area of the base multiplied by the height. What happens if the base of the prism is not directly below the top?

Cavalieri's first principle states that if the cross-sections of two solids, taken at the same distance above the base, have the same area, then the solids have the same volume.



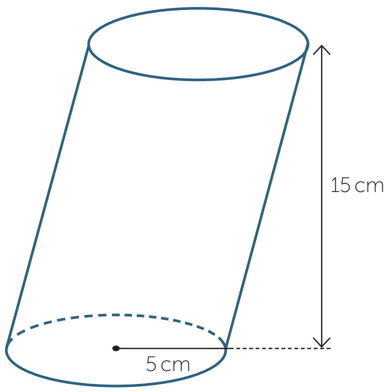
We will not give a proof of Cavalieri's principle here. To present a rigorous proof requires integration and slicing ideas.

It allows us to say that the volume of any rectangular prism, right or oblique, is given by the area of the base multiplied by the height.

The same applies to oblique cylinders and cones.

EXAMPLE

Find the volume of the cylinder shown in the diagram.



SOLUTION

The volume of the cylinder is $V = \pi \times 5^2 \times 15 = 375\pi \text{ cm}^2$.

Cavalieri's second principle states that if the cross-sections of two solids, taken at the same distance above the base, have areas in the ratio a:b, then the solids have the volumes in the ratio a:b.

EXERCISE 6

We showed earlier that the volume of a square based pyramid with base length $2x$ and height x has volume $\frac{1}{3} \times 2x \times 2x \times x$. Use Cavalieri's second principle to show that the volume of a pyramid whose base is a rectangle with side lengths c and d and height h is $\frac{1}{3} \times cd \times x$.

THE SPHERE

A **sphere** is the set of all points in three-dimensional space whose distance from a fixed point O (the centre), is less than or equal to r (the radius).

Every point on the surface of the sphere lies at distance r from the centre of the sphere.

We will derive the surface area formula from the volume formula.

Volume of a sphere

The easiest and most natural modern derivation for the formula of the volume of a sphere uses calculus and will be done in senior mathematics. A derivation using a clever application of Cavalieri's principle is discussed in the History section of this module.

The volume of a sphere radius is given by

Volume of a sphere of radius $r = \frac{4}{3}\pi r^3$.

EXAMPLE

a Calculate the volume of the a sphere with a diameter of 30 m.

b A sphere has volume 2800 cm^3. Find the radius of the sphere, correct to the nearest millimetre.

SOLUTION

a The radius is 15 m.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 15^3 \\ &= 4500\pi \text{ m}^3 \end{aligned}$$

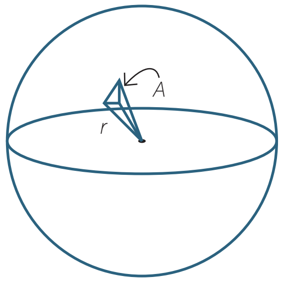
b

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ 2800 &= \frac{4}{3}\pi r^3 \\ \text{so } r^3 &= \frac{2800 \times 3}{4\pi} \\ r &= \sqrt[3]{\frac{2800 \times 3}{4\pi}} \\ &\approx 8.7 \text{ cm} \qquad \text{(correct to 1 decimal place).} \end{aligned}$$

The radius is approximately 87 mm.

Surface area of a sphere

Calculus is needed to derive the formula for the surface area of a sphere rigorously. Here is a interesting formula that uses the idea of approximating the sphere by pyramids with a common vertex at the centre of the sphere.



Consider a sphere of radius r split up into very small pyramids, as shown. The volume of each pyramid is equal to $\frac{1}{3}Ar$, where A is the area of the base. Suppose there are n of these pyramids in the sphere, each with base area A.

Hence the total volume of these pyramids is $\frac{1}{3}rnA$.

The more pyramids we take, the closer this will be to the volume of the sphere.

Also the sum of the areas of the bases, nA will get closer to the surface area of the sphere, S. Hence, using the formula for the volume of the sphere, we have

$$\frac{1}{3}rS = \frac{4}{3}\pi r^3, \text{ giving } S = 4\pi r^2.$$

Hence the surface area of a sphere radius r is

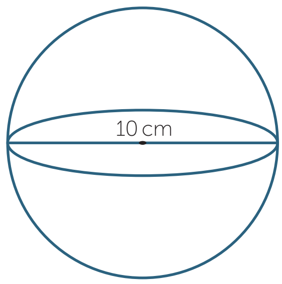
Surface area $= 4\pi r^2$.

EXAMPLE

Calculate, correct to 2 decimal places, the surface area of a sphere with diameter 10 cm.

SOLUTION

The diameter is 10 cm, so $r = 5$ cm.



Then $S = 4\pi r^2$

$= 4 \times \pi \times 5^2$

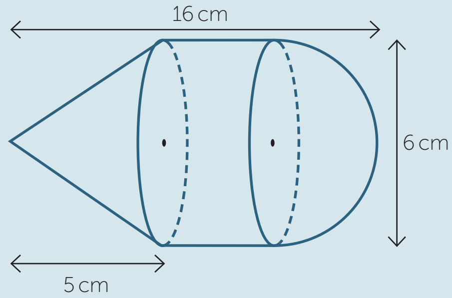
$= 100\pi \text{ cm}^2$

$\approx 314.16 \text{ cm}^2$ (correct to 2 decimal places).

The surface area is approximately 314.16 cm^2.

EXERCISE 7

Find the surface area and volume of the following:

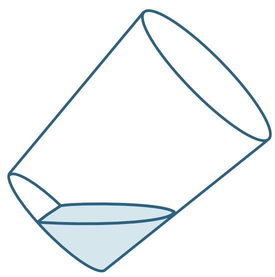


LINKS FORWARD

As promised in the Motivation, we have now completed the mensuration formulae of all the standard two and three-dimensional objects. Nevertheless, there are other objects that occur in everyday life whose areas and volumes we cannot find using these formulae and methods alone. Once such example are the `sails' on the Opera House in Sydney.

Here is another.

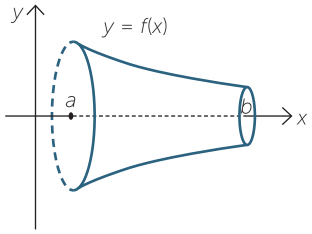
Suppose we 'half-fill' a glass with water. Supposing the glass is a cylinder of radius r and height h, and suppose the water covers exactly half of the circular base, what is the volume of the water?



This problem was posed and solved by Archimedes. It is usually solved today using slicing techniques from integral calculus.

Slicing techniques in calculus exploit the idea we saw when finding the volume of a prism or a cylinder. If we can find the volume of a typical slice of the solid, then, assuming the solid has uniform cross-section, we can add all the slices to find the volume. In calculus, we find the volume, in terms of a variable x, of a typical slice whose thickness, written as δx, is very small. We can then integrate this to obtain the total volume. This method can be used very effectively to find the volume of solids which do not have uniform cross-section, and may have curved boundaries.

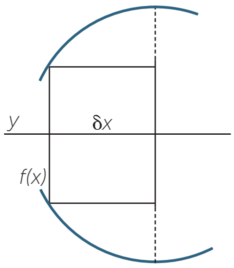
One simple application of this method allows us to find the volume of any solid that is formed by rotating part of a curve, $y = f(x)$, in the xy-plane, between $x = a$ and $x = b$, around the axis.



Those familiar with integral calculus will recognize the following formula for the volume of such a solid of revolution.

Volume $= \pi\int_a^b (f(x))^2 dx$.

Here, the thin slice of thickness δx is approximately a cylinder with base radius $f(x)$ and so the volume of the slice is approximately $\pi\,(f(x))^2\delta x$.



If we take the circle centre the origin and radius r, then its Cartesian equation is $x^2 + y^2 = r^2$. A sphere of radius can then be formed by rotating this circle about the x-axis. Thus, its volume is found by computing the integral

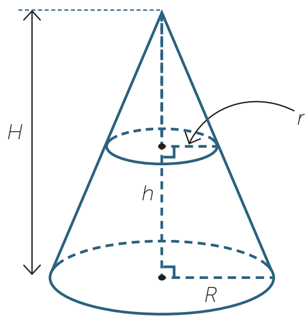
$$\pi\int_{-r}^{r} y^2 dx = \pi\int_{-r}^{r} y^2(r^2 - x^2)dx = \pi\left[xr^2 - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3$$

after simplification.

More on cones

The portion of a right cone remaining after a smaller cone is cut off is called a **frustum**. Suppose the top and bottom of a frustum are circles of radius R and r, respectively, and that the height of the frustum is h, while the height of the original cone is H. The volume of the frustum is by the difference of the volumes of he two cones and is given by

Volume of a frustum $= \frac{1}{3}\pi[H(R^2 - r^2) + r^2h]$.



Using similar triangles, we can eliminate H and the formula can be rewritten as

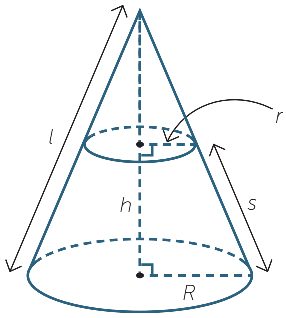
$$\text{Volume of a frustum} = \tfrac{1}{3}\pi h(R^2 + r^2 + rR).$$

EXERCISE 8

Derive these results.

Similarly, it can be shown that the surface area of the frustum of a cone with base radii r *and* R and slant height s, is given by

$$\text{Surface Area of a frustum} = \pi(r^2 + R^2) + \pi(r + R)s.$$



Note that when $r = l$ we obtain the surface area of a cylinder.

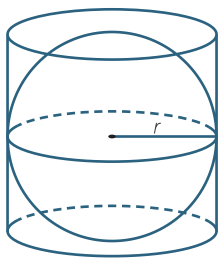
EXERCISE 9

Derive this formula.

The concepts and insights developed in finding the formulas for areas and volumes are used in Physics and Engineering to find such quantities as the *centre of mass* and the *moment of inertia* of a solid body. Thus the development of volume formulas are important for students, as is the careful memorizing of the key formulas, such as the volume of a sphere.

HISTORY AND DEVELOPMENT

Archimedes gave a geometric demonstration that the surface area of a sphere with radius r was equal to the area of the curved surface of the cylinder into which the sphere exactly fits.

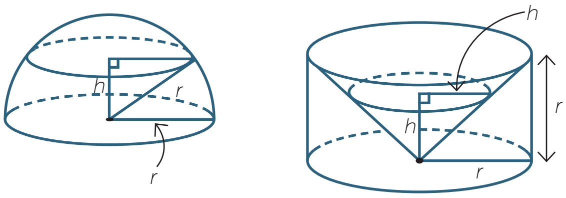


EXERCISE 10

Assuming Archimedes' result, derive the formula for the surface area of the sphere.

The formula for the volume of a sphere can be found with a striking application of Cavalieri's first principle as follows:

Take a hemisphere of radius and look at the area of a typical cross-section at height above the base.



Also consider a cylinder of height h and radius r, with a cone of the same height and radius removed.

EXERCISE 11

a Show that the area of the circular cross-section of the hemisphere at height h is given by $\pi(r^2 - h^2)$.

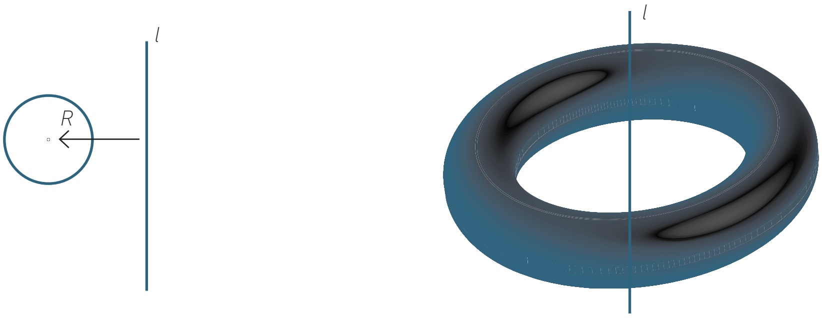
b Show that the area of the cross-section of the cylinder with the cone removed at height h is also $\pi(r^2 - h^2)$.

c Use Cavalieri's principle to deduce that these solids have the same volume and hence find derive the formula for the volume of the sphere.

PAPPUS' THEOREM

The later Greek Mathematician Pappus (290-350 AD) discovered the following remarkable method for finding the volume of a **solid of revolution** generated by rotating a plane region with area A about a fixed axis.

The volume of the solid is equal to the product of the area A of the plane region and the distance travelled by its geometric centroid as it moves through one revolution. The *centroid* is another word for *centre of mass*. This method can be used to find the volume of a *torus* (a Latin word meaning *couch*), which is the solid obtained by rotating a circle about a line external to the circle.



Suppose that we rotate a circle of radius r about a vertical line whose distance from the centre of the circle is R. The centroid of the circle is simply the centre of the circle and so as the circle is rotated about the line, the centroid will trace out another circle of radius R and which has circumference $2\pi R$. Since the area of the rotated circle is πr^2, Pappus' theorem tells us that the volume of the torus is

$$2\pi R \pi r^2 = 2\pi^2 r^2 R.$$

Pappus also showed that the surface area of a solid of revolution is equal to the product of the perimeter of the plane region being rotated and the distance travelled by its geometric centroid as it moves through one revolution.

EXERCISE 12

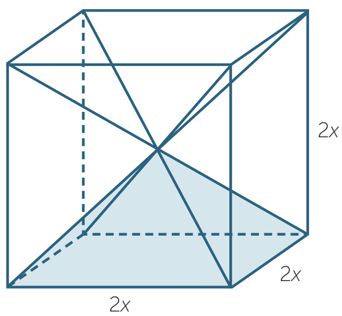
Show that the surface area of the torus is given by $4\pi^2 rR$.

Little further progress was made with volumes and surface areas until the development of the Calculus. The theory of several variable calculus enables more complicated volumes and surface areas to be calculated and the concept of area and volume to be generalised to higher dimensions.

Modern analysis is that branch of mathematics and studies and develops ideas from calculus. Using calculus, the theory of integration generalises the notions of area and volume. Measure Theory, a branch of modern analysis beginning the work of Henri Lebesgue, (d. 1910), generalises the notions of the integral.

APPENDIX ON THE PYRAMID

In the section Pyramids, we showed that a pyramid whose base is a square of sides $2x \times 2x$ and height x has volume $\frac{1}{3} \times 2x \times 2x \times x$ which is $\frac{1}{3}$ base $\times$ height.



In this appendix, we will show how to extend this result to any rectangular based pyramid.

It can also be shown that the volume of any pyramid is given by $\frac{1}{3}$ × area of base × perpendicular height.

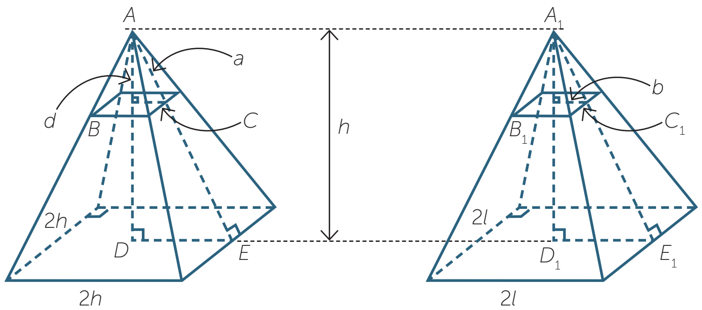
Cavalieri's first principle states that given two solids of the same height, whose cross-sections, taken at the same distance above the base, are of equal area, then the solids have the same volume.

Cavalieri's second principle states that if the cross-sections of two solids taken at the same distance above the base have areas in the ratio $a{:}b$, then the solids have their volumes in the ratio $a{:}b$.

We will use the second principle to show that the formula holds for any square pyramid of height h.

Take two square pyramids of height h, one with base square length $2h$ and one with base square length $2l$. From our earlier discussion, we know that the volume of the first pyramid is $V_1 = \frac{1}{3} \times 4h^2 \times h$. Let V_2 be the volume of the second pyramid.

Take a slices at B and B_1, distance d from the vertices, as shown in the diagram, so that $AB = A_1B_1 = d$.



Let $BC = a$, $B_1C_1 = b$. Referring to the diagrams, ΔABC is similar to ΔADE.

Hence $\frac{d}{h} = \frac{a}{h}$ implies $a = d$.

Hence the area of the cross-section for the first pyramid is given by $4a^2$ but this equals $4d^2$.

Now in the second pyramid, $\Delta A_1B_1C_1$ is similar to $\Delta A_1D_1E_1$.

Hence $\frac{d}{h} = \frac{b}{l}$ implies $b = \frac{ld}{h}$

Thus the area of the cross-section is $4b^2$ but this equals $4l^2d^2$. Now the ratios of the areas is $4a^2{:}4b^2 = 4d^2{:}\frac{4l^2d^2}{h^2} = h^2{:}l^2$.

This ratio is independent of d and so from Cavalieri's second principle, the ratio of the volumes is also $h^2{:}l^2$. This gives us

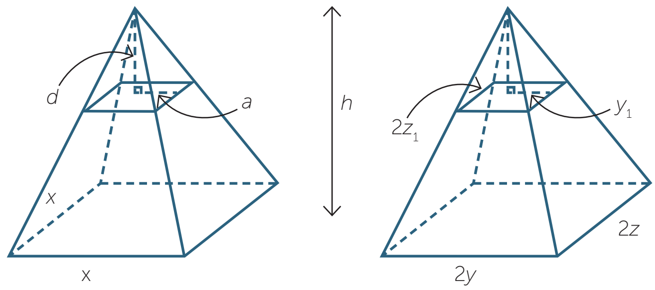
$$\frac{V_1}{V_2} = \frac{h^2}{l^2}$$

Hence $V_2 = \frac{l^2V_1}{h^2} = \frac{1}{3} \times 4l^2 \times h$ which is $\frac{1}{3} \times$ base $\times$ perpendicular height.

This tells us that the formula $\frac{1}{3} \times$ base $\times$ perpendicular height is valid for a square pyramid of height h.

Suppose we have two pyramids, one with a square base side length $2x$ and height h and one whose base is a rectangle with sides $2y \times 2z$, also of height h. The volume of the first pyramid is $\frac{1}{3} \times 4h^2 \times h$ by our discussion above.

We can choose x such that $4x^2 = 4yz$, so that the area of the rectangle equals the area of the square.



Take slices at a distance d from the vertex in each pyramid. Using similar triangles, we can easily show that so $\frac{d}{a} = \frac{h}{x}$ and $a = \frac{dx}{h}$. Thus the area of the cross-section in the first pyramid is $A_1 = 4a^2 = \frac{4d^2x^2}{h^2}$.

In the second pyramid, we can again use similar triangles, as we did above, to show that $y_1 = \frac{dy}{h}$. Similarly, if z_1 is half the width of the rectangular slice, then $z_1 = \frac{dz}{h}$ giving the area of the cross-section to be $4y_1z_1 = \frac{4d^2zy}{h^2}$.

Since $zy = x^2$, the cross-sections are equal in area and since the pyramids have the same height, their volumes are equal by Cavalieri's first principle. Thus the volume of the rectangular pyramid is $\frac{1}{3} \times 2y \times 2z \times h$, or $\frac{1}{3} \times$ base area x height.

It remains to say that a similar method can be used to progress from a pyramid with rectangular base to one whose base is a regular polygon, although the technical details are more complicated and will not be given here.

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A History of Mathematics: An Introduction, 3rd Edition, Victor J. Katz, Addison-Wesley, (2008)

History of Mathematics, D. E. Smith, Dover Publications, New York, (1958)

ANSWERS TO EXERCISES

EXERCISE 1

137 000 m^2

EXERCISE 2

2 500 000 m^3

EXERCISE 3

800 cm^3

EXERCISE 4

726π m^2

EXERCISE 5

$(1800 + 36\sqrt{21})\pi$ m^3

EXERCISE 6

The ratios of the areas of the cross-sections taken at the same heights is $4x^2$: cd

From Cavalieri's second principle:

Volume of pyramid $= \frac{4}{3}x^3 \times \frac{cd}{4x^2} = \frac{1}{3}\,cdx$

EXERCISE 7

Surface area $=(3\sqrt{34} + 66)\pi$ cm^2

Volume $= 105\pi$ cm^3

EXERCISE 8

$$\begin{aligned}\text{Volume of frustrum} &= \text{volume of large cone} - \text{Volume of small cone}\\ &= \frac{1}{3}\,\pi R^2 H - \frac{1}{3}\,\pi r^2(H-h)\\ &= \frac{1}{3}\,\pi((R^2 - r^2)H + r^2 h)\end{aligned}$$

By similarity, $H = \frac{hR}{R-r}$. Substituting in the above formula gives the result.

EXERCISE 9

Surface area of the frustrum = $\pi(r^2 + R^2) + \pi Rl - \pi r(l - s)$

By similar triangles, $Rs = Rl - lr$

$\pi(r^2 + R^2) + \pi Rl - \pi r(l - s) = \pi(r2 + R2) + \pi(r + R)s$

EXERCISE 10

$2\pi \times r \times 2r = 4\pi r^2$

EXERCISE 11

a The square of the radius of the cross-section = $r^2 - h^2$ (By Pythagoras' theorem)

Area of cross-section= $\pi(r^2 - h^2)$

b Area of cross-section = $\pi r^2 - \pi h^2$

c Volume of cylinder – volume of cone = $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$

EXERCISE 12

Surface area = $2\pi r \times 2\pi R = 4\pi^2 rR$

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