THE CHANGING HISTORY OF MATHS

Ancient Tablet Reveals Babylonian Trigonometry Beginnings

By Laura Watson

You'd be forgiven for thinking that it was an airport thriller, perhaps the latest Dan Brown: an ancient civilisation, a broken tablet, mysterious numbers, and a long-forgotten form of trigonometry.

However, you won't find this story on the shelves but in *Historica Mathematica*. Two University of New South Wales mathematicians have challenged the origins of trigonometry in an explosive paper, *Plimpton 322 is Babylonian exact sexagesimal trigonometry*.

With 15 rows of complicated Pythagorean triples written in the Babylonian sexagesimal (base 60) arithmetical system, missing columns and rows, and curious errors, Plimpton 322 has been the subject of decades of intense scholarship. Lecturer Dr Daniel Mansfield and Professor Norman Wildberger are among the many drawn to its mysteries. Their theory was born from a routine discussion following Daniel's explanation of the tablet to his first year Linear Algebra class.

"Daniel and I started talking about Plimpton 322 and it occurred to us that it might have something to do with the Rational Trigonometry theory that I had written a book about earlier. It turns out that we were much more correct than I could have wished," says Professor Wildberger.

FIGURE 1

To understand the enormity of the pair's theory and its potential impact, you need to first understand the geometrical thinking behind Plimpton 322. The world's first trigonometry bears little resemblance to the more familiar application of angles to triangles.

"It is unlike anything we have today. There are no angles and no approximations. The Babylonians thought of a right triangle as half a rectangle, and used the ratio of sides to measure steepness. Concepts such as sin, cos and tan of an angle are out," explains Dr Daniel Mansfield.

"They measured fields, built temples, palaces and canals– so why should we presume that this advanced society did not have a trigonometry of some kind?" he says.

Wildberger agrees, citing the Old Babylonian notion of *ukullu*.

"This measure is similar to *seked*, a ratio based on the Egyptian length measure of the royal cubit. Run over rise, the reciprocal to our slope, it was used to describe the steepness of pyramids," Wildberger explains.

But this is not the first time that Plimpton 322 has made headlines. In 1945 the tablet showed us something new about the origins of another ancient discovery.

"Plimpton 322 showed that the Babylonians understood Pythagoras' theorem more than 1000 years before Pythagoras. This suggests that a lot of other Greek innovations may have had their origins in earlier Mesopotamian understandings," says Wildberger.

The impact of these findings on our understanding of mathematics is significant. Trigonometry as we know it began with Hipparchus in the 2nd century BC. But Plimpton 322 shows us that there is a much older, more precise way of thinking about trigonometry.

"The traditional way to study triangles uses angles. In his book, Norman showed that triangles can also be studied using ratios alone. Then we discovered that this is what the Babylonians were doing in this tablet," says Mansfield.

The ordering of the Plimpton table is also significant, however, Mansfield urges caution in using this to draw conclusions the table is trigonometric. An argument must give careful thought to the historical context.

"Plimpton 322 enumerates shapes of right triangles which are decreasing at roughly uniform intervals, so it is easy to conclude the table is trigonometric. But saying it is about sin, cos and tan is fatal because the concept of angle did not exist at the time it, Plimpton 322, was written," Mansfield explains.

One of the challenges for many mathematicians, according to Wildberger, is that the table brings into question so much that is assumed not only about geometry but arithmetic and computation.

"Plimpton 322 is not only the world's first trig table, but remarkably the only complete exact table! We are so used to thinking of trigonometry as an approximate science that it is hard for us to fathom an exact approach," he says.

With less than 200 years of discovery to draw from, there is still a great deal we don't know about this mysterious ancient culture and its contribution to mathematical and scientific understanding. Like many of our favourite fiction series, what's in the sequel may prove even more exciting.

"We have some good ideas about other directions for Old Babylonian research,

THE BABYLONIANS UNDERSTOOD PYTHAGORAS' THEOREM MORE THAN 1000 YEARS BEFORE PYTHAGORAS

particularly into the question of how they managed to perform such advanced arithmetic," says Wildberger.

With the application of mathematics to history a rich and under-developed research area, Mansfield confirms the pair have their sights on how Babylonian computation might have something to teach us.

"I've always been attracted to logic in both history and mathematics. However, logic alone is not sufficient. You also need to understand the spirit of the subject to guide your reasoning – then the subject comes to life," says Mansfield.

It is clear that in this quest to solve the mysteries of the Plimpton 322, the spirt of this ancient culture is as vital to understand as the mathematical legacy they left behind. □

You can read the full paper published in Historia Mathematica. Daniel F. Mansfield, N. J. Wildberger. Plimpton 322 is Babylonian exact sexagesimal trigonometry. Historia Mathematica, 44(4), November 2017, Pages 395-419.

THE UNSW MATHEMATICIANS

DR DANIEL MANSFIELD

Daniel Mansfield is a Lecturer in the School of Mathematics and Statistics at UNSW Sydney. He is an award-winning educator who was voted as UNSW's most inspiring first year lecturer in 2017. He has a passion for improving mathematics education, especially in schools.

PROFESSOR NORMAN WILDBERGER

Professor Norman Wildberger of the School of Mathematics and Statistics at UNSW Sydney has done research in harmonic analysis, combinatorics and hyperbolic geometry. He has written a book on Rational Trigonometry, and is also a keen YouTube educator, with his own channel Insights into Mathematics.

FIGURE 1: WHAT IS PLIMPTON 322?

The clay tablet Plimpton 322 is named after George Arthur Plimpton, who bought it from Edgar Banks in about 1922 and later bequeathed it to Columbia University. It measures 12.7 cm by 8.8 cm, and based on a comparison of writing styles with other Babylonian tablets, has been dated to between 1822 and 1762 BC, which is the time of the Babylonian king Hammurabi.

It has four columns and 15 rows of numbers. Vertical column lines are also drawn on the back of the tablet, which is otherwise empty.

In 1945, the Austrian mathematician Otto Neugebauer and his associate Abraham Sachs noted that Plimpton 322 has 15 pairs of numbers forming parts of Pythagorean triples, meaning three whole numbers a, b and c such that $a^2 + b^2 = c^2$. The integers 3, 4 and 5 are a well-known example of a Pythagorean triple, but the values on Plimpton 322 are often considerably larger with, for example, the first row referencing the triple 119, 120 and 169.

FIGURE 2: TRANSLATING PLIMPTON 322 INTO MODERN MATHEMATICAL TERMS

Mansfield and Wildberger have built on previous research to present new mathematical evidence suggesting there were originally 6 columns, and that the tablet was meant to be completed to 38 rows. They also present the most likely method of construction of the tablet.

The table below lists the numbers in the first 11 rows of this completed form of Plimpton 322, written in our decimal number system, approximated to eight decimal points. (For more rows see the published paper).

The completed tablet contains the three ratios of sides of right-angle triangles – but written exactly using the Babylonian base 60 place value system. A known ratio of sides of a right-angle triangle could be compared to these lists to work out the unknown ratios.

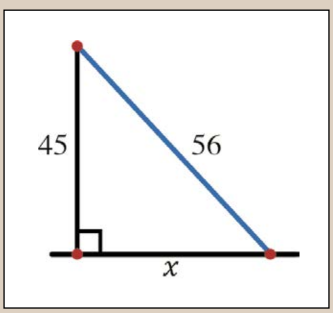
The table describes the shapes of right-angle triangles with base b, long side l, and diagonal d, where in all cases l is a regular number in their system, with only factors 2, 3 and 5.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
ß = b/l	$\delta = d/l$	δ^2	base b	diagonal d	Row number
0.99166666	1.40833333	1.98340277	119	169	1
0.97424768	1.39612268	1.94915855	3367	4825	2
0.95854166	1.38520833	1.91880212	4601	6649	3
0.94140740	1.37340740	1.88624790	12709	18541	4
0.90277777	1.34722222	1.81500771	65	97	5
0.88611111	1.33611111	1.78519290	319	481	6
0.84851851	1.31148148	1.71998367	2291	3541	7
0.83229166	1.30104166	1.69270941	799	1249	8
0.80166666	1.28166666	1.64266944	481	769	9
0.76558641	1.25941358	1.58612256	4961	8161	10
0.75	1.25	1.5625	45	75	11

COLUMN 1 - contains the exact ratio b/l. | COLUMN 2 - contains the exact ratio d/l. | A modern trigonometric table would also contain the ratio b/d or d/b. But this ratio could not be written exactly by the Babylonians in base 60. So, to maintain the exact nature of the table, they split this ratio into three parts. | COLUMN 3 - contains the exact ratio $(d/l)^2$ and was used as an index into the table when b/d or d/b was already known. | COLUMNS 4 & 5 - contain the numbers b and d with common factors removed and was used so the scribe could construct their own approximation to b/d or d/b. | COLUMN 6 is just the row number

FIGURE 3: USING THE PLIMPTON 322 TABLE

Suppose that a ramp leading to the top of a ziggurat, or stepped pyramid, is 56 cubits long, and the vertical height of the ziggurat is 45 cubits. What is the distance x from the outside base of the ramp to the point directly below the top?



Solution using Plimpton 322:

For this right-angle triangle, l = 45 and d = 56.

This means $\delta = d/l = 56/45$ = about 1.2444.

On the tablet, the closest value of δ in column 2 is found in row 11, where $\delta = 1.25$.

From columns 4 and 5 in row 11 you can work out that the corresponding ratio for this triangle is b/d = 45/75 = 3/5 = 0.6.

Thus x approximately equals 56 x 0.6 = 33.6

Using a calculator today, we can determine that x approximately equals the square root of (562 – 452) which is approximately 33.3317.

The UNSW mathematicians show the Babylonian approach, which avoids calculating square roots, was more accurate than a trigonometric sine table approach devised by the Indian mathematician Madhava more than 3000 years later.