Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Calculus: Module 17

Motion in a straight line

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Motion in a straight line – A guide for teachers (Years 11–12)

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Motion in a straight line

Assumed knowledge

The content of the modules:

- *Introduction to differential calculus*
- *Applications of differentiation*
- *Integration*.

Motivation

> I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.
>
> — Isaac Newton (1642–1727)

Mathematics is now applied to all branches of science, engineering, commerce and the social sciences. But traditionally, mathematics was applied to the study of science, and principally to physics. The study of physics depends on mathematical methods and models, particularly calculus and differential equations.

From the 17th century onwards, mathematics developed in two directions: pure and applied. One of the first areas of applied mathematics to be studied in the 17th century was motion in a straight line, often referred to as the **kinematics** of a particle moving in a straight line.

Sir Isaac Newton first developed both the differential and integral calculus in the 1660s, then shortly afterwards turned his attention to kinematics. In his 1671 work *Treatise on methods*, Newton listed the two central problems of calculus as:

- 'Given the length of the space continuously (that is, at every time), to find the speed of the motion at any time proposed.' (In other words, find the speed of a particle given the position.)
- 'Given the speed of motion continuously, to find the length of the space described at any time proposed.' (In other words, find the position given the speed.)

These problems were the motivation for Newton's work on calculus.

In this module, we deal with the quantities: position, displacement, distance, velocity, speed, acceleration and time. We establish mathematical models for the motion of a particle in a straight line. Initially, we deal with the special case of constant acceleration.

The study of motion in a straight line should be included in any introductory course in calculus. It serves both as a context for understanding the basic concepts of calculus and as an important application of calculus.

Portrait of Isaac Newton when undertaking a B.A. at Trinity College, 1677. Engraved by Burnet Reading (1749/50–1838) from a head painted by Sir Peter Lely (1618–1680), published 1799. Source: University of Cambridge, Institute of Astronomy Library.

Content

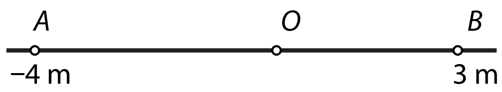
Position, displacement and distance

In this module, we are only talking about motion in a straight line. For a horizontal line, there are only two directions to consider: right and left. For a vertical line, the two directions are up and down. We choose a point O on the line, which we call the reference point or **origin**. For convenience, we will measure distance in metres and time in seconds.

Position

The line is coordinatised and referenced from a point O, the origin. For a horizontal line, the convention is that positions to the right of O are positive, and positions to the left are negative.

For example:



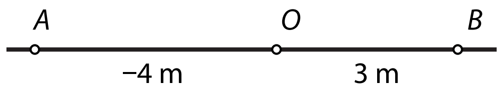
- The position of the particle at B is 3 m.
- The position of the particle at A is -4 m.

The position of a particle is often thought of as a function of time, and we write $x(t)$ for the position of the particle at time t.

Displacement

The **displacement** of a particle moving in a straight line is the change in its position. If the particle moves from the position $x(t_1)$ to the position $x(t_2)$, then its displacement is $x(t_2) - x(t_1)$ over the time interval $[t_1, t_2]$. In particular, the position of a particle is its displacement from the origin.

For example:



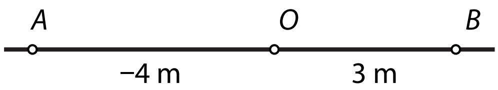
- If a particle moves from O to B, its displacement is 3 m.
- If a particle moves from O to A, its displacement is -4 m.
- If a particle moves from A to B, its displacement is 7 m.
- If a particle moves from B to A, its displacement is -7 m.

Position and displacement are vector quantities, that is, they have both magnitude and direction. In this module, we are dealing with vectors in one dimension. The sign of the quantity (positive or negative) indicates its direction. The absolute value of the quantity is its magnitude.

Distance

The **distance** is the 'actual distance' travelled. Distances are always positive or zero.

For example, given the following diagram, if a particle moves from A to B and then to O, the displacement of the particle is 4 m, but the distance travelled is 10 m.



Example

A particle moves along a straight line so that its position at time t seconds is $x(t)$ metres, relative to the origin. Assume that $x(0) = 0$, $x(3) = 2$ and $x(6) = -5$, and that the particle only changes direction when $t = 3$. Find the distance travelled by the particle from time $t = 0$ to time $t = 6$.

Solution

The distance travelled is $2 + 7 = 9$ metres.

Summary

- The **position** of a particle moving in a straight line is a vector which represents a point P on the line in relation to the origin O. The position of a particle is often thought of as a function of time, and we write $x(t)$ for the position of the particle at time t.
- The **displacement** of a particle moving in a straight line is a vector defined as the change in its position. If the particle moves from the position $x(t_1)$ to the position $x(t_2)$, its displacement is $x(t_2) - x(t_1)$ for the time interval $[t_1, t_2]$.
- The **distance** travelled by a particle is the 'actual distance' travelled.

Constant velocity

The rate of change of the position of a particle with respect to time is called the **velocity** of the particle. Velocity is a vector quantity, with magnitude and direction. The **speed** of a particle is the magnitude of its velocity.

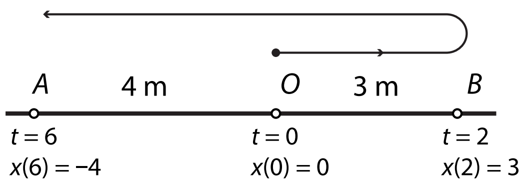
Students have learnt in earlier years that, for motion at a constant speed,

$$\text{Speed} = \frac{\text{Distance travelled}}{\text{Time taken}}.$$

Similarly, for motion at a constant velocity, we have

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time taken}}.$$

For example, consider a particle which starts at the origin O, moves to a point B at a constant velocity, and then moves to a point A at a different constant velocity, as shown in the following diagram.



- The particle moves from O to B in two seconds. During this time, it has a constant velocity of $\frac{3}{2} = 1.5$ metres per second.
- The particle moves from B to A in four seconds. During this time, it has a constant velocity of $-\frac{7}{4} = -1.75$ metres per second.

In this module, we abbreviate 'metres per second' as m/s. The alternative abbreviation ms^{-1} is also very common.

- If a particle is moving with a constant velocity of 10 m/s, then its displacement over 5 seconds is 50 metres.
- If a particle is moving with a constant velocity of −10 m/s, then its displacement over 5 seconds is −50 metres. The distance travelled is 50 metres.

If a particle is moving with constant velocity, it does not change direction. If the particle is moving to the right, it has positive velocity, and if the particle is moving to the left, it has negative velocity. In general, if a particle is moving at a constant velocity (rate), then its constant velocity v is determined by the formula

$$v = \frac{x(t_2) - x(t_1)}{t_2 - t_1},$$

where $x(t_i)$ is the position at time t_i.

Example

A particle is moving in a straight line with constant velocity, and its position at time t seconds is $x(t)$ metres. If $x(1) = 6$ and $x(5) = -12$, find the velocity of the particle.

Solution

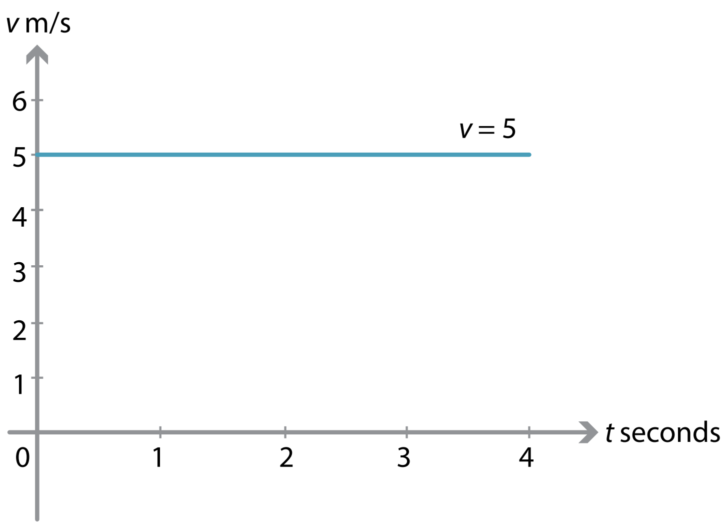
The velocity is

$$\begin{aligned} \frac{x(5) - x(1)}{5 - 1} &= \frac{-12 - 6}{4} \\ &= -\frac{9}{2} \text{ m/s.} \end{aligned}$$

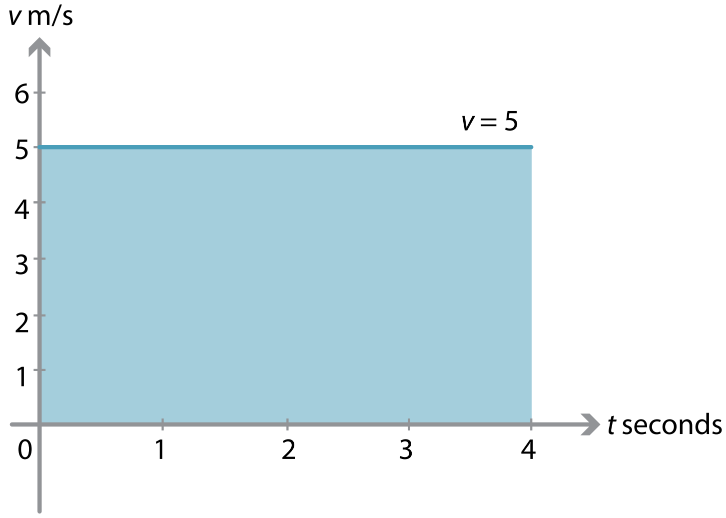
There are two important graphical representations for constant velocity. The first is the graph of velocity against time. The second is the graph of position against time.

Velocity–time graphs

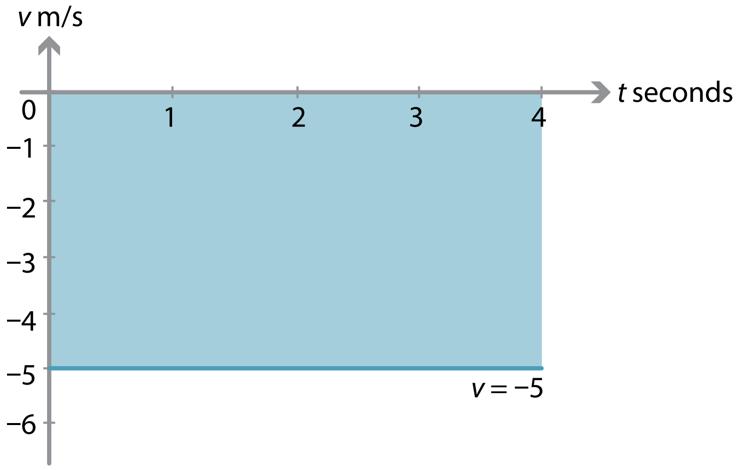
The following diagram shows the velocity–time graph for a particle moving at 5 m/s for 4 seconds. The constant velocity is plotted against time. Naturally, it gives a line segment parallel to the t-axis. The gradient of this line is zero.



The region between the line segment and the t-axis is shaded in the following diagram. This region is a rectangle of area $5 \times 4 = 20$, which is the product of the velocity and the time taken. So this area represents the displacement. The area between the line segment and the t-axis is the displacement of the particle from $t = 0$ to $t = 4$.



If the particle has a constant velocity of -5 m/s for 4 seconds, as shown in the following diagram, then the region lies below the t-axis and represents a displacement of -20 m. The region has a **signed area** of -20. The notion of signed area is introduced in the module *Integration*.



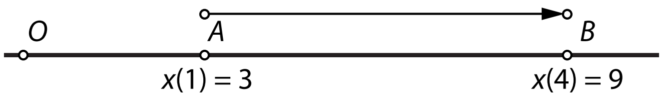
Finding the displacement of a particle from the velocity–time graph using integration will be discussed in a later section of this module.

Position–time graphs

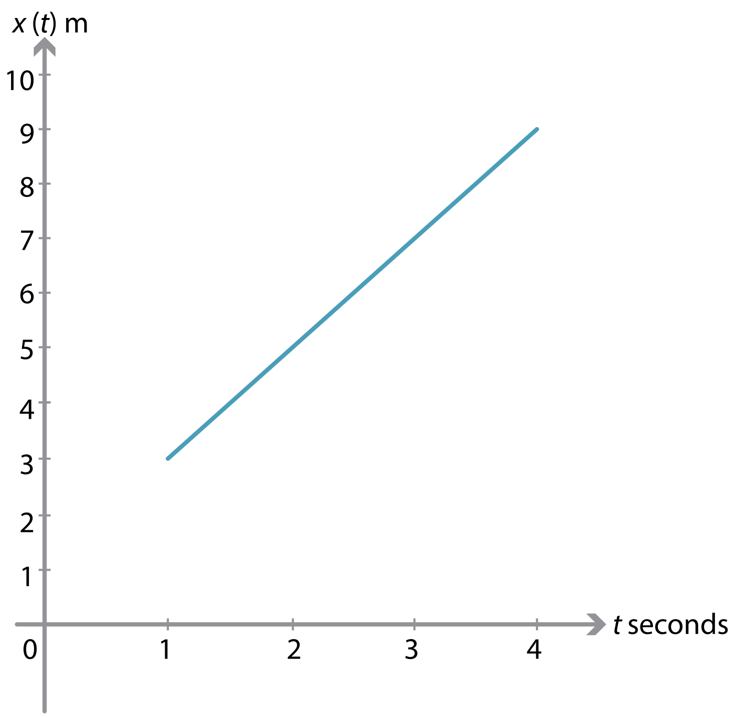
We can also plot a graph of position against time. In this case, it is the gradient of the graph that is of interest.

Positive velocity

Assume that a particle moves with constant velocity from point A to point B, as shown in the following diagram. At time $t = 1$, the position of the particle is 3 m to the right of O, that is, $x(1) = 3$. At time $t = 4$, its position is 9 m to the right of O, that is, $x(4) = 9$.



The displacement (change of position) of the particle is 6 metres over the time interval $[1,4]$. The duration of the motion is 3 seconds. Therefore the constant velocity is 2 m/s. The position–time graph for this motion is as follows.



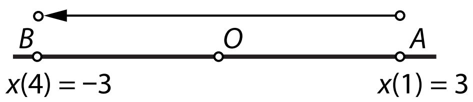
The gradient of this line is

$$\frac{x(4) - x(1)}{4 - 1} = \frac{9 - 3}{4 - 1} = \frac{6}{3} = 2.$$

Since this is the displacement divided by the time taken, the gradient of the line is equal to the velocity. It is always the case that, for a particle moving with constant velocity, the gradient of the position–time graph gives the velocity of the particle.

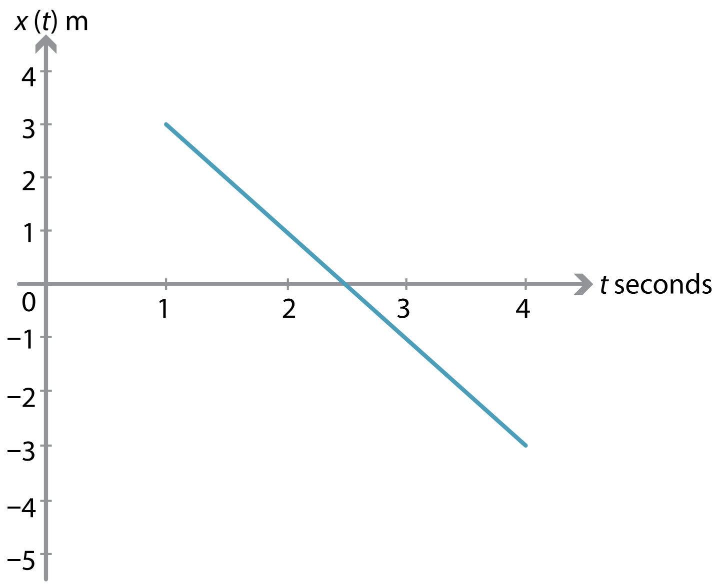
Negative velocity

Assume that a particle moves with constant velocity from A to B, as in the following diagram. At time $t = 1$, the position of the particle is 3 m to the right of O, that is, $x(1) = 3$. At time $t = 4$, its position is 3 m to the left of O, that is, $x(4) = -3$.



The displacement is −6 metres over 3 seconds. Therefore the constant velocity is −2 m/s. The gradient of the position–time graph, shown below, is

$$\frac{x(4) - x(1)}{4 - 1} = \frac{-3 - 3}{4 - 1} = \frac{-6}{3} = -2.$$



Throughout this module, we often use the gradient of a position–time graph to determine velocity, and use the area under a velocity–time graph to determine displacement. These are both important ideas, which will be further developed using calculus.

Example

A particle moves in a straight line with constant velocity so that, at time t seconds, the position of the particle is $x(t)$ metres, with respect to the origin O. Assume that $x(2) = -3$ and $x(5) = 6$.

1 Find the displacement over the time interval $[2, 5]$.

2 Find the constant velocity.

Solution

1 The displacement is $x(5) - x(2) = 6 - (-3) = 9$ m.

2 The velocity is $\dfrac{x(5) - x(2)}{5-2} = \dfrac{9}{3} = 3$ m/s.

Example

A particle moves in a straight line with constant velocity so that, at time t seconds, the position of the particle is $x(t)$ metres, with respect to the origin O. Assume that $x(2) = 6$ and $x(5) = -5$.

1 Find the displacement over the time interval $[2, 5]$.

2 Find the constant velocity.

Solution

1 The displacement is $x(5) - x(2) = -5 - 6 = -11$ m.

2 The velocity is $\dfrac{x(5) - x(2)}{5-2} = \dfrac{-11}{3} = -3\frac{2}{3}$ m/s.

Exercise 1

a A particle is moving with a constant velocity of 4 m/s. Sketch the position–time graph of the particle for $t \in [0, 4]$, if $x(0) = 3$.

b A particle is moving with a constant velocity of -4 m/s. Sketch the position–time graph of the particle for $t \in [0, 4]$, if $x(0) = 3$.

Constant acceleration

We are all familiar with the fact that a car speeds up when we put our foot down on the accelerator. The rate of change of the velocity of a particle with respect to time is called its **acceleration**. If the velocity of the particle changes at a constant rate, then this rate is called the **constant acceleration**.

Since we are using metres and seconds as our basic units, we will measure acceleration in metres per second per second. This will be abbreviated as $\mathrm{m/s^2}$. It is also commonly abbreviated as $\mathrm{ms^{-2}}$.

For example, if the velocity of a particle moving in a straight line changes uniformly (at a constant rate of change) from 2 m/s to 5 m/s over one second, then its constant acceleration is 3 m/s^2.

Example

Let t be the time in seconds from the beginning of the motion of a particle. If the particle has a velocity of 4 m/s initially (at $t = 0$) and has a constant acceleration of 2 m/s^2, find the velocity of the particle:

1 when $t = 1$

2 when $t = 2$

3 after t seconds.

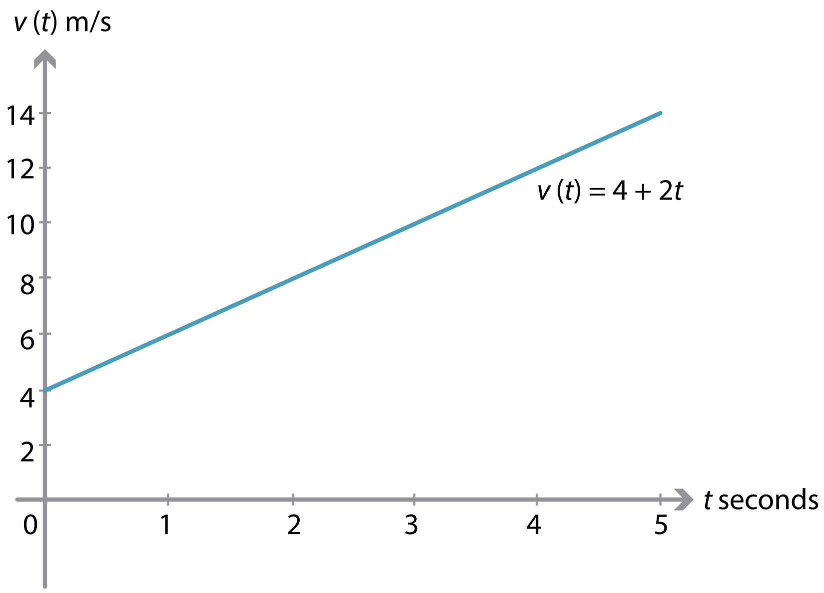
Draw the velocity–time graph for the motion.

Solution

1 When $t = 1$, the velocity of the particle is $4 + 2 = 6$ m/s.

2 When $t = 2$, the velocity of the particle is $4 + 2 \times 2 = 8$ m/s.

3 After t seconds, the velocity of the particle is $4 + 2t$ m/s.



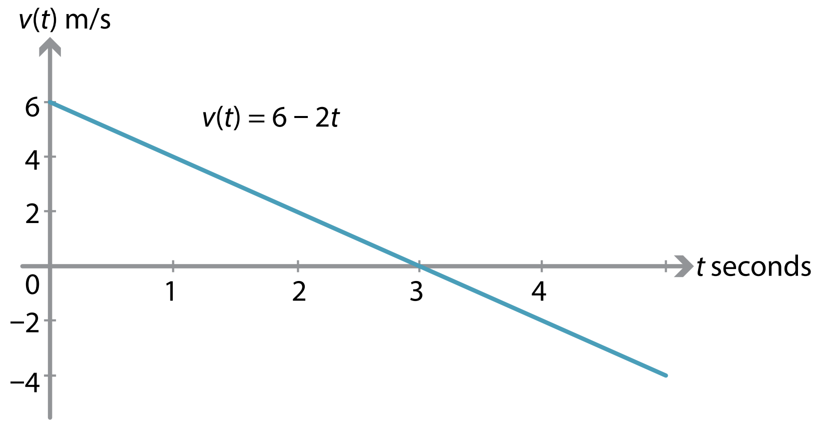
Decreasing velocity

If the velocity of a particle moving in a straight line changes uniformly (at a constant rate of change) from 5 m/s to 2 m/s over one second, its constant acceleration is −3 m/s^2.

If a particle has an initial velocity of 6 m/s and a constant acceleration of -2 m/s^2, then:

- when $t = 1$, the velocity of the particle is 4 m/s
- when $t = 2$, the velocity of the particle is 2 m/s
- when $t = 3$, the velocity of the particle is 0 m/s
- when $t = 4$, the velocity of the particle is -2 m/s
- when $t = 10$, the velocity of the particle is -14 m/s.

In general, the velocity of the particle is $6 - 2t$ m/s after t seconds. The velocity–time graph for this motion is shown below; it is the graph of $v(t) = 6 - 2t$.



Over the first three seconds, the particle's speed is decreasing (the particle is slowing down). At three seconds, the particle is momentarily at rest. After three seconds, the velocity is still decreasing, but the speed is increasing (the particle is going faster and faster).

Summary

If we assume that the rate of change of velocity (acceleration) is a constant, then the constant acceleration is given by

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Change in time}}.$$

More precisely, the constant acceleration a is given by the formula

$$a = \frac{v(t_2) - v(t_1)}{t_2 - t_1},$$

where $v(t_i)$ is the velocity at time t_i. Since velocity is a vector, so is acceleration.

Example

A particle is moving in a straight line with constant acceleration of 1.5 m/s^2. Initially its velocity is 4.5 m/s. Find the velocity of the particle:

1. after 1 second
2. after 3 seconds
3. after t seconds.

Solution

1. After 1 second, the velocity is $4.5 + 1.5 = 6$ m/s.
2. After 3 seconds, the velocity is $4.5 + 3 \times 1.5 = 9$ m/s.
3. After t seconds, the velocity is $4.5 + 1.5t$ m/s.

Example

A car is travelling at 100 km/h $= \frac{250}{9}$ m/s, and applies its brakes to stop. The acceleration is -10 m/s^2. How long does it take for the car to stop?

Solution

After one second, the car's velocity is $\frac{250}{9} - 10$ m/s. After t seconds, its velocity is

$$v(t) = \frac{250}{9} - 10t \text{ m/s.}$$

The car stops when $v(t) = 0$. Solving this equation gives

$$\begin{aligned} \frac{250}{9} - 10t &= 0 \\ t &= \frac{25}{9}. \end{aligned}$$

The car takes approximately 2.8 seconds to stop.

(In exercise 6, we will find out how far the car travels during this time.)

The constant-acceleration formulas for motion in a straight line

Throughout this section, we have been considering motion in a straight line with constant acceleration. This situation is very common; for example, a body moving under the influence of gravity travels with a constant acceleration.

There are five frequently used formulas for motion in a straight line with constant acceleration. The formulas are given in terms of the initial velocity u, the final velocity v, the displacement (position) x, the acceleration a and the time elapsed t. Of course, they require consistent systems of units to be used.

It is assumed that the motion begins when $t = 0$, and that the initial position is taken as the origin, that is, $x(0) = 0$.

The five equations of motion

1. $v = u + at$
2. $x = \dfrac{(u+v)t}{2}$
3. $x = ut + \frac{1}{2}at^2$
4. $v^2 = u^2 + 2ax$
5. $x = vt - \frac{1}{2}at^2$

Note. Each of the five equations involve four of the five variables u, v, x, a, t. If the values of three of the variables are known, then the remaining values can be found by using two of the equations.

Deriving the constant-acceleration formulas

The first equation of motion

Since the acceleration is constant, we have $a = \dfrac{v-u}{t}$. This gives the first equation of motion, $v = u + at$.

The second equation of motion

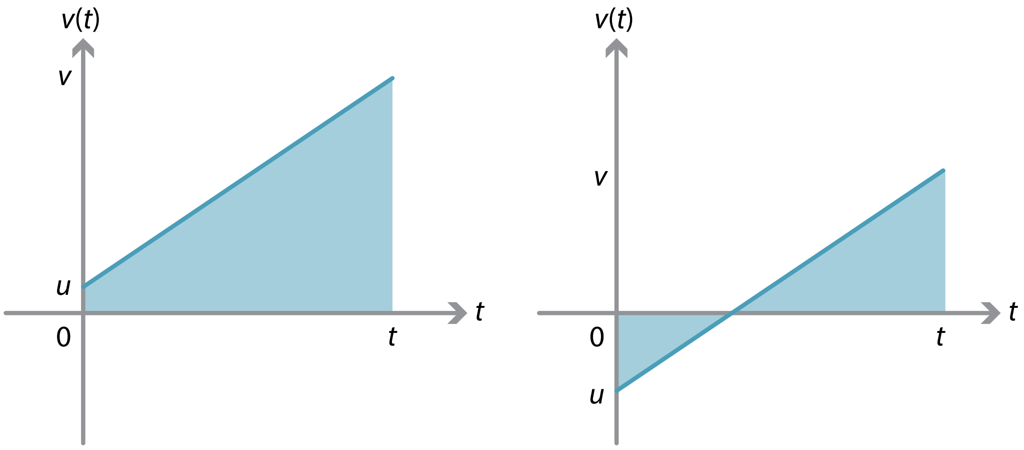
The second equation,

$$x = \frac{(u+v)t}{2},$$

says that displacement is obtained by multiplying the average of the initial and final velocities by the time elapsed during the motion. More simply:

Displacement = Average velocity × Time taken.

We can derive this equation using the fact that the displacement is equal to the signed area under the velocity–time graph.



For the velocity–time graph on the left, the region under the graph is a trapezium. The displacement x is equal to the area of the trapezium, which is $\frac{1}{2}(u+v)t$. So the second equation of motion holds in this case.

For the graph on the right, the displacement can be found by considering the two triangles between the graph and the t-axis. One of the triangles has positive signed area and the other has negative signed area.

Finding the displacement of a particle from the velocity–time graph using integration will be discussed in a later section of this module.

The third equation of motion

Substituting for v from the first equation into the second equation gives

$$\begin{aligned} x &= \frac{(u+v)t}{2} \\ &= \frac{(u+u+at)t}{2} \\ &= \frac{2ut+at^2}{2} \\ &= ut+\frac{1}{2}at^2, \end{aligned}$$

which is the third equation. Thus x is a quadratic in t, and hence the graph of x against t is a parabola.

The fourth equation of motion

From the first equation, we have $t = \dfrac{v-u}{a}$. Substituting this into the second equation gives

$$\begin{aligned} x &= \frac{(u+v)t}{2} \\ &= \frac{(u+v)(v-u)}{2a} \\ &= \frac{v^2-u^2}{2a}. \end{aligned}$$

Rearranging to make v^2 the subject produces the fourth equation: $v^2 = u^2 + 2ax$.

The fifth equation of motion

From the first equation, we have $u = v - at$. Using the second equation, we obtain

$$\begin{aligned} x &= \frac{(u+v)t}{2} \\ &= \frac{(v-at+v)t}{2} \\ &= \frac{2vt-at^2}{2} \\ &= vt - \frac{1}{2}at^2, \end{aligned}$$

which is the fifth equation.

Exercise 2

Derive the third and fifth equations of motion from a velocity–time graph. (For simplicity, you may assume that both u and v are positive.)

Vertical motion

Motion due to gravity is a good context in which to demonstrate the use of the constant-acceleration formulas. As discussed earlier, our two directions in vertical motion are up and down, and a decision has to be made as to which of the two directions is positive. Acceleration due to gravity is a constant, with magnitude denoted by g. In the following example, we take the upwards direction to be positive and take $g = 10\ \text{m/s}^2$.

Example

A stone is launched vertically upwards from ground level with the initial velocity 30 m/s. Assume that the acceleration is $-10\ \text{m/s}^2$.

1 Find the time taken for the stone to return to the ground again.

2 Find the maximum height reached by the stone.

3 Find the velocity with which it hits the ground.

4 Sketch the position–time graph and the velocity–time graph for the motion.

5 Find the distance covered by the stone from the launch to when it returns to earth.

Solution

In this example, we have $u = 30$ and $a = -10$, with up as the positive direction.

1 Using the third equation of motion, $x = ut + \frac{1}{2}at^2$, we have $x = 30t - 5t^2$. To find the time taken to return to the ground, we substitute $x = 0$ to obtain

$$0 = 30t - 5t^2$$

$$0 = 5t(6 - t).$$

Therefore $t = 0$ or $t = 6$. Thus the time taken to return to the ground is 6 seconds.

2 We can find the maximum height using the fourth equation of motion, $v^2 = u^2 + 2ax$. When the stone reaches the highest point, $v = 0$. So we have

$$0 = 900 - 20x$$

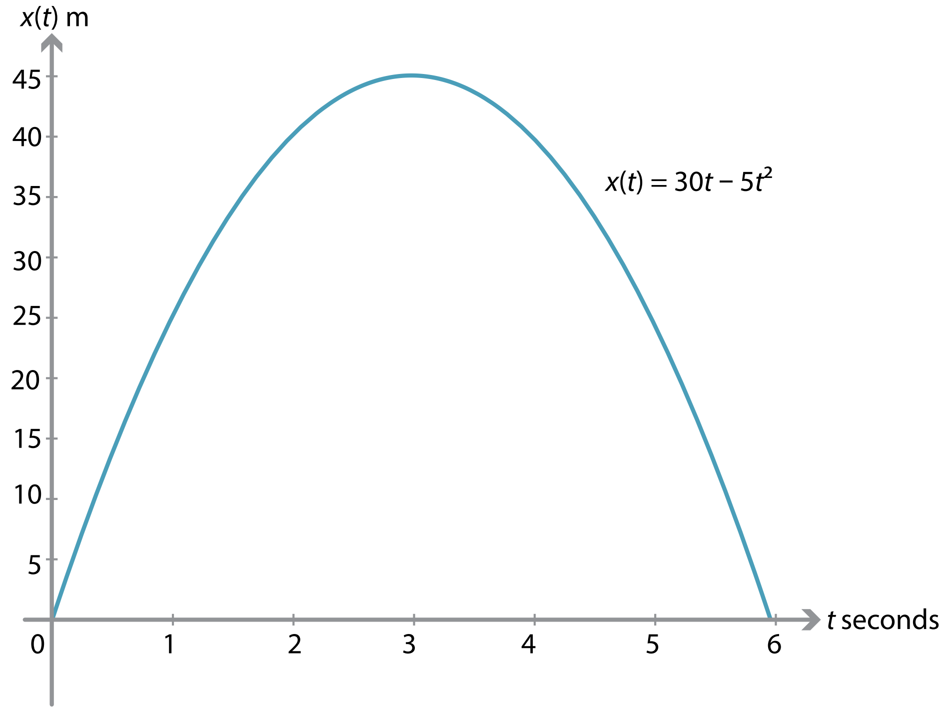
$$x = 45.$$

Hence, the maximum height reached by the stone is 45 metres.

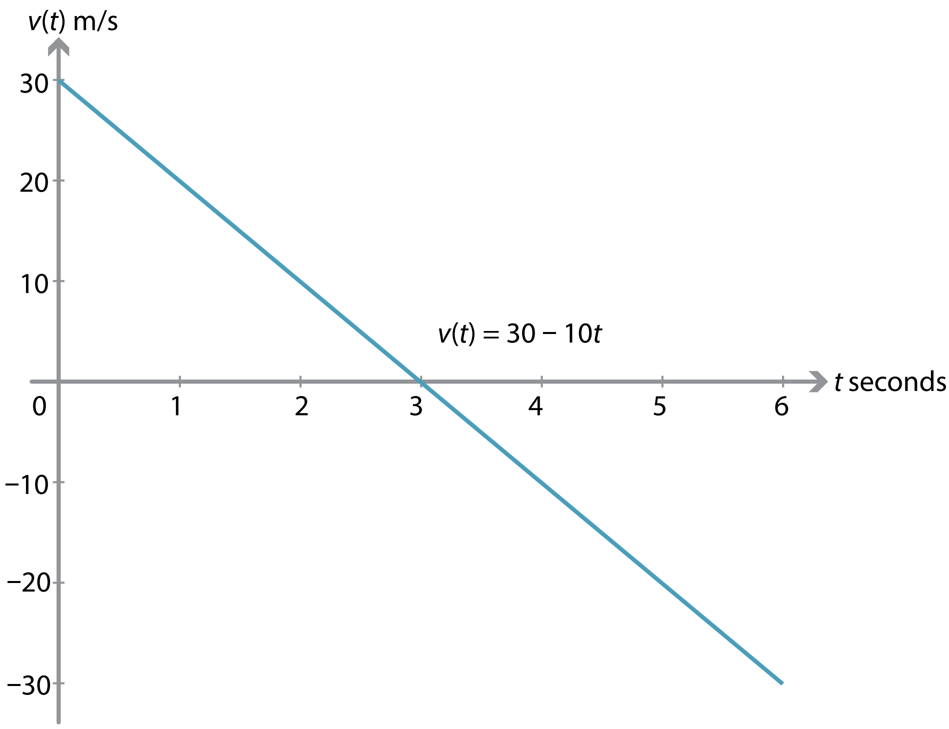
3 The stone hits the ground when $t = 6$. Using the first equation of motion,

$$v = 30 - 10 \times 6 = -30\ \text{m/s}.$$

4 From the third equation of motion, we have $x(t) = 30t - 5t^2$.



The first equation of motion gives $v(t) = 30 - 10t$.



5 The maximum height reached by the stone is 45 m, and so the stone travels a distance of 90 m.

Notes.

- Another way to find the maximum height is by using the position–time graph. The graph of $x(t)$ is a parabola with t-intercepts 0 and 6. Hence, the maximum value of $x(t)$ occurs when $t = 3$. It follows that the maximum height reached by the stone is $x(3) = 30 \times 3 - 5 \times 3^2 = 45$ metres.
- Looking at the velocity–time graph, we can see that the triangle above the t-axis has a signed area of 45 and the triangle below the t-axis has a signed area of -45. Thus, when the stone has returned to ground level, it has travelled 90 m but its displacement is zero.

In the previous example, we took $-10\ \text{m/s}^2$ as the approximate acceleration due to gravity. A more accurate value is $-9.8\ \text{m/s}^2$. Use this value in the following exercise.

Exercise 3

A man dives from a springboard where his centre of gravity is initially 12 metres above the water, and his initial velocity is 4.9 m/s upwards. Regard the diver as a particle at his centre of gravity, and assume that the diver's motion is vertical.

a Find the diver's velocity after t seconds (up to when he hits the water).

b Find the diver's height above the water after t seconds (up to when he hits the water).

c Find the maximum height of the diver above the water.

d Find the time taken for the diver to reach the water.

e Sketch the velocity–time graph for this motion (up to when he hits the water).

f Sketch the position–time graph for this motion (up to when he hits the water).

Exercise 4

A body projected upwards from the top of a tower reaches the ground in t_1 seconds. If projected downwards with the same speed, it reaches the ground in t_2 seconds. Prove that, if simply let drop, it would reach the ground in $\sqrt{t_1 t_2}$ seconds.

Exercise 5

A particle is dropped from a tower of height h. At the same time, a particle is projected upwards with a velocity from the bottom the tower. They meet when the particle dropped from the top has travelled a distance of $\dfrac{h}{n}$. Show that the velocities when they meet are in the ratio $2 : 2 - n$, and that the initial velocity of the second particle is $\sqrt{\dfrac{ngh}{2}}$.

Further use of the equations of motion

Example

A car accelerates uniformly (constant acceleration) from 0 m/s to 30 m/s in 12 seconds, and continues to accelerate at the same rate. Find

1. the acceleration
2. the time it will take for the car to increase its velocity from 30 m/s to 80 m/s
3. the distance the car travels in the first 30 seconds of motion.

Solution

1. Use the first equation of motion, $v = u + at$. Substituting $u = 0$, $v = 30$ and $t = 12$ gives $30 = 12a$. Hence, the acceleration is $a = \frac{5}{2}$ m/s^2.
2. Use the first equation of motion again, with $u = 30$ and $v = 80$:

$$80 = 30 + \frac{5}{2}t.$$

Thus $t = 20$ seconds.

3. Use the third equation of motion, $x = ut + \frac{1}{2}at^2$. Substitute $u = 0$, $a = \frac{5}{2}$ and $t = 30$:

$$x = \frac{1}{2} \times \frac{5}{2} \times 30^2 = 1125 \text{ metres.}$$

The car travels 1125 metres in the first 30 seconds of motion.

Exercise 6

A car is travelling at 100 km/h $= \dfrac{250}{9}$ m/s. It brakes and slows down with an acceleration of -10 m/s^2. How far, to the nearest metre, does it go before it stops?

Exercise 7

A car accelerates from 0 km/h to 100 km/h in 10 seconds, and continues for 40 seconds at 100 km/h. The driver then brakes strongly to stop in 38 metres.

a. Convert 100 km/h to m/s.
b. Find the constant acceleration of the car for the first 10 seconds in m/s^2.
c. Find the total distance travelled by the car in metres.
d. Find the acceleration for the braking phase in m/s^2.
e. How long does it take the car to stop from when the brakes are first applied?
f. Sketch a velocity–time graph for the motion of the car.

Average velocity and average speed

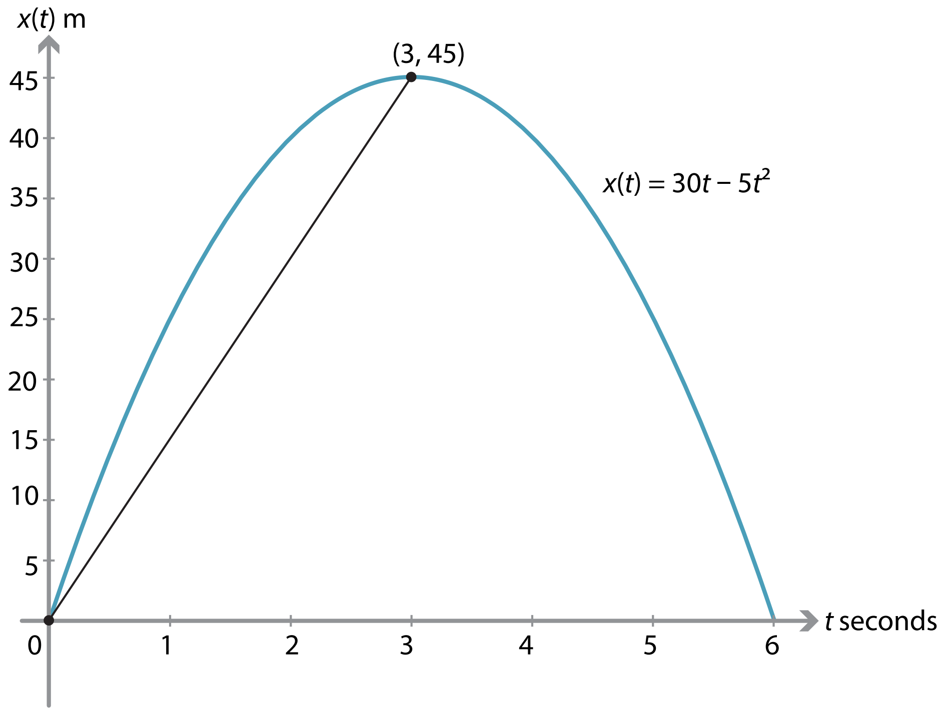
Before turning our attention to problems involving non-constant acceleration and using calculus, we consider the concept of average velocity. We will look at examples involving motion with constant acceleration.

Average velocity

We return to the example of the stone being launched vertically upwards from ground level with an initial velocity of 30 m/s. The acceleration is taken as -10 m/s^2. We found that the stone's velocity is 0 m/s after three seconds, when it is at a height of 45 metres. The average velocity for the first three seconds of motion is

$$\frac{x(3)-x(0)}{3-0} = \frac{45}{3} = 15 \text{ m/s}.$$

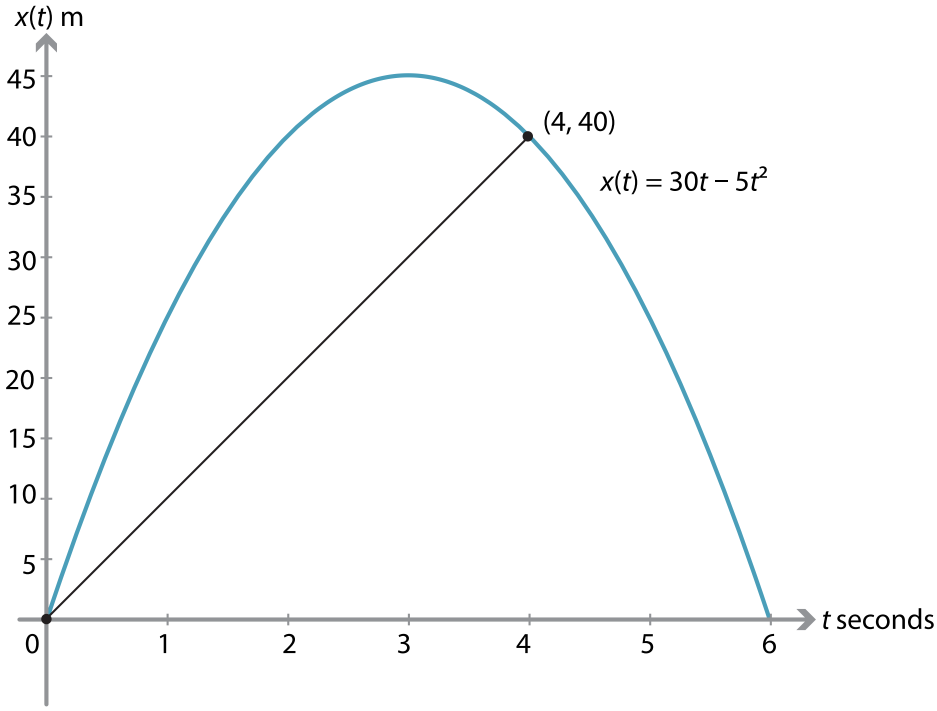
This can be illustrated on the position–time graph by the chord from the origin to the point $(3,45)$. The gradient of this chord is 15.



After a further second, the height (position) of the stone is 40 m. That is, $x(4) = 40$. The average velocity for the first four seconds is

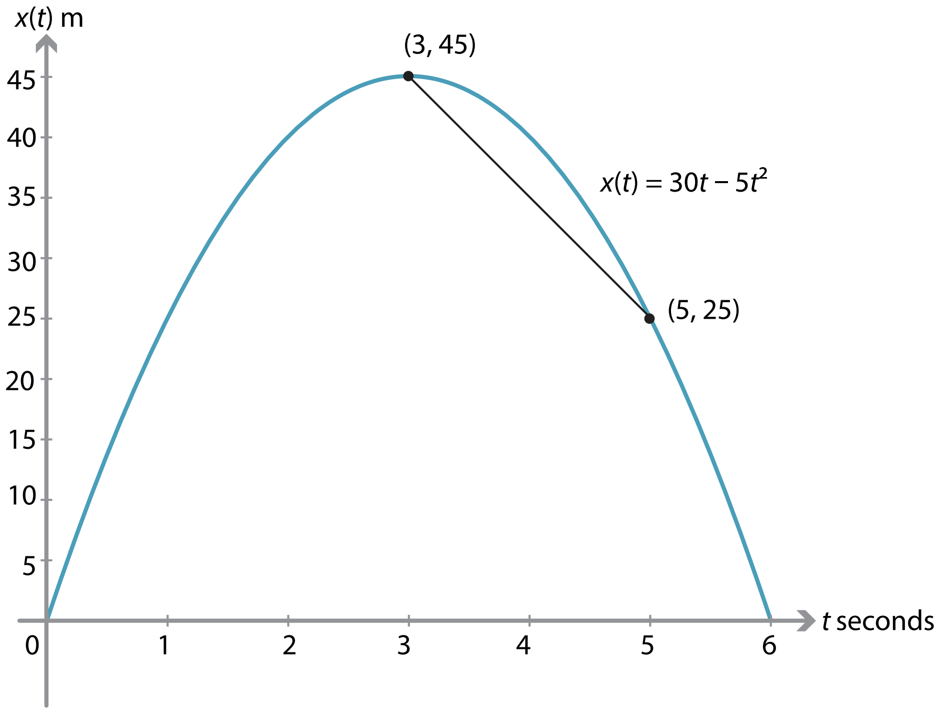
$$\frac{x(4)-x(0)}{4-0} = \frac{40}{4} = 10 \text{ m/s}.$$

This can be illustrated by the chord from the origin to the point $(4,40)$. This chord has gradient 10.



The average velocity from the time the stone reaches its highest point to when the total time elapsed is 5 seconds is

$$\frac{x(5)-x(3)}{5-3} = \frac{25-45}{2} = -10 \text{ m/s}.$$



The average velocity from the time the stone is launched to the time it returns to ground level is

$$\frac{x(6) - x(0)}{6 - 0} = 0 \text{ m/s}.$$

In general, we can write

$$\text{Average velocity} = \frac{x(t_2) - x(t_1)}{t_2 - t_1},$$

where $x(t_i)$ is the position of the particle at time t_i.

Average speed

The average speed is the distance travelled divided by the time taken:

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time taken}}.$$

Returning to the example of the stone launched upwards with initial velocity 30 m/s:

- the average speed for the first three seconds is $\frac{45}{3} = 15$ m/s
- the average speed for the first four seconds is $\frac{50}{4} = 12.5$ m/s
- the average speed from the time the stone reaches its highest point to when the total time elapsed is 5 seconds is $\frac{20}{2} = 10$ m/s
- the average speed from the time the stone is launched to the time it returns to ground level is $\frac{90}{6} = 15$ m/s.

Differential calculus and motion in a straight line

If I drive 180 kilometres in two hours, then my average velocity is 90 kilometres per hour. However, my instantaneous velocity during the journey is displayed on the speedometer, and my velocity may range from 0 km/h to 110 km/h, if the latter is the speed limit. Neither my velocity nor my acceleration will be constant.

Just as an average velocity corresponds to the gradient of a chord on the position–time graph, so an instantaneous velocity corresponds to the gradient of a tangent. We know that, if the motion is described by a nicely behaved position function $x(t)$, then the function for the instantaneous velocity can be found as the derivative of $x(t)$ with respect to time. This is discussed in the module *Introduction to differential calculus*.

Similarly, if the velocity of the particle is described by a nicely behaved function $v(t)$, the function for the instantaneous acceleration can be found as the derivative of $v(t)$ with respect to time.

For motion in a straight line with constant acceleration, we have seen that the displacement of the particle can be determined from a velocity–time graph by finding the signed area between the graph and the t-axis. This leads us to a more general approach.

Integration is the inverse process of differentiation. Therefore, if the velocity function is known, the position function can be found by integration. Similarly, if the acceleration function is known, the velocity function can be found.

Instantaneous velocity and speed

From now on in this module, the words velocity and speed alone will mean instantaneous velocity and speed.

The **instantaneous velocity** $v(t)$ of a particle is the derivative of the position with respect to time. That is,

$$v(t) = \frac{dx}{dt}.$$

This derivative is often written as $\dot{x}(t)$, or simply as $\dot{x}$. From here on, the dot is used to denote the derivative with respect to t. This is Newton's notation.

Example

The position function of a particle is $x(t) = 30t - 5t^2$.

1. Find the velocity function $\dot{x}(t)$.
2. Find the velocity when $t = 2$.
3. Find the velocity when $t = 4$.
4. When is the particle stationary?

Solution

1. If $x(t) = 30t - 5t^2$, then $\dot{x}(t) = 30 - 10t$.
2. $\dot{x}(2) = 30 - 10 \times 2 = 10$.
3. $\dot{x}(4) = 30 - 10 \times 4 = -10$.
4. $\dot{x}(t) = 0$ implies $30 - 10t = 0$, and so $t = 3$. The particle is stationary when $t = 3$.

Example

The position function of a particle is $x(t) = 4\sin(2\pi t)$.

1 Find the velocity function $\dot{x}(t)$.

2 Find the velocity when $t = 0$.

3 Find the velocity when $t = \frac{1}{4}$.

4 Find the velocity when $t = \frac{1}{2}$.

5 Find the velocity when $t = 1$.

Solution

1 If $x(t) = 4\sin(2\pi t)$, then $\dot{x}(t) = 8\pi\cos(2\pi t)$.

2 $\dot{x}(0) = 8\pi$.

3 $\dot{x}(\frac{1}{4}) = 8\pi\cos\frac{\pi}{2} = 0$.

4 $\dot{x}(\frac{1}{2}) = 8\pi\cos\pi = -8\pi$.

5 $\dot{x}(1) = 8\pi\cos 2\pi = 8\pi$.

A particle moving in a straight line is **stationary** when its velocity is zero, that is, $\dfrac{dx}{dt} = 0$. This is the origin of the term 'stationary point' introduced in the module *Applications of differentiation*.

When a stone is thrown into the air with a velocity of 30 m/s, the motion of the stone can be modelled by the position function $x(t) = 30t - 5t^2$. The velocity function for this motion is $\dot{x}(t) = 30 - 10t$. The velocity is zero when $t = 3$. This occurs when the stone is at its maximum height.

The **speed** of a particle is the absolute value of its velocity. For example, if a particle is moving with a velocity of -5 m/s, then its speed is $|-5| = 5$ m/s.

Acceleration

A particle whose velocity is changing is said to be accelerating. Acceleration is defined to be the rate of change of velocity. So the acceleration $a(t)$ is the derivative of the velocity with respect to time:

$$a(t) = \frac{dv}{dt} = \dot{v}(t).$$

The velocity is itself the derivative of the position, and so the acceleration is the second derivative of the displacement:

$$a(t) = \frac{d^2x}{dt^2} = \ddot{x}(t).$$

Example

A large stone is falling through a layer of mud. At time t seconds, the depth of the stone in metres below the surface is given by $x(t) = 20(1 - e^{-\frac{t}{2}})$.

1 Find $\dot{x}(t)$.

2 Find $\ddot{x}(t)$.

3 Find the position, velocity and acceleration when $t = 1$.

4 What happens to the position, velocity and acceleration as $t \to \infty$?

Solution

We take downwards to be the positive direction.

1 $\dot{x}(t) = 10e^{-\frac{t}{2}}$.

2 $\ddot{x}(t) = -5e^{-\frac{t}{2}}$.

3 $x(1) = 20(1 - e^{-\frac{1}{2}}), \quad \dot{x}(1) = 10e^{-\frac{1}{2}}, \quad \ddot{x}(1) = -5e^{-\frac{1}{2}}$.

The direction of the velocity is down, as it has a positive sign, and the direction of the acceleration is up, as it has a negative sign. The velocity and the acceleration are in opposite directions, so the particle is slowing down.

4 As $t \to \infty$, we have $x(t) \to 20$, $\dot{x}(t) \to 0$ and $\ddot{x}(t) \to 0$.

Exercise 8

A particle moves in a straight line so that its position $x(t)$ metres at time t seconds, relative to a fixed position O, is given by $x(t) = t(t-4)^2$. Find

a the velocity at time t

b the values of t when the particle is instantaneously at rest

c the acceleration after four seconds.

Integral calculus and motion in a straight line

We can move from the acceleration function to the velocity function and from the velocity function to the position function through integration. Integrating a function always introduces an arbitrary constant. Hence, one or more boundary conditions are required to determine the motion completely.

Example

A particle moves in a straight line. It is initially at rest at the origin. The acceleration of the particle is given by $\ddot{x}(t) = \frac{1}{3}\cos 3t$.

1 Find the velocity at time t.

2 Find the position at time t.

3 Sketch the position–time graph for $t \in [0, 2\pi]$.

Solution

1 Starting from $\ddot{x}(t) = \frac{1}{3}\cos 3t$ and integrating once, we obtain

$$\dot{x}(t) = \frac{1}{9}\sin 3t + c_1.$$

Since $\dot{x}(0) = 0$, we have $c_1 = 0$. Hence,

$$\dot{x}(t) = \frac{1}{9}\sin 3t.$$

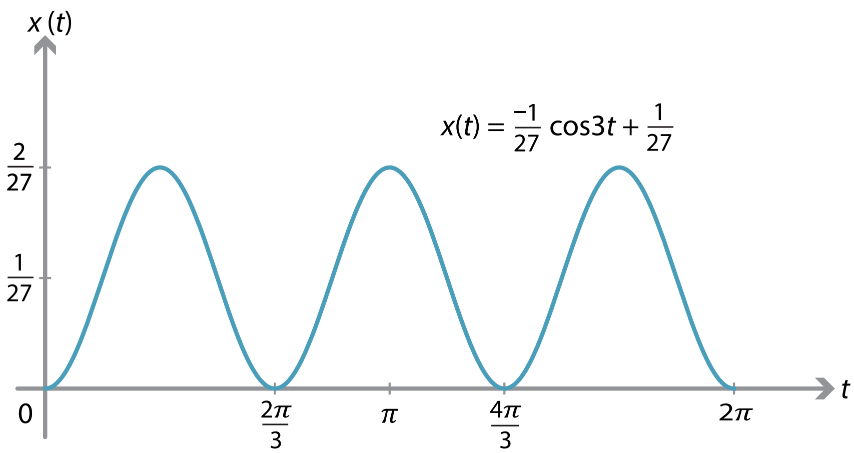
2 Integrating again gives

$$x(t) = -\frac{1}{27}\cos 3t + c_2.$$

Since $x(0) = 0$, we have $c_2 = \frac{1}{27}$. Hence,

$$x(t) = -\frac{1}{27}\cos 3t + \frac{1}{27}.$$

3



Note. We have $\ddot{x}(t) = -9x(t) + \frac{1}{3}$. Motion that satisfies a differential equation of this form is called *simple harmonic motion*. The particle oscillates along a straight line between 0 and $\frac{2}{27}$. The centre of the oscillation is at $x = \frac{1}{27}$.

It is often convenient to use calculus to solve problems of motion in a straight line, particularly when the boundary conditions vary. In addition, calculus is often used when we are considering the motion of two or more particles with different starting times and initial positions.

Example

Two particles A and B are moving to the right along a straight line. For particle A, we have $\ddot{x}_A(t) = \frac{1}{2}$, with boundary conditions $x_A(0) = \frac{1}{4}$ and $\dot{x}_A(0) = \frac{1}{2}$. For particle B, we have $\ddot{x}_B(t) = \frac{2}{3}$, with boundary conditions $x_B(1) = \frac{1}{2}$ and $\dot{x}_B(1) = \frac{5}{6}$.

1 Find when and where the two particles have the same position.

2 Sketch the position–time graphs for A and B on the one set of axes.

Solution

1 *Particle A.* Starting from $\ddot{x}_A(t) = \frac{1}{2}$ and integrating once, we obtain $\dot{x}_A(t) = \frac{1}{2}t + c_1$. Since $\dot{x}_A(0) = \frac{1}{2}$, we have $c_1 = \frac{1}{2}$, and so

$$\dot{x}_A(t) = \frac{1}{2}t + \frac{1}{2}.$$

Integrating again gives

$$x_A(t) = \frac{1}{4}t^2 + \frac{1}{2}t + c_2.$$

Since $x_A(0) = \frac{1}{4}$, we have $c_2 = \frac{1}{4}$. Hence,

$$x_A(t) = \frac{1}{4}\big(t^2 + 2t + 1\big).$$

Particle B. Integrating $\ddot{x}_B(t) = \frac{2}{3}$, we obtain $\dot{x}_B(t) = \frac{2}{3}t + c_3$. Since $\dot{x}_B(1) = \frac{5}{6}$, we have $c_3 = \frac{1}{6}$ and so

$$\dot{x}_B(t) = \frac{1}{6}\big(4t + 1\big).$$

Integrating again:

$$x_B(t) = \frac{1}{6}\big(2t^2 + t\big) + c_4.$$

Since $x_B(1) = \frac{1}{2}$, we have $c_4 = 0$ and so

$$x_B(t) = \frac{1}{6}\big(2t^2 + t\big).$$

Particles meet. The particles meet when $x_A(t) = x_B(t)$:

$$\frac{1}{4}\left(t^2 + 2t + 1\right) = \frac{1}{6}\left(2t^2 + t\right)$$

$$t^2 - 4t - 3 = 0$$

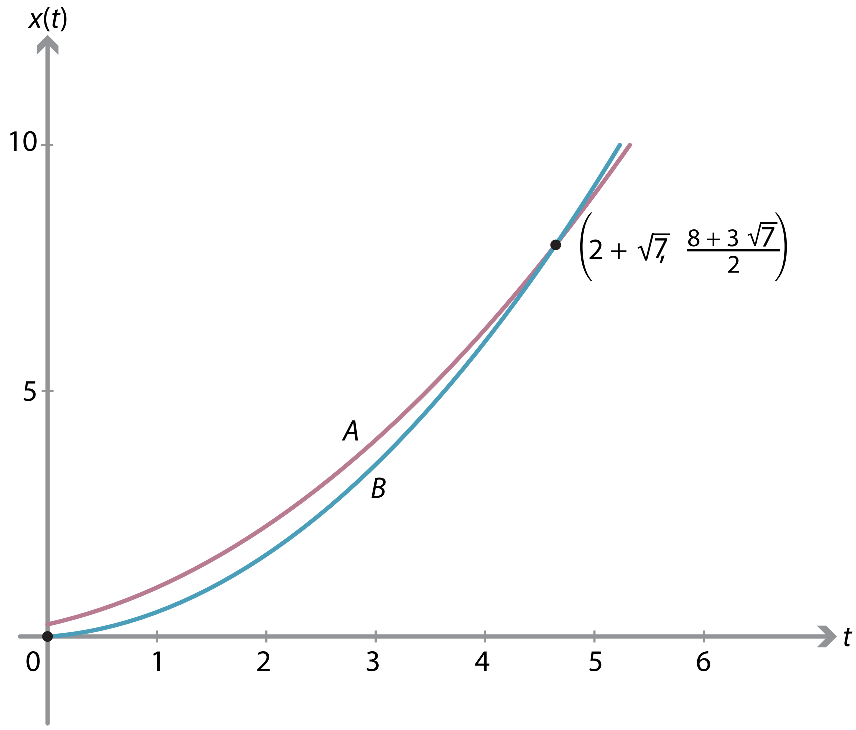
$$t = 2 \pm \sqrt{7}.$$

Since $t \geq 0$, the particles meet when $t = 2 + \sqrt{7}$. Substituting in $x_B(t) = \frac{1}{6}(2t^2 + t)$, we have

$$x_B(2+\sqrt{7}) = \frac{1}{6}\left(2(2+\sqrt{7})^2 + (2+\sqrt{7})\right) = \frac{1}{2}\left(8 + 3\sqrt{7}\right).$$

The particles meet at $\left(2+\sqrt{7}, \frac{1}{2}(8+3\sqrt{7})\right)$.

2



Exercise 9

Two particles A and B are moving in a straight line. The positions of the particles, with respect to an origin, are given by $x_A(t) = \frac{1}{2}t^2 - t + 3$ and $x_B(t) = -\frac{1}{4}t^2 + t + 1$ at time $t \geq 0$.

a Prove that A and B do not collide.

b How close do A and B get to one another?

c During which intervals of time are they moving in opposite directions?

Revisiting the constant-acceleration formulas

We can use calculus to derive the five equations of motion from the section *Constant acceleration*.

We assume that $\ddot{x}(t) = a$, where a is a constant, and that $x(0) = 0$ and $\dot{x}(0) = u$. Integrating $\ddot{x}(t) = a$ with respect to t for the first time gives

$$v(t) = \dot{x}(t) = at + c_1.$$

Since $v(0) = \dot{x}(0) = u$, we have $c_1 = u$. Hence, we obtain the first equation of motion:

$$v(t) = u + at.$$

Integrating again gives

$$x(t) = ut + \frac{1}{2}at^2 + c_2.$$

Since $x(0) = 0$, we have $c_2 = 0$. This yields the third equation of motion:

$$x(t) = ut + \frac{1}{2}at^2.$$

The other equations of motion can easily be derived from these two using only algebra.

Using definite integrals to find displacement and change in velocity

The displacement (that is, the change in position) over some time interval can be found directly using a definite integral of the velocity.

If we are given the velocity $v(t)$ as a function of time, then from $t = t_1$ to $t = t_2$, the displacement is given by

$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\, dt.$$

If we are given the acceleration $\ddot{x}(t)$ as a function of time, then from $t = t_1$ to $t = t_2$, the change in velocity is given by

$$\text{Change in velocity} = \int_{t_1}^{t_2} \ddot{x}(t)\, dt.$$

Example

A particle is moving with constant acceleration of 5 m/s^2. When $t = 0$, the velocity of the particle is 4 m/s. Find the displacement of the particle from $t = 4$ to $t = 8$.

Solution

The velocity is given by $v(t) = 4 + 5t$. Thus, the displacement of the particle from $t = 4$ to $t = 8$ is

$$\int_4^8 (4 + 5t)\, dt = \left[4t + \frac{5t^2}{2} \right]_4^8 = 136 \text{ m.}$$

Exercise 10

A particle is moving with velocity $v(t) = \cos(2\pi t)$ at time t. Find the displacement of the particle from

a $t = 0$ to $t = \frac{1}{4}$

b $t = \frac{1}{4}$ to $t = 1$

c $t = \frac{3}{4}$ to $t = \frac{5}{4}$.

Links forward

Simple harmonic motion

Simple harmonic motion is a form of motion in a straight line. It is discussed in the module *The calculus of trigonometric functions*.

Vector calculus

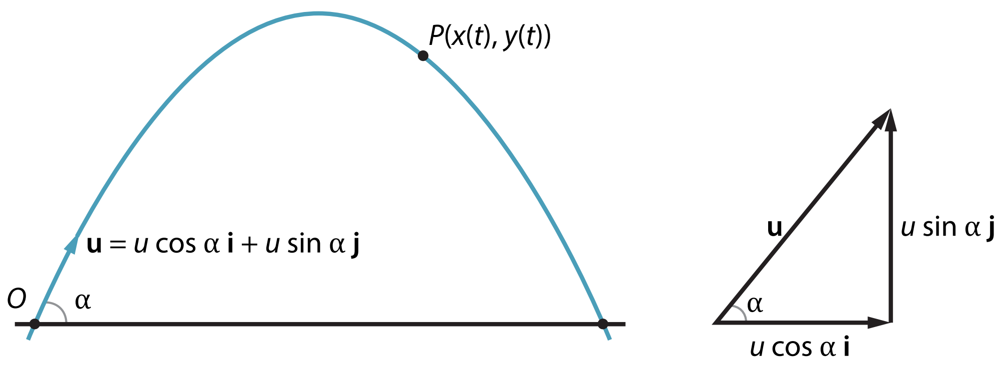
The ideas which have been introduced in this module can be extended to motion in two and three dimensions. We can illustrate this with the example of the projectile motion of a particle in a plane.

Projectile motion

We choose two orthogonal unit vectors **i** and **j**. The convention is to take **i** horizontal and pointing to the right, and to take **j** vertical and pointing up. We can describe projectile motion in the plane of **i** and **j**. The underlying idea is that, in two-dimensional space, you can describe any vector as a sum of scalar multiples of **i** and **j**.

In the following, we consider the motion of a particle which is projected at an angle α to the horizontal with an initial speed of u m/s. The initial velocity vector is

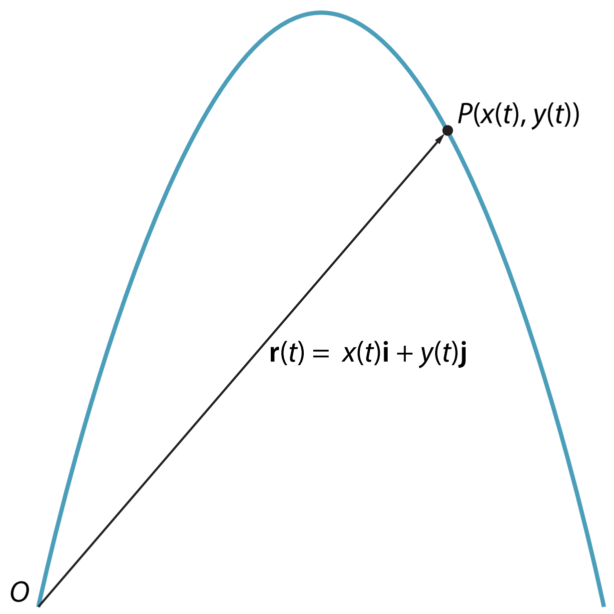
$$\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}.$$



Denote the position vector of the particle at time t, with respect to an origin O, by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}.$$

Here $x(t)$ is a function of t describing the position of the particle in the $\mathbf{i}$-direction, and $y(t)$ is a function of t describing the position of the particle in the $\mathbf{j}$-direction. Assume the particle starts its motion from the origin.



The velocity vector of the particle at time t is given by

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j}.$$

Here $\dot{x}(t)$ is the velocity of the particle in the **i**-direction, and $\dot{y}(t)$ the velocity of the particle in the **j**-direction.

The acceleration vector of the particle at time t is given by

$$\ddot{\mathbf{r}}(t) = \ddot{x}(t)\,\mathbf{i} + \ddot{y}(t)\,\mathbf{j}.$$

We start off as we did for motion in a straight line. Since the acceleration is due to gravity, we have

$$\ddot{\mathbf{r}}(t) = -g\,\mathbf{j}.$$

Integrating once gives

$$\dot{\mathbf{r}}(t) = -gt\,\mathbf{j} + \mathbf{c}_1,$$

where $\mathbf{c}_1$ is a vector constant. We are assuming that the initial velocity vector $\dot{\mathbf{r}}(0)$ is equal to $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$, and hence

$$\dot{\mathbf{r}}(t) = u\cos\alpha\,\mathbf{i} + (u\sin\alpha - gt)\,\mathbf{j}.$$

Integrating again with respect to t gives

$$\mathbf{r}(t) = ut\cos\alpha\,\mathbf{i} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{c}_2,$$

where $\mathbf{c}_2$ is a vector constant. Since we are assuming that the particle starts at the origin, we have $\mathbf{r}(0) = \mathbf{0}$ and so $\mathbf{c}_2 = \mathbf{0}$. Hence,

$$\mathbf{r}(t) = ut\cos\alpha\,\mathbf{i} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j}.$$

Since $y(t)$ can be written as a quadratic function of $x(t)$, the path of the projectile is a parabola.

These ideas can easily be extended to three dimensions.

Vector calculus was developed by J. Willard Gibbs and Oliver Heaviside near the end of the 19th century, and most of the standard notation and terminology was established by Gibbs and Edwin Bidwell Wilson in their 1901 book *Vector analysis*. This book is now available free online at `http://archive.org/details/117714283`.

History and applications

Kinematics before Newton

There is no surviving evidence that Archimedes, or the other truly great Greek mathematicians Euclid and Apollonius, studied kinematics. But so much of Archimedes' work has been lost that it is hard to know.

The representation of quantities by lengths and areas has a long history. In the 14th century, Nicholas Oresme represented time and velocity by lengths. He invented a type of coordinate geometry before Descartes.

The need for mathematical descriptions of velocity contributed to the development of the concept of the derivative. In the 14th century, scholastic philosophers at Merton College, Oxford, studied motion with constant acceleration and deduced what is now known as the *Merton rule*:

> An object with constant acceleration travels the same distance as it would have if it had constant velocity equal to the average of its initial and final velocities.

This rule corresponds to the second equation of motion from the section *Constant acceleration*.

In the 17th century, Galileo Galilei (1564–1642) and others discovered that, in a void, all falling objects have the same constant acceleration, and so their motion may be determined by using the Merton rule. Galileo may well best be remembered for his battle with the Catholic Church over his support for Copernicus' heliocentric model of the solar system, but perhaps he deserves to be remembered as the founder of modern physics. As early as 1604, Galileo discovered that falling bodies are uniformly accelerated. He then worked out a number of mathematical consequences of this fact, some of which could be confirmed by experiment. His work was held back by primitive equipment — for example, there were no stop watches available. Part of his work concerned projectiles, and he was aware that their paths are parabolic.

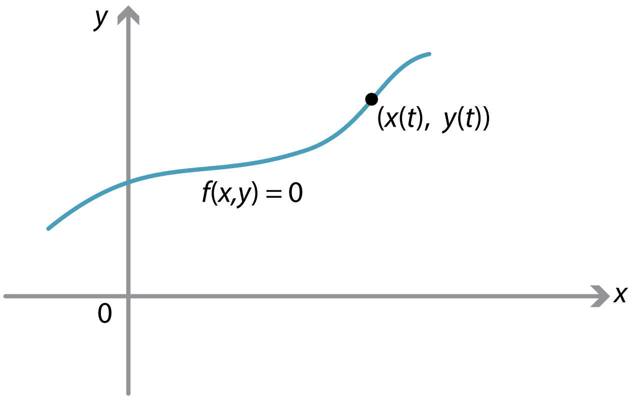
Sir Isaac Newton (1642–1727)

Newton was a central figure in the scientific revolution, and arguably the greatest mathematician and greatest physicist of all time. On one occasion, he claimed with false modesty: 'If I have seen further it is by standing on the shoulders of giants.' Certainly, he studied the work of Galileo, Copernicus, Kepler, Fermat and Descartes, among others.

He began his serious study of calculus in the mid 1660s, partly at his family home in Wolsthorpe when he was forced to leave Cambridge because of the great plague of 1665. He instigated a study of *fluxions* of a given *fluent*. The fluxion is the derivative of a function, which Newton called the fluent. He began his study by finding power-series representations of many functions. For example,

$$\sqrt{1+t^2} = 1 + \frac{t^2}{4} - \frac{t^4}{8} + \frac{t^6}{16} - \frac{5t^8}{128} + \cdots, \qquad \text{for } -1 < t < 1.$$

To obtain the fluxion, all he had to do was to differentiate the power series term-by-term. Newton, however, was much more interested in curves than in functions.



He always thought of t as time, although in modern language, t is a parameter. Thus we may think of motion in a straight line, $x(t)$, as the principal subject of Newton's calculus. It is this context that motivated Newton to study both differential and integral calculus. Newton listed two central problems in his *Treatise on methods*:

- 'Given the length of the space continuously (that is, at every time), to find the speed of the motion at any time proposed.'
- 'Given the speed of motion continuously, to find the length of the space described at any time proposed.'

In modern terms, these problems are:

- Given $x(t)$, find $v(t) = \dfrac{dx(t)}{dt}$.
- Given $v(t)$, find $x(t) = \displaystyle\int v(t)\,dt$.

As noted earlier, Newton's independent variable is always t. As an example of his ideas, Newton took a curve $f(x,y) = 0$ with $x = x(t)$ and $y = y(t)$, and invented a procedure equivalent to what is today written as

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

He also wrote down a simplified table of integrals which did not include logarithmic, exponential or trigonometric functions.

By the 1670s, Newton had started work on kinematics, and clarified the ideas of position (displacement), velocity and acceleration, using calculus to relate them. He also introduced the concepts of force, momentum, energy and power. In 1687, he published the most important mathematical book to appear up to that time. It is known as *Principia*, but its full title is *Philosophiae naturalis principia mathematica* (Mathematical principles of natural philosophy). This book includes Newton's famous three laws of motion, which he took as axioms:

1. Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed. (This is the law of conservation of momentum.)
2. A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed. (This is the law $F = ma$ or, expressed as a vector equation, $\mathbf{F} = m\mathbf{a}$.)
3. To any action there is always an equal and opposite reaction.

From these three laws, Newton derived Kepler's second law: the area swept out is proportional to time. He then effectively stated his universal law of gravitation,

$$F = \frac{Gm_1 m_2}{d^2},$$

and derived Kepler's first and third laws. Kepler's laws were actually conjectures based on astronomical observations. Newton's genius was to derive them from his axioms.

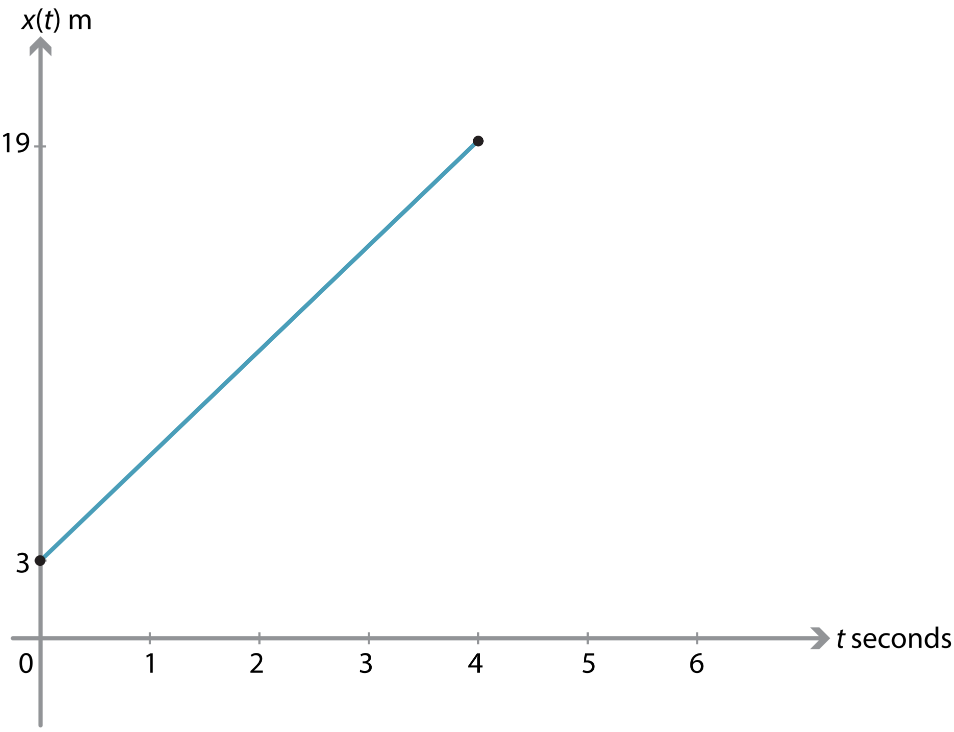
Kinematics after Newton

It can be argued that, for nearly two hundred years, all of the developments in kinematics and astronomy were simply applications of Newton's ideas. This all changed in 1905, when Einstein published his special theory of relativity.

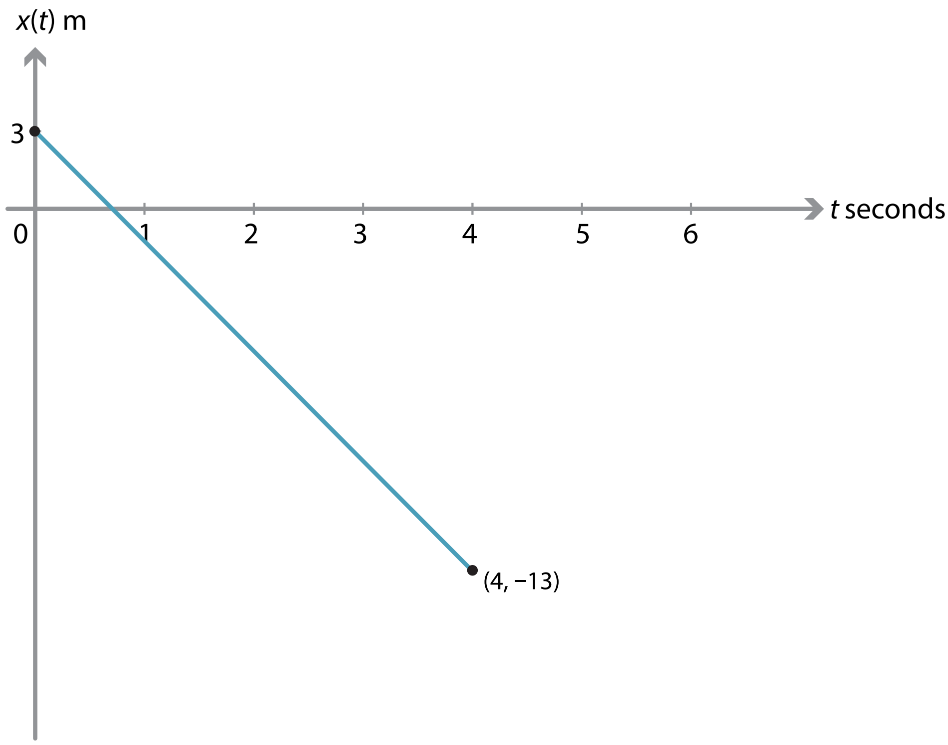
Answers to exercises

Exercise 1

a

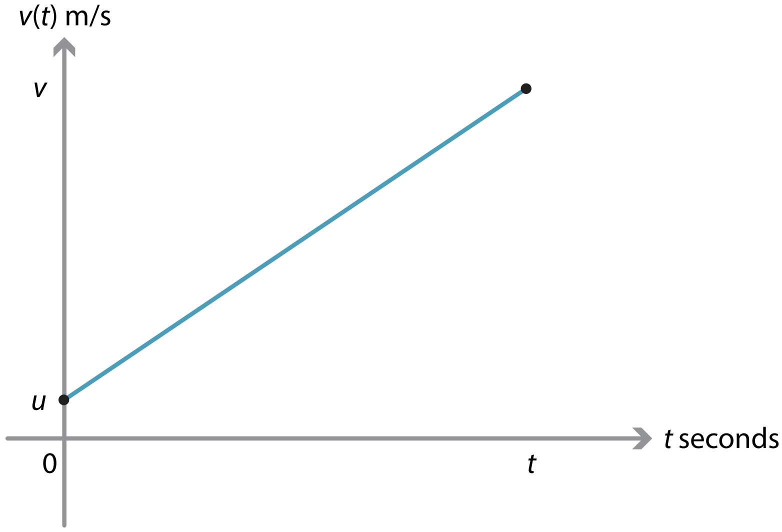


b



Exercise 2

From the first equation of motion, the velocity–time graph is given by $v = u + at$. We assume u and v are positive.



The displacement is equal to the area of the region under the velocity–time graph, which is a trapezium. So $x = \frac{1}{2}(u + u + at)t = ut + \frac{1}{2}at^2$. This is the third equation of motion. If we substitute $u = v - at$ into the third equation, we obtain the fifth equation.

Exercise 3

a $v = 4.9 - 9.8t$ m/s.

b $x = 4.9t - 4.9t^2 + 12$ metres. (This is the position of the diver taking the origin to be the point on the water vertically below the diver's initial position.)

c Maximum height is reached when the velocity is zero. From the fourth equation of motion, we have

$$v^2 = 4.9^2 - 19.6d,$$

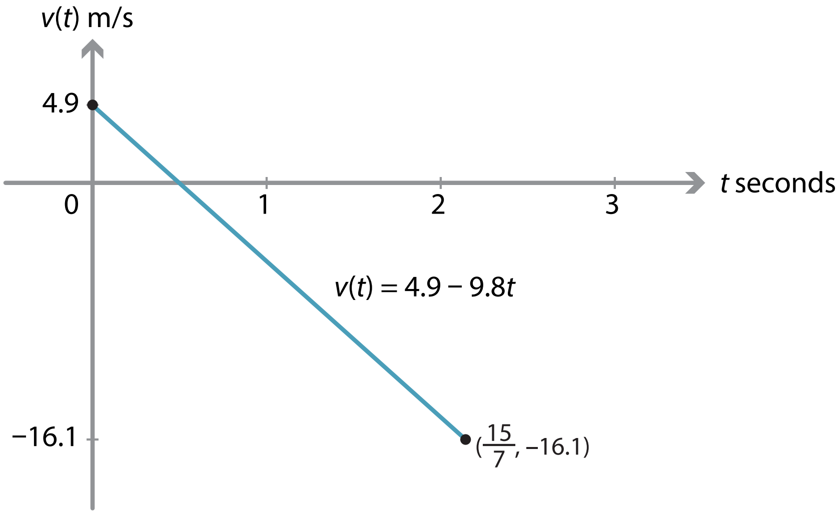
where d is the displacement (that is, the position of the diver relative to the initial position). If $v = 0$, then

$$d = \frac{4.9^2}{19.6} = 1.225 \text{ metres}.$$

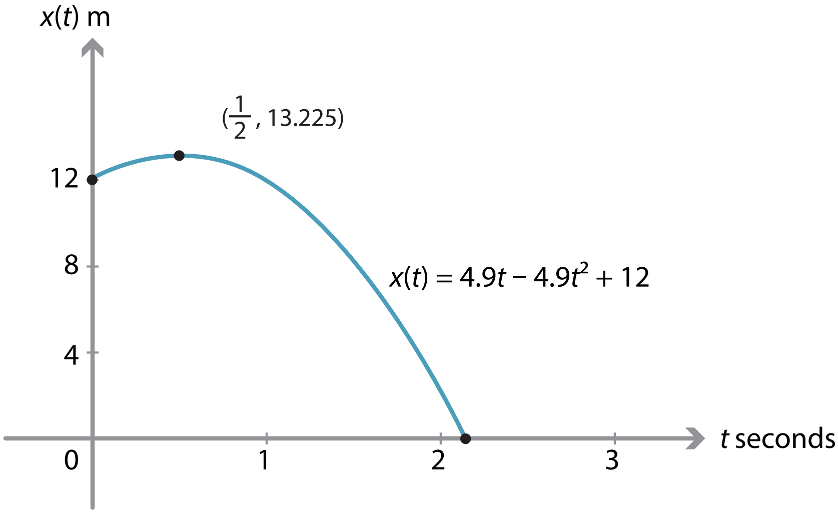
Hence, the maximum height above the water is 13.225 metres.

d The diver reaches the water when $x = 0$. This gives $4.9t - 4.9t^2 + 12 = 0$, and therefore $t = \frac{15}{7}$ seconds.

e



f



Exercise 4

Let h be the height of the tower. We measure positions from the base of the tower, taking the upwards direction as positive.

For the particle that initially goes up, we have $x_{up} = ut - \frac{1}{2}gt^2 + h$, and so

$$0 = ut_1 - \frac{1}{2}g(t_1)^2 + h, \qquad (1)$$

where t_1 is the time taken to reach the ground. For the particle that initially goes down, we have $x_{down} = -ut - \frac{1}{2}gt^2 + h$, and so

$$0 = -ut_2 - \frac{1}{2}g(t_2)^2 + h, \qquad (2)$$

where t_2 is the time taken to reach the ground.

We eliminate h by subtracting equation 2 from equation 1, and solve for u:

$$u = \frac{g}{2}(t_1 - t_2).$$

Substituting in equation 1 and solving for h gives

$$h = \frac{g}{2} t_1 t_2.$$

For the particle dropped from the tower, we have $x_{\text{drop}} = -\frac{1}{2} g t^2 + h$. So

$$0 = -\frac{1}{2} g t^2 + h,$$

where t is the time taken for the dropped particle to reach the ground. Substituting for h and solving for t gives $t = \sqrt{t_1 t_2}$.

Exercise 5

The two particles meet when $-\frac{1}{2} g t^2 + h = ut - \frac{1}{2} g t^2$. Thus, the particles meet when

$$h = ut. \qquad (1)$$

When the dropped particle has travelled a distance of $\dfrac{h}{n}$, we have

$$-\frac{h}{n} = -\frac{1}{2} g t^2.$$

Substitute for t from equation 1:

$$-\frac{h}{n} = -\frac{g}{2}\left(\frac{h}{u}\right)^2.$$

Now solve for u:

$$u = \sqrt{\frac{ngh}{2}}. \qquad (2)$$

The ratio of the velocities is

$$-gt : u - gt.$$

Substituting from equation 1 and simplifying, this ratio is

$$gh : gh - u^2.$$

Now, substituting from equation 2 and simplifying, the ratio is

$$2 : 2 - n.$$

Exercise 6

39 metres.

Exercise 7

a 100 km/h $= \frac{250}{9}$ m/s.

b The acceleration is $\frac{25}{9}$ m/s^2.

c The car travels 1288 metres.

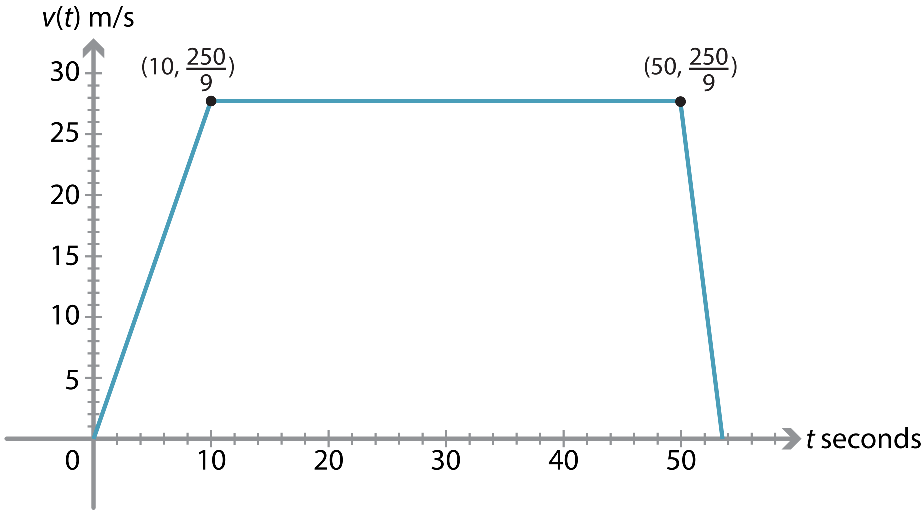
d The car goes from 100 km/h to 0 km/h in 38 m. Thus

$$0 = \left(\frac{250}{9}\right)^2 + 2a \times 38$$

$$a = -\frac{15625}{1539} \text{ m/s}^2.$$

e $\dfrac{342}{125}$ seconds.

f



Exercise 8

a $\dot{x}(t) = (t-4)(3t-4)$.

b The particle is instantaneously at rest when $\dot{x}(t) = 0$, so $t = 4$ or $t = \frac{4}{3}$.

c We have $\ddot{x}(t) = 6t - 16$, and so $\ddot{x}(4) = 8$. The particle has an acceleration of 8 m/s^2.

Exercise 9

a If $x_A(t) = x_B(t)$, then $3t^2 - 8t + 8 = 0$. The discriminant of $3t^2 - 8t + 8$ is $-32 < 0$. Thus there are no solutions, and the particles do not collide.

b The closest distance is $\frac{2}{3}$.

c Over the time interval $[0,1)$, particle A is moving to the left and particle B to the right. Over $(2,\infty)$, particle A is moving to the right and particle B to the left.

Exercise 10

a $\dfrac{1}{2\pi}$ **b** $-\dfrac{1}{2\pi}$ **c** $\dfrac{1}{\pi}$

0 1 2 3 4 5 6 7 8 9 10 11 12