Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Calculus: Module 8

Limits and continuity

Education Services Australia

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Limits and continuity – A guide for teachers (Years 11–12)

Principal author: Peter Brown, University of NSW

Dr Michael Evans, AMSI
Associate Professor David Hunt, University of NSW
Dr Daniel Mathews, Monash University

Editor: Dr Jane Pitkethly, La Trobe University

Illustrations and web design: Catherine Tan, Michael Shaw

Full bibliographic details are available from Education Services Australia.

Published by Education Services Australia
PO Box 177
Carlton South Vic 3053
Australia

Tel: (03) 9207 9600
Fax: (03) 9910 9800
Email: info@esa.edu.au
Website: www.esa.edu.au

This publication is funded by the Australian Government Department of Education, Employment and Workplace Relations.

Supporting Australian Mathematics Project

Australian Mathematical Sciences Institute
Building 161
The University of Melbourne
VIC 3010
Email: enquiries@amsi.org.au
Website: www.amsi.org.au

Limits and continuity

Assumed knowledge

The content of the modules:

- *Algebra review*
- *Functions I*
- *Functions II*
- *Coordinate geometry*.

Motivation

Functions are the heart of modelling real-world phenomena. They show explicitly the relationship between two (or more) quantities. Once we have such a relationship, various questions naturally arise.

For example, if we consider the function

$$f(x) = \frac{\sin x}{x},$$

we know that the value $x = 0$ is not part of the function's domain. However, it is natural to ask: What happens *near* the value $x = 0$? If we substitute small values for x (in radians), then we find that the value of $f(x)$ is approximately 1. In the module *The calculus of trigonometric functions*, this is examined in some detail. The closer that x gets to 0, the closer the value of the function $f(x) = \dfrac{\sin x}{x}$ gets to 1.

Another important question to ask when looking at functions is: What happens when the independent variable becomes very large? For example, the function $f(t) = e^{-t} \sin t$ is used to model damped simple harmonic motion. As t becomes very large, $f(t)$ becomes very small. We say that $f(t)$ approaches zero as t goes to infinity.

Both of these examples involve the concept of **limits**, which we will investigate in this module. The formal definition of a limit is generally not covered in secondary school mathematics. This definition is given in the *Links forward* section. At school level, the notion of limit is best dealt with informally, with a strong reliance on graphical as well as algebraic arguments.

When we first begin to teach students how to sketch the graph of a function, we usually begin by plotting points in the plane. We generally just take a small number of (generally integer) values to substitute and plot the resulting points. We then 'join the dots'. That we can 'join the dots' relies on a subtle yet crucial property possessed by many, but not all, functions; this property is called **continuity**. In this module, we briefly examine the idea of continuity.

Content

Limit of a sequence

Consider the sequence whose terms begin

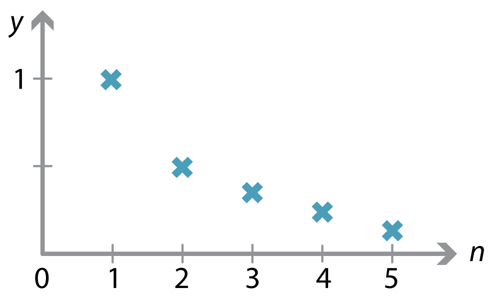
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

and whose general term is $\frac{1}{n}$. As we take more and more terms, each term is getting smaller in size. Indeed, we can make the terms as small as we like, provided we go far enough along the sequence. Thus, although no term in the sequence is 0, the terms can be made as close as we like to 0 by going far enough.

We say that the limit of the sequence $\left(\frac{1}{n} : n = 1, 2, 3, \dots\right)$ is 0 and we write

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

It is important to emphasise that we are not putting n equal to ∞ in the sequence, since infinity is not a number — it should be thought of as a convenient idea. The statement above says that the terms in the sequence $\frac{1}{n}$ get as close to 0 as we please (and continue to be close to 0), by allowing n to be large enough.



Graph of the sequence $\frac{1}{n}$.

In a similar spirit, it is true that we can write

$$\lim_{n\to\infty} \frac{1}{n^a} = 0,$$

for any positive real number a. We can use this, and some algebra, to find more complicated limits.

Example

Find

$$\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2-2}.$$

Solution

Intuitively, we can argue that, if n is very large, then the largest term (sometimes called the **dominant** term) in the numerator is $3n^2$, while the dominant term in the denominator is n^2. Thus, ignoring the other terms for the moment, for very large n the expression $\dfrac{3n^2+2n+1}{n^2-2}$ is close to 3.

The best method of writing this algebraically is to divide by the highest power of n in the denominator:

$$\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2-2} = \lim_{n\to\infty} \frac{3+\frac{2}{n}+\frac{1}{n^2}}{1-\frac{2}{n^2}}.$$

Now, as n becomes as large as we like, the terms $\frac{2}{n}$, $\frac{1}{n^2}$ and $\frac{2}{n^2}$ approach 0, so we can complete the calculation and write

$$\begin{aligned}\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2-2} &= \lim_{n\to\infty} \frac{3+\frac{2}{n}+\frac{1}{n^2}}{1-\frac{2}{n^2}} \\ &= \frac{\lim_{n\to\infty}\left(3+\frac{2}{n}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)} \\ &= \frac{3}{1} = 3.\end{aligned}$$

Exercise 1

Find

$$\lim_{n\to\infty} \frac{5n^3+(-1)^n}{4n^3+2}.$$

Limiting sums

A full study of infinite series is beyond the scope of the secondary school curriculum. But one infinite series, which was studied in antiquity, is of particular importance here.

Suppose we take a unit length and divide it into two equal pieces. Now repeat the process on the second of the two pieces, and continue in this way as long as you like.

Dividing a unit length into smaller and smaller pieces.

This generates the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots.$$

Intuitively, the sum of all these pieces should be 1.

After n steps, the distance from 1 is $\frac{1}{2^n}$. This can be written as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

The value of the sum approaches 1 as n becomes larger and larger. We can write this as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} \to 1 \quad \text{as} \quad n \to \infty.$$

We also write this as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1.$$

This is an example of an infinite geometric series.

A series is simply the sum of the terms in a sequence. A **geometric sequence** is one in which each term is a constant multiple of the previous one, and the sum of such a sequence is called a **geometric series**. In the example considered above, each term is $\frac{1}{2}$ times the previous term.

A typical geometric sequence has the form

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}$$

where $r \neq 0$. Here a is the **first term**, r is the constant multiplier (often called the **common ratio**) and n is the number of terms.

The terms in a geometric sequence can be added to produce a geometric series:

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}. \qquad (1)$$

We can easily find a simple formula for S_n. First multiply equation (1) by r to obtain

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n. \qquad (2)$$

Subtracting equation (2) from equation (1) gives

$$S_n - rS_n = a - ar^n$$

from which we have

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad \text{for } r \neq 1.$$

Now, if the common ratio r is less than 1 in magnitude, the term r^n will become very small as n becomes very large. This produces a **limiting sum**, sometimes written as S_∞. Thus, if $|r| < 1$,

$$\begin{aligned} S_\infty &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}. \end{aligned}$$

In the example considered at the start of this section, we have $a = \frac{1}{2}$ and $r = \frac{1}{2}$, hence the value of the limiting sum is $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$, as expected.

Exercise 2

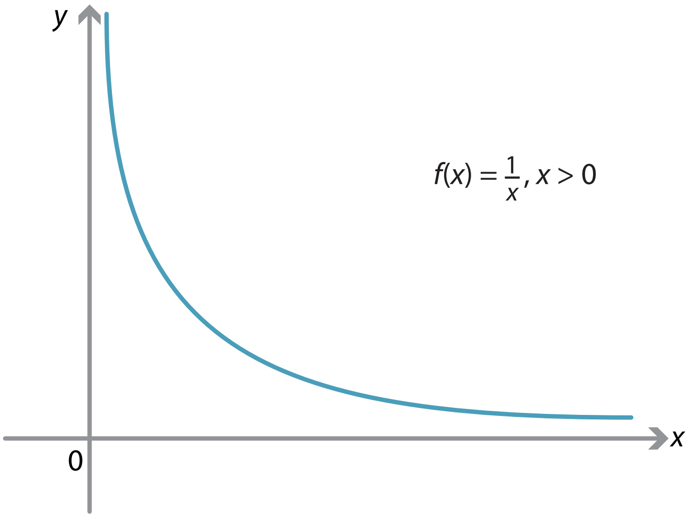
Find the limiting sum for the geometric series

$$\frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \cdots.$$

Limit of a function at infinity

Just as we examined the limit of a sequence, we can apply the same idea to examine the behaviour of a function $f(x)$ as x becomes very large.

For example, the following diagram shows the graph of $f(x) = \dfrac{1}{x}$, for $x > 0$. The value of the function $f(x)$ becomes very small as x becomes very large.

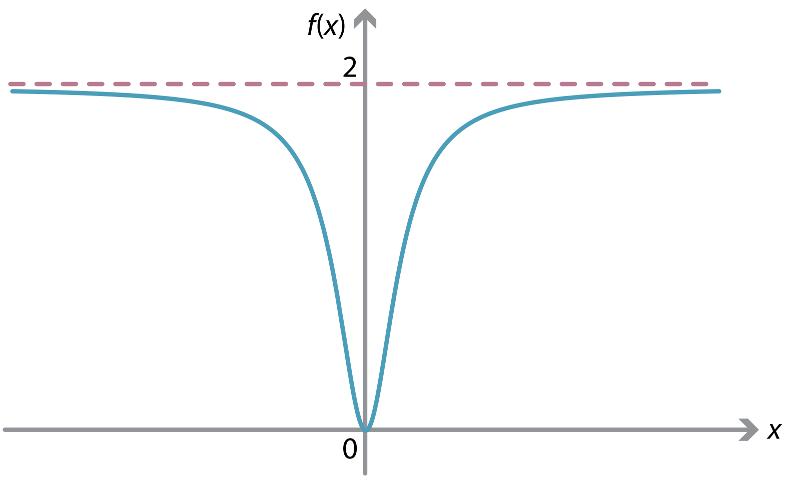


We write

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

One of the steps involved in sketching the graph of a function is to consider the behaviour of the function for large values of x. This will be covered in the module *Applications of differentiation*.

The following graph is of the function $f(x) = \dfrac{2x^2}{1+x^2}$. We can see that, as x becomes very large, the graph levels out and approaches, but does not reach, a height of 2.



We can analyse this behaviour in terms of limits. Using the idea we saw in the section *Limit of a sequence*, we divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2}{1+x^2} = \lim_{x\to\infty} \frac{2}{\frac{1}{x^2}+1} = 2.$$

Note that as x goes to negative infinity we obtain the same limit. That is,

$$\lim_{x\to-\infty} \frac{2x^2}{1+x^2} = 2.$$

This means that the function approaches, but does not reach, the value 2 as x becomes very large. The line $y = 2$ is called a **horizontal asymptote** for the function.

Exercise 3

Find the horizontal asymptote for the function $f(x) = \dfrac{x^2-1}{3x^2+1}$.

Examining the *long-term* behaviour of a function is a very important idea. For example, an object moving up and down under gravity on a spring, taking account of the inelasticity of the spring, is sometimes referred to as **damped simple harmonic motion**. The displacement, $x(t)$, of the object from the centre of motion at time t can be shown to have the form

$$x(t) = Ae^{-\alpha t} \sin \beta t,$$

where A, α and β are positive constants. The factor $Ae^{-\alpha t}$ gives the amplitude of the motion. As t increases, this factor $Ae^{-\alpha t}$ diminishes, as we would expect. Since the factor $\sin \beta t$ remains bounded, we can write

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} Ae^{-\alpha t} \sin \beta t = 0.$$

In the long term, the object returns to its original position.

Limit at a point

As well as looking at the values of a function for large values of x, we can also look at what is happening to a function near a particular point.

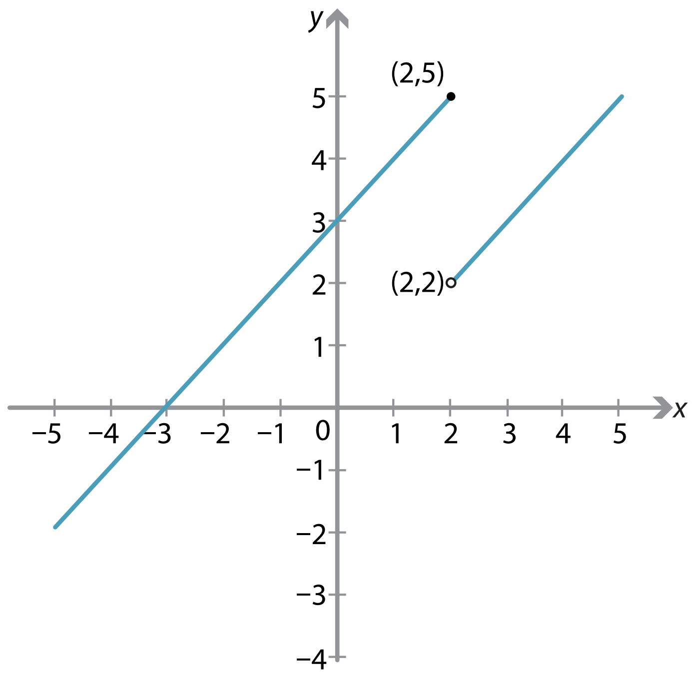
For example, as x gets close to the real number 2, the value of the function $f(x) = x^2$ gets close to 4. Hence we write

$$\lim_{x\to 2} x^2 = 4.$$

Sometimes we are given a function which is defined piecewise, such as

$$f(x) = \begin{cases} x+3 & \text{if } x \le 2 \\ x & \text{if } x > 2. \end{cases}$$

The graph of this function is as follows.



We can see from the 'jump' in the graph that the function does not have a limit at 2:

- as the x-values get closer to 2 from the left, the y-values approach 5
- but as the x-values get closer to 2 from the right, the y-values do not approach the same number 5 (instead they approach 2).

In this case, we say that

$$\lim_{x\to 2} f(x) \text{ does not exist.}$$

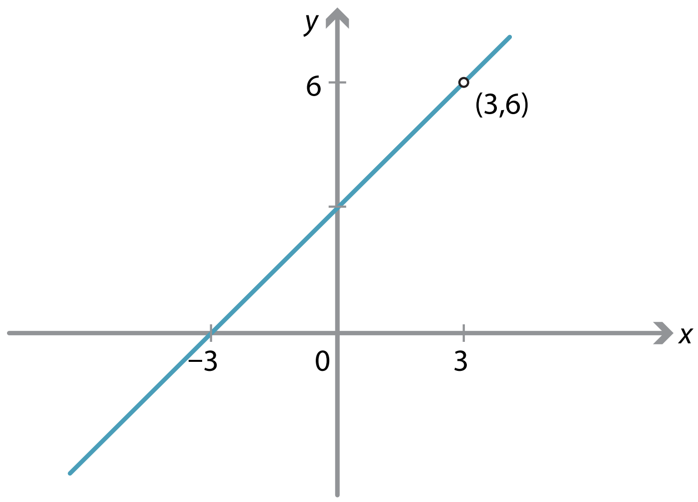
Sometimes we are asked to analyse the limit of a function at a point which is not in the domain of the function. For example, the value $x = 3$ is not part of the domain of the function $f(x) = \dfrac{x^2-9}{x-3}$. However, if $x \ne 3$, we can simplify the function by using the difference of two squares and cancelling the (non-zero) factor $x-3$:

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3, \qquad \text{for } x \ne 3.$$

Now, when x is near the value 3, the value of $f(x)$ is near $3+3=6$. Hence, near the x-value 3, the function takes values near 6. We can write this as

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6.$$

The graph of the function $f(x) = \dfrac{x^2-9}{x-3}$ is a straight line with a hole at the point $(3,6)$.



Example

Find

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4}.$$

Solution

We cannot substitute $x=2$, as this produces 0 in the denominator. We therefore factorise and cancel the factor $x-2$:

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{x-1}{x+2} = \frac{1}{4}.$$

Even where the limit of a function at a point does not exist, we may be able to obtain useful information regarding the behaviour of the function near that point, which can assist us in drawing its graph.

For example, the function

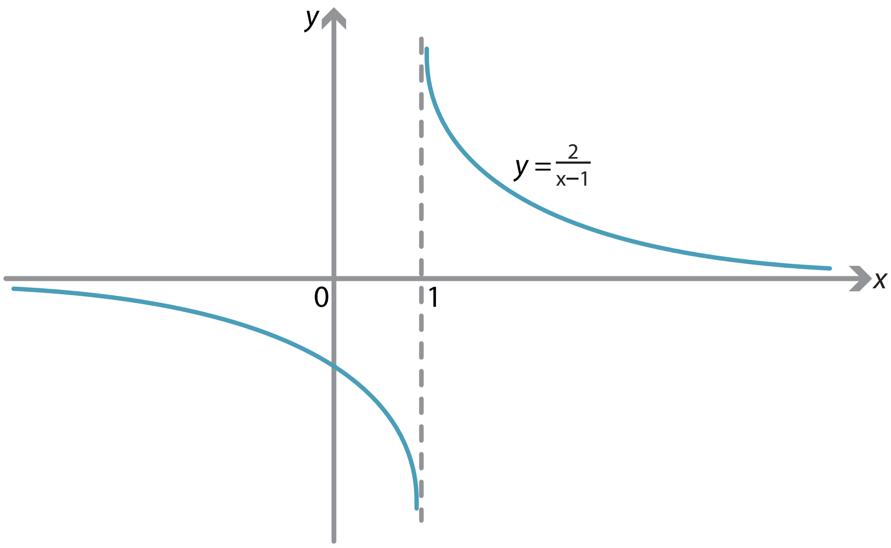
$$f(x) = \frac{2}{x-1}$$

is not defined at the point $x = 1$. As x takes values close to, but *greater than* 1, the values of $f(x)$ are very large and positive, while if x takes values close to, but *less than* 1, the values of $f(x)$ are very large and negative. We can write this as

$$\frac{2}{x-1} \to \infty \text{ as } x \to 1^+ \qquad \text{and} \qquad \frac{2}{x-1} \to -\infty \text{ as } x \to 1^-.$$

The notation $x \to 1^+$ means that 'x approaches 1 from above' and $x \to 1^-$ means 'x approaches 1 from below'.

Thus, the function $f(x) = \frac{2}{x-1}$ has a vertical asymptote at $x = 1$, and the limit as $x \to 1$ does not exist. The following diagram shows the graph of the function $f(x)$. The line $y = 0$ is a horizontal asymptote.



Exercise 4

Discuss the limit of $f(x) = \frac{x^2}{x^2-1}$ at the points $x = 1$, $x = -1$ and as $x \to \pm\infty$.

Exercise 5

Discuss the limit of the function

$$f(x) = \frac{x^2 - 2x - 15}{2x^2 + 5x - 3} = \frac{(x-5)(x+3)}{(2x-1)(x+3)}$$

as

a $x \to \infty$ **b** $x \to 5$ **c** $x \to -3$ **d** $x \to \frac{1}{2}$ **e** $x \to 0$.

Further examples

There are some examples of limits that require some 'tricks'.

For example, consider the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}.$$

We cannot substitute $x = 0$, since then the denominator will be 0. To find this limit, we need to rationalise the numerator:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\left(\frac{\sqrt{x^2+4}-2}{x^2} \times \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\right) \\
&= \lim_{x\to 0} \frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+4}+2} = \frac{1}{4}.
\end{aligned}$$

Exercise 6

Find

a $\displaystyle\lim_{x\to 1} \frac{\sqrt{x^2+15}-4}{x-1}$ **b** $\displaystyle\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4}$.

So far in this module, we have implicitly assumed the following facts — none of which we can prove without a more formal definition of limit.

Algebra of limits

Suppose that $f(x)$ and $g(x)$ are functions and that a and k are real numbers. If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

a $\displaystyle\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

b $\displaystyle\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$

c $\displaystyle\lim_{x\to a} f(x)g(x) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big)$

d $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $\lim_{x\to a} g(x)$ *is not equal to* 0.

Continuity

When first showing students the graph of $y = x^2$, we generally calculate the squares of a number of x-values and plot the ordered pairs (x, y) to get the basic shape of the curve. We then 'join the dots' to produce a *connected* curve.

We can justify this either by plotting intermediary points to show that our plot is reasonable or by using technology to plot the graph. That we can 'join the dots' is really the consequence of the mathematical notion of **continuity**.

A formal definition of continuity is not usually covered in secondary school mathematics. For most students, a sufficient understanding of *continuity* will simply be that they can draw the graph of a continuous function without taking their pen off the page. So, in particular, for a function to be continuous at a point a, it must be defined at that point.

Almost all of the functions encountered in secondary school are continuous everywhere, unless they have a good reason not to be. For example, the function $f(x) = \frac{1}{x}$ is continuous everywhere, except at the point $x = 0$, where the function is not defined.

A point at which a given function is not continuous is called a **discontinuity** of that function.

Here are more examples of functions that are continuous everywhere they are defined:

- polynomials (for instance, $3x^2 + 2x - 1$)
- the trigonometric functions $\sin x$, $\cos x$ and $\tan x$
- the exponential function a^x and logarithmic function $\log_b x$ (for any bases $a > 0$ and $b > 1$).

Starting from two such functions, we can build a more complicated function by either adding, subtracting, multiplying, dividing or composing them: the new function will also be continuous everywhere it is defined.

Example

Where is the function $f(x) = \dfrac{1}{x^2 - 16}$ continuous?

Solution

The function $f(x) = \dfrac{1}{x^2 - 16}$ is a quotient of two polynomials. So this function is continuous everywhere, except at the points $x = 4$ and $x = -4$, where it is not defined.

Continuity of piecewise-defined functions

Since functions are often used to model real-world phenomena, sometimes a function may arise which consists of two separate pieces joined together. Questions of continuity can arise in these case at the point where the two functions are joined. For example, consider the function

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3. \end{cases}$$

This function is continuous everywhere, except possibly at $x = 3$. We can see whether or not this function is continuous at $x = 3$ by looking at the limit as x approaches 3. Using the ideas from the section *Limit at a point*, we can write

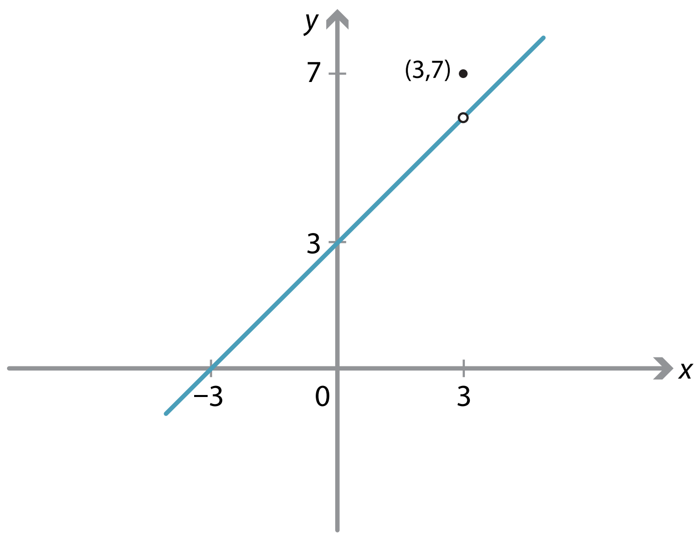
$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = 6.$$

Since 6 is also the value of the function at $x = 3$, we see that this function is continuous. Indeed, this function is identical with the function $f(x) = x+3$, for all x.

Now consider the function

$$g(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3. \end{cases}$$

The value of the function at $x = 3$ is different from the limit of the function as we approach 3, and hence this function is not continuous at $x = 3$. We can see the discontinuity at $x = 3$ in the following graph of $g(x)$.



We can thus give a slightly more precise definition of a function $f(x)$ being continuous at a point a. We can say that $f(x)$ is **continuous** at $x = a$ if

- $f(a)$ is defined, and
- $\lim_{x \to a} f(x) = f(a)$.

Example

Examine whether or not the function

$$f(x) = \begin{cases} x^3 - 2x + 1 & \text{if } x \leq 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

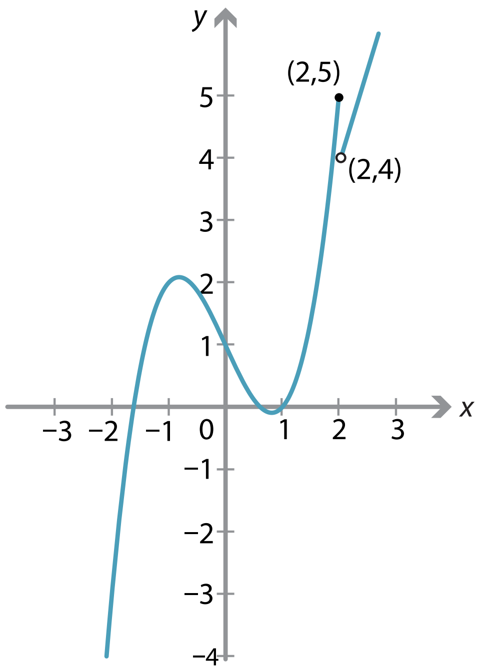
is continuous at $x = 2$.

Solution

Notice that $f(2) = 2^3 - 2 \times 2 + 1 = 5$. We need to look at the limit from the right-hand side at $x = 2$. For $x > 2$, the function is given by $3x - 2$ and so

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2) = 4.$$

In this case, the limit from the right at $x = 2$ does not equal the function value, and so the function is not continuous at $x = 2$ (although it is continuous everywhere else).



Exercise 7

Examine whether or not the function

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \le 0 \\ 4 + x & \text{if } x > 0 \end{cases}$$

is continuous at $x = 0$.

Links forward

Formal definition of a limit

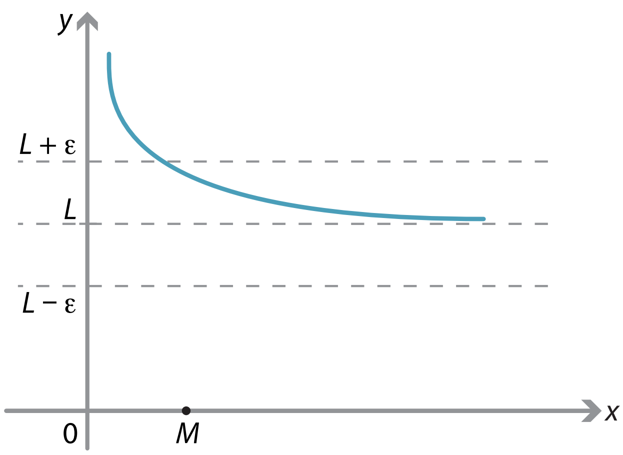
In this module, the notion of limit has been discussed in a fairly informal manner. To be able to prove results about limits and capture the concept logically, we need a formal definition of what we mean by a limit. We will only look here at the precise meaning of $\lim_{x\to\infty} f(x) = L$, but there is a similar definition for the limit at a point.

In words, the statement $\lim_{x\to\infty} f(x) = L$ says that $f(x)$ gets (and stays) as close as we please to L, provided we take sufficiently large x. We now try to pin down this notion of *closeness*.

Another way of expressing the statement above is that, if we are given any small positive number ε (the Greek letter *epsilon*), then the distance between $f(x)$ and L is less than ε provided we make x large enough. We can use absolute value to measure the distance between $f(x)$ and L as $|f(x) - L|$.

How large does x have to be? Well, that depends on how small ε is.

The formal definition of $\lim_{x\to\infty} f(x) = L$ is that, given any $\varepsilon > 0$, there is a number M such that, if we take x to be larger than M, then the distance $|f(x) - L|$ is less than ε.

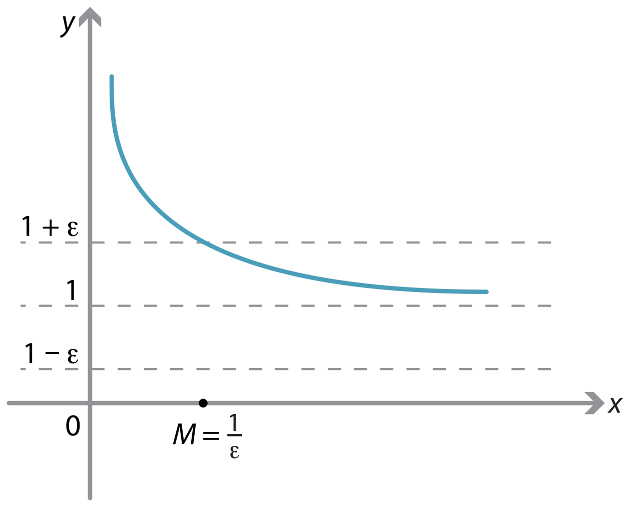


The value of $f(x)$ stays within ε of L from the point $x = M$ onwards.

For example, consider the function $f(x) = \dfrac{x+1}{x}$. We know from our basic work on limits that $\lim_{x\to\infty} f(x) = 1$. For $x > 0$, the distance is

$$|f(x) - 1| = \left|\frac{x+1}{x} - 1\right| = \frac{1}{x}.$$

So, given any positive real number ε, we need to find a real number M such that, if $x > M$, then $\frac{1}{x} < \varepsilon$. For $x > 0$, this inequality can be rearranged to give $x > \frac{1}{\varepsilon}$. Hence we can choose M to be $\frac{1}{\varepsilon}$.



Exercise 8

Let $f(x) = \dfrac{2x^2+3}{x^2}$. Given $\varepsilon > 0$, find M such that if $x > M$ we have $|f(x)-2| < \varepsilon$. Conclude that $f(x)$ has a limit of 2 as $x \to \infty$.

While the formal definition can be difficult to apply in some instances, it does give a very precise framework in which mathematicians can properly analyse limits and be certain about what they are doing.

The pinching theorem

One very useful argument used to find limits is called the **pinching theorem**. It essentially says that if we can 'pinch' our limit between two other limits which have a common value, then this common value is the value of our limit.

Thus, if we have

$$g(x) \le f(x) \le h(x), \quad \text{for all } x,$$

and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Here is a simple example of this.

To find $\lim_{n\to\infty} \frac{n!}{n^n}$, we can write

$$\begin{aligned}\frac{n!}{n^n} &= \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{3}{n} \times \frac{2}{n} \times \frac{1}{n}\\ &\le 1 \times 1 \times 1 \times \cdots \times 1 \times 1 \times \frac{1}{n} = \frac{1}{n},\end{aligned}$$

where we replaced every fraction by 1 except the last. Thus we have $0 \le \frac{n!}{n^n} \le \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we can conclude using the pinching theorem that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Other examples will be found in later modules. In particular, the very important limit

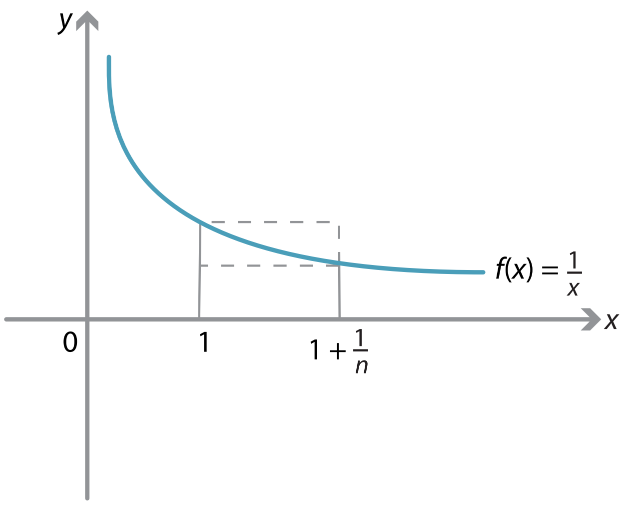
$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0$$

(where x is expressed in radians) will be proven using the pinching theorem in the module *The calculus of trigonometric functions*.

Finding limits using areas

One beautiful extension of the pinching theorem is to bound a limit using areas.

We begin by looking at the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 1 + \frac{1}{n}$.



The area under the curve is bounded above and below by areas of rectangles, so we have

$$\frac{1}{n} \times \frac{1}{1+\frac{1}{n}} \le \int_1^{1+\frac{1}{n}} \frac{1}{x}\,dx \le \frac{1}{n} \times 1.$$

Hence

$$\frac{1}{1+n} \le \log_e\left(1+\frac{1}{n}\right) \le \frac{1}{n}.$$

Now multiplying by n, we have

$$\frac{n}{1+n} \le n\log_e\left(1+\frac{1}{n}\right) \le 1.$$

Hence, if we take limits as $n \to \infty$, we conclude by the pinching theorem that

$$n\log_e\left(1+\frac{1}{n}\right) \to 1 \implies \log_e\left(1+\frac{1}{n}\right)^n \to 1$$

$$\implies \left(1+\frac{1}{n}\right)^n \to e \qquad \text{(!!)}$$

That is,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

History and applications

Paradoxes of the infinite

The ancient Greek philosophers appear to have been the first to contemplate the infinite in a formal way. The concept worried them somewhat, and Zeno came up with a number of paradoxes which they were not really able to explain properly. Here are two of them:

The dichotomy paradox

Suppose I travel from A to B along a straight line. In order to reach B, I must first travel half the distance AB_1 of AB. But to reach B_1 I must first travel half the distance AB_2 of AB_1, and so on ad infinitum. They then concluded that motion is impossible since, presumably, it is not possible to complete an infinite number of tasks.

The paradox of Achilles and the tortoise

A tortoise is racing against Achilles and is given a head start. Achilles is much faster than the tortoise, but in order to catch the tortoise he must reach the point P_1 where the tortoise started, but in the meantime the tortoise has moved to a point P_2 ahead of P_1. Then when Achilles has reached P_2 the tortoise has again moved ahead to P_3. So on ad infinitum, and so even though Achilles is faster, he cannot catch the tortoise.

In both of these supposed paradoxes, the problem lies in the idea of adding up infinitely many quantities whose size becomes infinitely small.

Pi as a limit

The mathematician François Vieta (1540–1603) gave the first theoretically precise expression for π, known as Vieta's formula:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \cdots.$$

This expresses π as the limit of an infinite product.

John Wallis (1616–1703), who was one of the most influential mathematicians in England in the time just prior to Newton, is also known for the following very beautiful infinite product formula for π:

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times \cdots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times \cdots}.$$

He also introduced the symbol ∞ into mathematics.

Infinitesimals

The notion of an infinitesimal essentially goes back to Archimedes, but became popular as a means to explain calculus. An infinitesimal was thought of as an infinitely small but non-zero quantity. Bishop Berkeley (1685–1753) described them as the 'ghosts of vanished quantities' and was opposed to their use.

Unfortunately, such quantities do not exist in the real number system, although the concept may be useful for discovering facts that can then be made precise using limits.

The real number system is an example of an **Archimedean system**: given any real number α, there is an integer n such that $n\alpha > 1$. This precludes the existence of infinitesimals in the real number system.

Non-archimedean systems can be defined, which contain elements which do not have this property. Indeed, all of calculus can be done using a system of mathematics known as **non-standard analysis**, which contains both infinitesimals and infinite numbers.

Cauchy and Weierstrass

Prior to the careful analysis of limits and their precise definition, mathematicians such as Euler were experimenting with more and more complicated limiting processes; sometimes finding correct answers — often for wrong reasons — and sometimes finding incorrect ones. A lack of rigour often led to paradoxes of the type we looked in the section *Paradoxes of the infinite*.

In the early 19th century, the need for a more formal and logical approach was beginning to dawn on mathematicians such as Cauchy and later Weierstrass.

The French mathematician Augustine-Louie Cauchy (pronounced Koshi, with a long o) (1789–1857) was one of the early pioneers in a more rigorous approach to limits and calculus. He was also responsible for the development of **complex analysis**, which applies the notions of limits and calculus to functions of a complex variable. Many theorems and equations in that subject bear his name. Cauchy is regarded by many as a pioneer of the branch of mathematics known as analysis, although he also made use of infinitesimals.

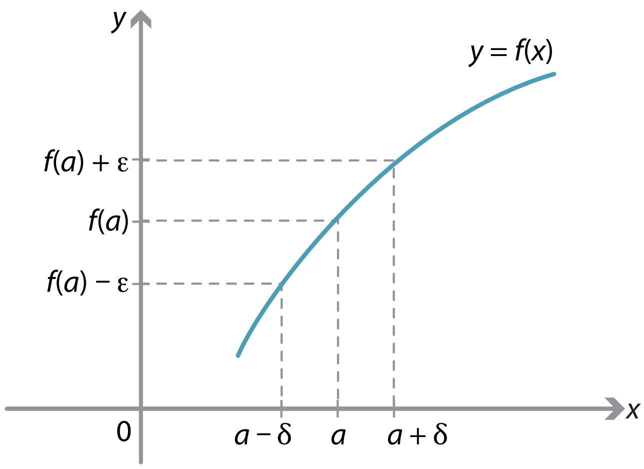
Analysis may be thought of as the theoretical side of limits and calculus. It is a very important branch of modern mathematics, and teaches us how to deal with calculus in ways that are rigorous and logically valid.

In 1821, Cauchy wrote *Cours d'Analyse*, which had a great impact on continental mathematics. In it he introduced proofs using the ε notation we saw in the section *Links forward* (*Formal definition of a limit*).

At roughly the same time, Bernard Bolzano (1781–1848) was attempting to deal with some of the classical paradoxes in his book *The paradoxes of the infinite*. He was the first to give a rigorous ε-δ definition of a limit, although much of his work was not widely disseminated at the time.

While Cauchy made mathematicians think more deeply about what they were doing, it was Karl Weierstrass (1815–1897) who is generally regarded as the father of modern analysis. He gave the first rigorous definition of continuity of a function $f(x)$ at a point a.

The definition states: Given $\varepsilon > 0$, there is a positive real number δ such that, if $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.



Continuity of $f(x)$ at $x = a$.

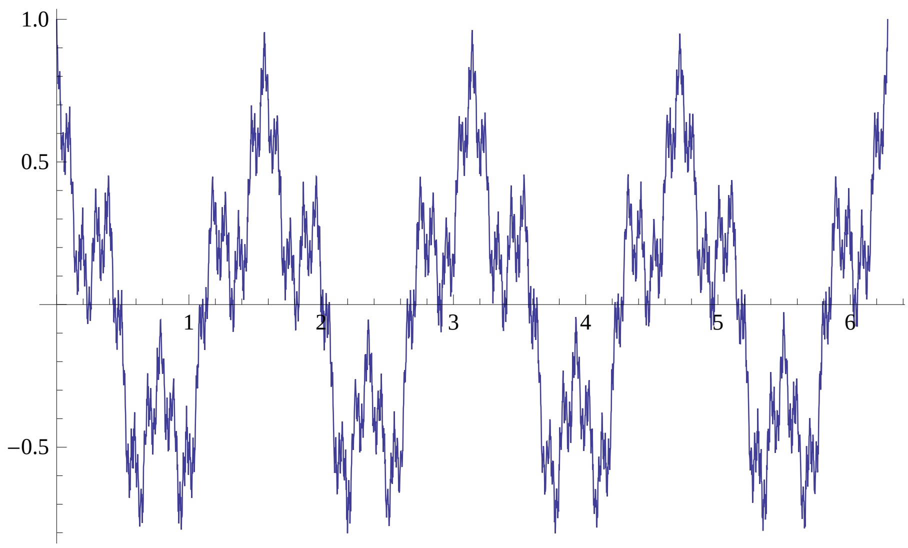
This basically says that a function $f(x)$ is continuous at a point a if x-values that are close to a (i.e., within δ of a) get mapped by f to y-values that are close to $f(a)$ (i.e., within ε of $f(a)$).

In fact, this definition of the continuity of $f(x)$ at a says exactly that $\lim_{x\to a} f(x) = f(a)$, using the formal definition of a limit. So it agrees with our definition of continuity in the section *Continuity of piecewise-defined functions*.

Weierstrass's work was very influential and formed a solid foundation for analysis for decades to come. He shocked the mathematical world by coming up with a function which is continuous everywhere but differentiable nowhere! That is, its graph could be drawn without lifting the pen, but it is not possible to draw a tangent to the curve at any point on it! The function he gave is expressed in terms of an infinite series of functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(4^n x).$$

The following diagram, created using Mathematica, gives the graph of $y = \sum_{n=1}^{50} \frac{1}{2^n} \cos(4^n x)$ for $0 \leq x \leq 2\pi$.



An approximation to Weierstrass's function.

Roughly speaking, this infinite series has trigonometric terms with amplitude $\frac{1}{2^n}$, which quickly approaches 0 as n gets larger. It is this aspect that is used to prove continuity. It can be formally proven that the infinite series converges for every value of x, and that the function so generated is continuous everywhere. If, however, we were to try to differentiate this function term-by-term, then the derivative of the general term is $-2^n \sin(4^n x)$ whose amplitude is 2^n, which becomes very large as n increases. So the series of derivatives does not converge. Hence the function is not differentiable anywhere.

Answers to exercises

Exercise 1

$$\lim_{n\to\infty} \frac{5n^3 + (-1)^n}{4n^3 + 2} = \lim_{n\to\infty} \frac{5 + \frac{(-1)^n}{n^3}}{4 + \frac{2}{n^3}} = \frac{5}{4}.$$

Exercise 2

In the geometric series $\frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \cdots$, the first term a is $\frac{3}{2}$ and the common ratio r is $\frac{3}{4}$ (which is less than 1 in magnitude). So the limiting sum is

$$S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{3}{4}} = 6.$$

Exercise 3

$$\lim_{x\to\infty} \frac{x^2 - 1}{3x^2 + 1} = \lim_{x\to\infty} \frac{1 - \frac{1}{x^2}}{3 + \frac{1}{x^2}} = \frac{1}{3}.$$

So the function $f(x) = \dfrac{x^2-1}{3x^2+1}$ has horizontal asymptote $y = \frac{1}{3}$.

Exercise 4

Define $f(x) = \dfrac{x^2}{x^2-1}$. Then

$$f(x) \to -\infty \text{ as } x \to 1^- \qquad \text{and} \qquad f(x) \to +\infty \text{ as } x \to 1^+,$$

and so $\lim_{x\to 1} f(x)$ does *not* exist. Also,

$$f(x) \to +\infty \text{ as } x \to -1^- \qquad \text{and} \qquad f(x) \to -\infty \text{ as } x \to -1^+,$$

hence $\lim_{x\to -1} f(x)$ does *not* exist. We can calculate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{1 - \frac{1}{x^2}} = 1 \qquad \text{and} \qquad \lim_{x\to -\infty} f(x) = 1.$$

Exercise 5

Define $f(x) = \dfrac{(x-5)(x+3)}{(2x-1)(x+3)}$. Then:

a $f(x) \to \frac{1}{2}$ as $x \to \infty$

b $f(x) \to 0$ as $x \to 5$

c $f(x) \to \frac{8}{7}$ as $x \to -3$

d $f(x) \to -\infty$ as $x \to \frac{1}{2}^+$, and $f(x) \to +\infty$ as $x \to \frac{1}{2}^-$, so $f(x)$ has *no* limit as $x \to \frac{1}{2}$

e $f(x) \to 5$ as $x \to 0$.

Exercise 6

a We rationalise the numerator:

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x^2+15}-4}{x-1} &= \lim_{x\to 1}\left(\frac{\sqrt{x^2+15}-4}{x-1}\times\frac{\sqrt{x^2+15}+4}{\sqrt{x^2+15}+4}\right)\\ &= \lim_{x\to 1}\frac{x^2-1}{(x-1)\big(\sqrt{x^2+15}+4\big)} = \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)\big(\sqrt{x^2+15}+4\big)}\\ &= \lim_{x\to 1}\frac{x+1}{\sqrt{x^2+15}+4} = \frac{1}{4}.\end{aligned}$$

b We get rid of the fractions in the numerator:

$$\begin{aligned}\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4} &= \lim_{x\to 4}\left(\frac{\frac{1}{x}-\frac{1}{4}}{x-4}\times\frac{4x}{4x}\right) = \lim_{x\to 4}\frac{4-x}{4x(x-4)}\\ &= \lim_{x\to 4}\frac{-(x-4)}{4x(x-4)} = \lim_{x\to 4}\left(-\frac{1}{4x}\right) = -\frac{1}{16}.\end{aligned}$$

Exercise 7

Clearly $f(0)=4$. We need to look at the limit at $x=0$ from above. For $x>0$, we have $f(x)=4+x$. So $f(x)\to 4$ as $x\to 0^+$. Since this limit is equal to $f(0)$, we conclude that f is continuous everywhere.

Exercise 8

Let $\varepsilon>0$. We want to find M such that, if $x>M$, then $|f(x)-2|<\varepsilon$. Note that

$$|f(x)-2| = \left|\frac{2x^2+3}{x^2}-2\right| = \frac{3}{x^2}.$$

We want $\frac{3}{x^2}<\varepsilon$, which is equivalent to $x^2>\frac{3}{\varepsilon}$. Hence, we take $M=\sqrt{\frac{3}{\varepsilon}}$. For all $x>M$, we now have $|f(x)-2|=\frac{3}{x^2}<\varepsilon$. This tells us that $f(x)$ has a limit of 2 as $x\to\infty$.

References

The following introductory calculus textbook begins with a very thorough discussion of functions, limits and continuity.

- Michael Spivak, *Calculus*, 4th edition, Publish or Perish, 2008.

0 1 2 3 4 5 6 7 8 9 10 11 12