Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Calculus: Module 13

Growth and decay

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Growth and decay – A guide for teachers (Years 11–12)

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Growth and decay

Assumed knowledge

This module builds upon the module *Exponential and logarithmic functions*. It assumes knowledge of that module and also the modules required for it:

- *Functions II*
- *Introduction to differential calculus*
- *Integration*.

Motivation

> And so from hour to hour we ripe and ripe,
> And then from hour to hour we rot and rot,
> And thereby hangs a tale.
>
> — William Shakespeare, *As You Like It*.

Almost everything in our world grows and decays. From the growth of organisms to their eventual death, from the growth of suns to their eventual supernovas, from the growth of crystals to the decay of radioactive isotopes, from the growth of animal populations to the decay of radio signals, from the growth of economic output to the depletion of resources, little in our world is stationary. As the ancient Greek philosopher Heraclitus said, 'all is in flux'.

Sometimes, we can describe processes of growth and decay — whether physical, chemical, biological or sociological — by mathematical models. These models are only approximations to the real world, but they can help us to understand it and make predictions about it.

One of the most common ways in which a quantity can grow or decay is *exponentially*. We primarily focus on exponential growth and decay in this module, but we also briefly consider other models and equations.

In the module *Exponential and logarithmic functions*, we developed exponential and logarithmic functions from a pure-mathematical point of view. We defined these functions rigorously and studied their properties, including derivatives, integrals and graphs. In this module, we take a more applied-mathematical point of view. We consider mathematical models of exponential growth and decay in other fields of science.

Content

A differential equation for exponential growth and decay

Consider the equation

$$\frac{dx}{dt} = kx,$$

where t and x are variables and k is a constant with $k \neq 0$. As an equation involving derivatives, this is an example of a **differential equation**. We often think of t as measuring time, and x as measuring some positive quantity over time. That is, x is a function of time.

The number k is called the **continuous growth rate** if it is positive, or the **continuous decay rate** if it is negative.[1]

There are many quantities in the real world that approximately obey an equation similar to this one, as we will see shortly. We will first solve the equation in general.

Taking the reciprocal of both sides and noting (via the chain rule) that $\frac{dx}{dt} \cdot \frac{dt}{dx} = 1$ gives

$$\frac{dt}{dx} = \frac{1}{kx}.$$

To solve this equation, we integrate $\frac{1}{kx}$. Recall that k is a constant and we are assuming that x is positive. So we obtain

$$\begin{aligned} t &= \int \frac{1}{kx}\,dx \\ &= \frac{1}{k}\int \frac{1}{x}\,dx \\ &= \frac{1}{k}\log_e x + c, \end{aligned}$$

where c is a constant of integration.

[1] There is another type of growth and decay, called *discrete growth and decay*, which we will discuss in the *Links forward* section. We say ‘continuous growth rate’ to distinguish from ‘discrete growth rate’.

Rearranging this equation to express x in terms of t, we have

$$x = e^{k(t-c)}.$$

We can simplify by noting (using index laws) that

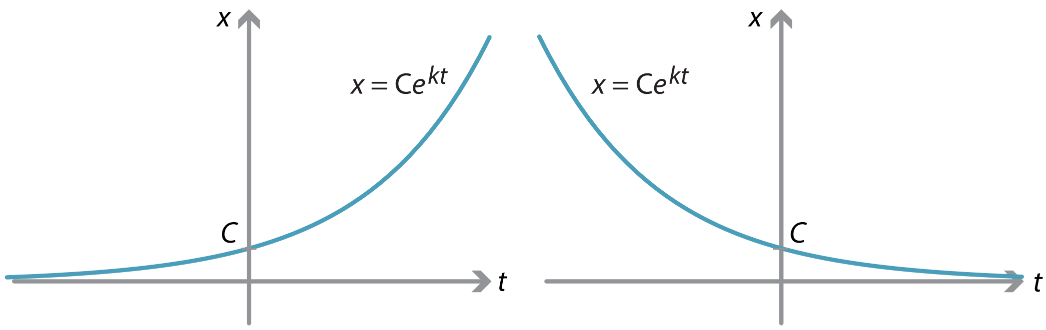
$$e^{k(t-c)} = e^{kt-kc} = e^{-kc}e^{kt}$$

and the first factor e^{-kc} is just a positive constant, which we call C. The general solution to the differential equation is then

$$x(t) = Ce^{kt},$$

where C is any positive constant. It's easy to verify that any such function is a solution: differentiating $x = Ce^{kt}$ with respect to t gives k times x.

Depending on whether k is positive or negative, the quantity x grows or decays with respect to t, as shown in the following graphs.



Graph of $x = Ce^{kt}$ when $k > 0$.

Graph of $x = Ce^{kt}$ when $k < 0$.

Note. If we do not assume that x is positive, then the general solution to the differential equation $\dfrac{dx}{dt} = kx$ is given by $x(t) = Ce^{kt}$, where C is any real constant.

We have found that the differential equation $\dfrac{dx}{dt} = kx$ has *infinitely many* solutions; any value of C gives a solution. If we have more information, such as the initial value of x, then the value of C can be determined.

Example

1 Find all solutions to the differential equation

$$\frac{dp}{dt} = 3p.$$

2 Find all solutions to this differential equation if we know that $p = 7$ when $t = 0$.

Solution

1 The variable p plays the role of x in the preceding discussion. With continuous growth rate $k = 3$, the general solution is

$$p(t) = Ce^{3t},$$

where C is any real constant. This is a solution for any real C.

2 Since $p(0) = 7$, we obtain $C = 7$, so there is a unique solution $p(t) = 7e^{3t}$.

Since derivatives measure rates of change, a differential equation tells us about how one variable changes with respect to another. In general, different starting values for the variables lead to different solutions. Thus, when we use differential equations to model situations in the real world, we often also have *initial conditions* for the variables, which determine a unique solution.

Radioactive decay and half-life

Decay of carbon-14

Carbon-14 is a radioactive isotope of carbon, containing 6 protons and 8 neutrons, that is present in the earth's atmosphere in extremely low concentrations.[2] It is naturally produced in the atmosphere by cosmic rays (and also artificially by nuclear weapons), and continually decays via nuclear processes into stable nitrogen atoms.

Suppose we have a sample of a substance containing some carbon-14. Let m be the mass of carbon-14 in nanograms after t years.[3] It turns out that, if the sample is isolated, then m and t approximately[4] satisfy the differential equation

$$\frac{dm}{dt} = -0.000121\, m.$$

Suppose our sample initially contains 100 nanograms of carbon-14. Let's investigate what happens to the sample over time.

[2] Almost all carbon on earth consists of the stable isotopes carbon-12 (98.9%) and carbon-13 (1.1%). About one in every 10^{12} carbon atoms on earth is carbon-14.

[3] One nanogram is 10^{-12} grams. This is a realistic unit here, as carbon-14 is rare! But there are still over 10^{13} atoms of carbon-14 in one nanogram.

[4] In reality, the atoms decay randomly, one by one, so the mass of carbon-14 does not really change continuously and deterministically, but in discrete random steps. However, for a large number of atoms this differential equation is a good approximation.

First, we can solve the differential equation. Since m has a continuous decay rate of -0.000121, a general solution to the differential equation is

$$m(t) = Ce^{-0.000121\,t},$$

where C is a constant. Substituting the initial condition $t = 0$, $m = 100$ gives $C = 100$, so

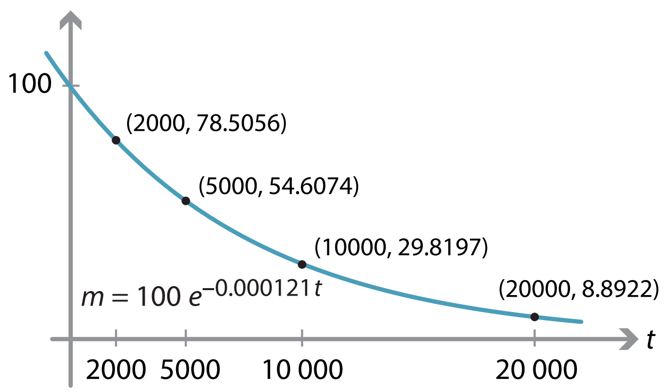
$$m(t) = 100\,e^{-0.000121\,t}.$$

With this formula, we can calculate the amount m of carbon-14 over the years.

Mass of carbon-14 in sample

t (years)	m (ng to 4 decimal places)
0	100.0000
100	98.7973
1000	88.6034
2000	78.5056
5000	54.6074
10 000	29.8197
20 000	8.8922

Every year, the mass m of carbon-14 is multiplied by $e^{-0.000121} \approx 0.999879$. After 100 years, 98.7973 nanograms still remain. After 1000 years, we still have 88.6034 nanograms. But after 5000 years, however, almost half of the carbon-14 has decayed.



Decay of 100 nanograms of Carbon-14.

Half-life of carbon-14

Example

How long does it take for precisely half of the carbon-14 in the sample to decay; that is, when does $m = 50$? Give the answer to three significant figures.

Solution

The mass of carbon-14 in our sample is given by

$$m(t) = 100\,e^{-0.000121\,t}.$$

So we solve $50 = 100\,e^{-0.000121\,t}$, which gives $e^{-0.000121\,t} = \frac{1}{2}$. Hence,

$$t = \frac{\log_e \frac{1}{2}}{-0.000121} \approx 5730 \text{ years} \quad \text{(to three significant figures).}$$

The time period calculated in this example is called the **half-life** of carbon-14. In fact over *any* period of 5730 years, the amount of carbon-14 in an isolated sample will decay by half. This fact is used in **radiocarbon dating** to determine the age of fossils up to 60 000 years old. Roughly speaking, while an organism is alive, its interactions with its environment maintain a constant ratio of carbon-14 to carbon-12 in the organism; but after it dies, the carbon-14 is no longer replenished, and the ratio of carbon-14 to carbon-12 decays in a predictable way. (See *Wikipedia* for more on radiocarbon dating.)

Exercise 1

Explain why the mass of carbon-14 in the sample is given (approximately) by

$$m(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}},$$

and hence explain why the amount of carbon-14 in the sample decays by half over *any* period of 5730 years.

Half-life in general

In general, whenever a quantity $x(t)$ obeys an exponential decay equation

$$x(t) = Ce^{kt},$$

where the continuous decay rate k is negative, then the quantity x has a half-life T. After any time period of length T, the quantity x decreases by half. Let us see why.

As k is negative, the factor e^{kt} decreases from 1 (at $t = 0$) towards 0 (as t approaches ∞). Therefore there is a time $t = T$ such that

$$e^{kT} = \frac{1}{2}.$$

We now solve for T and obtain

$$\begin{aligned} kT &= \log_e \frac{1}{2} \\ &= -\log_e 2, \end{aligned}$$

so

$$T = -\frac{1}{k} \log_e 2.$$

This T is the half-life. From time $t = 0$ to time $t = T$, the factor e^{kt} decreases from $e^0 = 1$ to $e^{kT} = \frac{1}{2}$, that is, decreases by half. Similarly, over *any* time period of length T, the term e^{kt} decreases by half.[5]

Note that, when $k = -0.000121$, we obtain $T = 5730$, in agreement with our calculation for carbon-14.

Population growth and doubling time

Suppose we have a population x of some organism, whether cells in a laboratory, animals in the wild, or humans in a society. Letting t denote time, a population sometimes approximately[6] obeys a differential equation

$$\frac{dx}{dt} = kx,$$

with constant continuous growth rate k. The population will then grow exponentially.

However, exponential population growth is usually unrealistic. The differential equation above expresses the idea that the rate of increase of the population is proportional to the population; and indeed, we might expect a larger population to produce more offspring. But increases or decreases in a population often depend on many other factors, including ecology, economics and culture. Obviously, on a finite planet, exponential population growth cannot continue indefinitely!

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[5] Rigorously: Over a period beginning at a particular time $t = t_0$ and ending at $t = t_0 + T$, the term e^{kt} decreases from e^{kt_0} to $e^{k(t_0+T)} = e^{kt_0} e^{kT} = \frac{1}{2} e^{kt_0}$, that is, decreases by half.

[6] Obviously, the number of organisms in a population is always an integer! Nonetheless, by treating x as a real-valued variable, we may obtain a useful approximation.

Since at present we are focusing on exponential growth, we investigate the differential equation

$$\frac{dx}{dt} = kx,$$

but always remembering that the results obtained for populations may be unrealistic. We will consider a more realistic model in the section *Links forward* (*Logistic growth*).

Example

Today there are 1000 birds on an island. They breed with a constant continuous growth rate of 10% per year. To three significant figures, how many birds will be on the island after seven years?

Solution

Let x be the number of birds on the island after t years. With continuous growth rate $k = 10\% = 0.1$, we have

$$\frac{dx}{dt} = 0.1\,x$$

and hence, solving the differential equation,

$$x(t) = Ce^{0.1\,t},$$

for some positive constant C. Since $x(0) = 1000$, we have $C = 1000$, so

$$x(t) = 1000\,e^{0.1\,t}.$$

After seven years, the number of birds on the island is $x(7) = 1000\,e^{0.7} \approx 2010$, to three significant figures.

In this example, the bird population more than doubled in seven years.

Exercise 2

For the bird population in the previous example, how long does it take for the population to double?

Just as an exponentially decaying quantity has a half-life, an exponentially growing quantity has a **doubling time** — the time it takes for the quantity to double. Calculating doubling time is very similar to calculating half-life, as we see in the next exercise.

Exercise 3

Prove that, for a population with constant continuous growth rate k, the doubling time T is given by

$$T = \frac{1}{k}\log_e 2.$$

For $k = 0.1$, the doubling time should agree with the answer to exercise 2. In fact, in the absence of ecological constraints, bird populations can grow far more rapidly!

Exercise 4

In 1937, eight pheasants were introduced to an island off the coast of the USA and proceeded to reproduce rapidly. For the next few years, the pheasant population grew with a continuous growth rate of over 100%.

a Assuming a constant continuous growth rate of 100%, what is the number of pheasants on the island after 60 years? (Give the answer to three significant figures.)

b Assume each pheasant weighs 1.5 kilograms. After 60 years, which has more mass: the pheasants or the earth? (The earth weighs approximately 5.98×10^{24} kilograms.)

c Discuss the realism of these questions. (In fact, the growth rate of 100% only continued for about three years.)

Reference

These events are discussed by David Lack in *The Natural Regulation of Animal Numbers*, Clarendon Press, 1954, pp. 11–12.

Another differential equation for growth and decay

Consider the differential equation

$$\frac{dx}{dt} = kx + m,$$

where k, m are real constants and $k \neq 0$. We can again think of this equation as describing the growth or decay of a quantity x with respect to time t. Now the growth or decay of x is not only at a rate proportional to itself; it also has a constant component m.

We solve this differential equation by a method similar to that used for our first differential equation. In the *Appendix*, we give an alternative, perhaps more elegant, method.

For simplicity, we consider the case that $kx+m$ is positive. Taking the reciprocal of both sides of the differential equation yields

$$\frac{dt}{dx} = \frac{1}{kx+m},$$

so that

$$t = \int \frac{1}{kx+m}\,dx.$$

Recalling that k and m are just constants, we can perform this integration and obtain

$$t = \frac{1}{k}\log_e(kx+m)+c,$$

where c is a constant of integration. Rearranging this equation gives

$$\begin{aligned} x &= \frac{e^{k(t-c)}-m}{k} \\ &= \frac{1}{k}e^{-kc}\,e^{kt} - \frac{m}{k}. \end{aligned}$$

Now the factor $\frac{1}{k}e^{-kc}$ of the first term is a constant, which we call C.

In general (making no simplifying assumptions), any solution is of the form

$$x(t) = Ce^{kt} - \frac{m}{k},$$

where C is a real constant. You can easily check that any such x is a solution.

Example

Given that $x = 3$ when $t = 0$, find the solution to the differential equation

$$\frac{dx}{dt} = 2x - 5.$$

Solution

The general solution of the differential equation is

$$x(t) = Ce^{2t} + \frac{5}{2},$$

where C is any constant. Substituting $t = 0$, $x = 3$ gives $C = \frac{1}{2}$, so the unique solution is

$$x(t) = \frac{1}{2}e^{2t} + \frac{5}{2}.$$

If $k < 0$ (that is, if x *decays*), then e^{kt} approaches 0 as t becomes very large, so

$$\lim_{t\to\infty} x = \lim_{t\to\infty}\left(Ce^{kt} - \frac{m}{k}\right)$$
$$= -\frac{m}{k}.$$

Thus x approaches $-\dfrac{m}{k}$, which is called the **equilibrium value**. On the other hand, if $k > 0$ (that is, if x *grows*), then x approaches the equilibrium value $-\dfrac{m}{k}$ as $t \to -\infty$.

Example

Suppose x is given by the differential equation

$$\frac{dx}{dt} = -3x + 7,$$

and $x = 11$ when $t = 3$. Find the equilibrium value of x as $t \to \infty$.

Solution

The general solution for x has the form

$$x(t) = Ce^{-3t} + \frac{7}{3}.$$

The first term approaches 0 for large t, so the equilibrium value of x is $\frac{7}{3}$.

Note that we do not need to compute C or use the initial condition in order to calculate the equilibrium value.

The equilibrium value can also be computed by setting $\dfrac{dx}{dt} = 0$. So, for the previous example, the differential equation reduces to $0 = -3x + 7$. At equilibrium, x is in a 'steady state' and the rate of change of x is 0.

Population growth with migration

The constant term m in the differential equation can be considered as the **rate of migration** of a population, as in the following example — remembering, as always, that sustained exponential population growth is usually unrealistic.

Example

Suppose again that there are 1000 birds on an island, breeding with a constant continuous growth rate of 10% per year. But now birds migrate to the island at a constant rate of 100 new arrivals per year. To three significant figures, how many birds are on the island after seven years?

Solution

Let x be the number of birds on the island after t years. With $k = 0.1$ and $m = 100$, we have

$$\frac{dx}{dt} = 0.1\,x + 100.$$

The general solution to this differential equation is

$$x(t) = Ce^{0.1\,t} - \frac{100}{0.1} = Ce^{0.1\,t} - 1000.$$

Substituting $t = 0$, $x = 1000$ gives $C = 2000$, so we have

$$x(t) = 2000\,e^{0.1\,t} - 1000.$$

We obtain $x(7) = 2000\,e^{0.7} - 1000 \approx 3030$ birds, to three significant figures.

Similar differential equations can be used to model quantities which decay but are replenished at a constant rate. In the *History and applications* section we will discuss the example of the HIV virus.

Exercise 5

We've now seen two examples of birds on an island, with initial population 1000 and constant continuous growth rate 10%. Consider the following three scenarios:

a 700 birds migrate to the island all at once, at time $t = 0$, and no further migration occurs.

b Birds migrate to the island at a constant rate of 100 per year, for seven years (as in the previous example).

c 700 birds migrate to the island all at once, at time $t = 7$, and no further migration occurs.

For each scenario, calculate the number of birds on the island after seven years, to three significant figures. (You may use the two relevant previous examples.) Discuss the differences between the answers.

Exercise 6

Nicotine patches are used by people who wish to discontinue smoking. The patches are applied to the skin and deliver nicotine to the blood stream. The concentration A of nicotine in blood plasma (in μg/L) can be modelled by differential equations:

$$\frac{dA}{dt} = R_0 + kA \qquad \text{(when the patch is on)}$$

$$\frac{dA}{dt} = kA \qquad \text{(when the patch is off),}$$

where t is time (in hours), R_0 is the infusion rate of nicotine, and k is the (negative) continuous decay rate of nicotine in the bloodstream. A particular brand of nicotine patch has infusion rate $R_0 = 1$. It is to be applied for 16 hours, then removed for 8 hours (while asleep) each day.

In this exercise, assume $k = -0.12$ and give answers to three significant figures.

a Assume at time $t = 0$ there is no nicotine in the bloodstream; then the patch is applied for 16 hours. What is the concentration of nicotine at $t = 16$?

b The patch is then removed for 8 hours. What is the nicotine concentration at $t = 24$?

References

This exercise was suggested by Geoffrey Kong. See the *References* section of this module for details of a relevant textbook and the research paper.

Growth and decay with input and output

The population of a country is not really ever just growing or decaying. People are always being born, dying, immigrating and emigrating. In general, a quantity may change according to various influences which cause it to grow or decay.

Example

A cell culture in a biology lab currently holds 1 million cells. The cells have a constant continuous birth rate of 1.5% and death rate of 0.5% per hour. Cells are extracted from the culture for an experiment at the rate of 5000 per hour.

How many cells will be in the culture 10 hours from now?

Solution

Let p be the population, and let t be the number of hours from now. Births, deaths and extraction respectively contribute $0.015\,p$, $-0.005\,p$ and -5000 to $\frac{dp}{dt}$. So we have

$$\frac{dp}{dt} = 0.015\,p - 0.005\,p - 5000 = 0.01\,p - 5000.$$

The general solution to this differential equation is

$$\begin{aligned} p(t) &= Ce^{0.01\,t} + \frac{5000}{0.01} \\ &= Ce^{0.01\,t} + 500\,000. \end{aligned}$$

Since $p = 1\,000\,000$ when $t = 0$, we have $C = 500\,000$, so

$$p(t) = 500\,000\,e^{0.01\,t} + 500\,000.$$

Substituting $t = 10$, the population in 10 hours will be $500\,000\,e^{0.1} + 500\,000 \approx 1\,052\,585$.

Exercise 7

a In the previous example, when will the population double (that is, reach 2 million)?

b If no cells were extracted from the culture, but the birth and death rate remained constant, what would be the population doubling time?

Give answers in hours to two decimal places.

Interest and discrete growth

When money is invested at a bank, or elsewhere, with a constant interest rate, the result is exponential growth of the money. Although most real investments involve complications such as varying interest rates, daily or monthly interest calculations, stock prices, or limited time periods, for present purposes let us consider a simplified hypothetical investment with a constant interest rate, with interest calculated after each year.

Suppose the interest rate is 5% per annum. Then the amount $M(0)$ of money invested will receive $0.05\,M(0)$ interest after a year, giving a total of $1.05\,M(0)$. After t years, the amount of money will be multiplied by 1.05, t times, for a resulting total of $M(t) = (1.05)^t\,M(0)$.

In general, let the interest rate be r per annum. (Above, $r = 5\% = 0.05$.) An initial amount $M(0)$ will be multiplied by $(1 + r)$ each year; after t years the total amount $M(t)$ is

$$M(t) = (1 + r)^t\,M(0).$$

As in our previous examples, we are measuring a quantity that varies over time. However, the processes considered in previous examples were approximately[7] *continuous*. The quantities were assumed to grow or decay continuously, and t could be any real number. With bank accounts, however, in practice interest is calculated at regular intervals, not continuously.

Processes which do not occur continuously over time, but rather in regular steps, are called **discrete processes**. Radioactive decay is a continuous process (at least to a good approximation); the regular addition of interest to a bank account is a discrete process.

Whereas continuous processes are described by differential equations such as $\frac{dx}{dt} = kx$, discrete processes are described by **difference equations** such as

$$x(t+1) - x(t) = k\,x(t).$$

Each time x changes, it changes by kx.

With an interest rate of r per annum, the change in the amount of money M in the account each year is given by

$$M(t+1) - M(t) = r\,M(t).$$

We can write this equation as $\Delta M = rM$; the interest rate r can be considered as a **discrete growth rate**.

In the *Links forward* section, we will discuss further the relationship between discrete and continuous processes.

Exercise 8

A bank account contains \$100 and accumulates interest at a rate of r per annum.

- **a** Write a formula for the amount M in the account after t years.
- **b** Assuming that this formula is valid for all real numbers t, when would the amount of money in the account double?
- **c** Assuming again that the formula is valid for all real numbers t, find $\frac{dM}{dt}$ and hence find the continuous growth rate of M.

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[7] Radioactive decays, human births and deaths, and cell divisions happen discretely, one at a time, not continuously; but with large enough populations of atoms, humans or cells, they happen often enough that we can usefully model them as happening continuously. These events can also happen at any time, rather than at regular intervals as with interest.

Non-exponential growth and decay

All the quantities we have seen so far, such as populations and radioactive samples, grow or decay exponentially. That is, they are exponential functions of time t. However, many quantities in the world grow or decay in a non-exponential fashion.

For instance, the electric field strength E of a charged particle decreases rapidly as you move away from the particle. At a distance of r from the particle, the field's strength is

$$E = \frac{C}{r^2},$$

where C is a constant. The electric field strength is said to obey an **inverse-square law**, rather than an exponential law; the 'decay' involves r^{-2} rather than e^{-r}.

Many other quantities in the world — such as gravitational force, sound intensity and radio signal strength — obey inverse-square laws.

Another interesting example is **Zipf's law**, which concerns spoken language. Suppose we rank all the words in a language according to their frequency as spoken. Zipf's law states that the frequency f in spoken language of the word ranked r is *inversely proportional* to r. That is,

$$f = \frac{C}{r},$$

for some constant C. In other words, word frequency decreases in inverse proportion to rank. (See *Wikipedia* for more on Zipf's law.)

In addition to these examples, many quantities *grow* non-exponentially. For instance, if you travel in a straight line at constant velocity v, then your displacement s grows *linearly* with time, $s = vt$. If you travel in a straight line, initially stationary, with constant acceleration a, then your velocity v grows linearly with time, $v = at$, and your displacement grows *quadratically* with time, $s = \frac{1}{2}at^2$. See the module *Motion in a straight line* for further details.

When two quantities x and y are related by an equality of the form $y = Cx^{\alpha}$, where C and α are non-zero constants, we say they obey a **power-law relationship**. The examples from this section are all power-law relationships:

- inverse-square decay ($\alpha = -2$)
- Zipf's law ($\alpha = -1$)
- linear growth ($\alpha = 1$)
- quadratic growth ($\alpha = 2$).

Exponential growth or decay is not a power-law relationship.

We will consider power-law relationships further in the section *History and applications* (*Logarithmic plots*).

Links forward

Discrete and continuous growth

We have seen many examples of a quantity x which continuously grows (or decays) over time by an amount proportional to x, with *continuous growth rate* k. Such an x obeys a differential equation with an exponential solution:

$$\frac{dx}{dt} = kx \quad \Longleftrightarrow \quad x(t) = Ce^{kt}, \text{ where } C \text{ is a constant.}$$

In discussing interest rates, however, we saw an example of a quantity x which grows in discrete steps by an amount proportional to x. Such an x obeys a difference equation with an exponential solution:

$$x(t+1) - x(t) = r\,x(t) \quad \Longleftrightarrow \quad x(t) = (1+r)^t\,x(0).$$

Thinking of this difference equation as $\Delta x = r\,x$, by analogy with the continuous case we call r the **discrete growth rate**. At each step, x is multiplied by $1+r$, and $x(t)$ is obtained from $x(0)$ by t such multiplications. As we have derived it, this equation only holds for integers t. However, if we imagine it is valid for all real t, then $x(t)$ is a continuous function. Rewriting using log laws,

$$x(t) = e^{t\log_e(1+r)}\,x(0).$$

We may then differentiate, noting that $x(0)$ is a constant:

$$\begin{aligned}\frac{dx(t)}{dt} &= \log_e(1+r)\,e^{t\log_e(1+r)}\,x(0)\\ &= \log_e(1+r)\,x(t).\end{aligned}$$

We conclude that the *discrete* growth rate r corresponds to a *continuous* growth rate of

$$k = \log_e(1+r).$$

We saw this fact in exercise 8.

Discrete and continuous interest

In the module *Exponential and logarithmic functions*, we discussed a related concept. We imagined that, instead of paying interest once a year at a rate of r, a bank might add interest twice a year at the rate $\frac{r}{2}$, or three times a year at the rate $\frac{r}{3}$. In general, we considered interest added n times a year at the rate of $\frac{r}{n}$. In this case, each year your money is multiplied by

$$\left(1+\frac{r}{n}\right)^n.$$

We saw that, as $n \to \infty$, this quantity approaches e^r:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

So the quantity $x(t)$ of money is multiplied by e^r each year, and

$$x(t) = e^{rt}\,x(0).$$

Therefore

$$\begin{aligned}\frac{dx(t)}{dt} &= r\,e^{rt}\,x(0)\\ &= r\,x(t),\end{aligned}$$

and the continuous growth rate is r.

Hence, if we take a discrete growth rate (or interest rate) of r, it corresponds to a continuous growth rate of $\log_e(1+r)$. But if that discrete growth is divided into an interest rate of $\frac{r}{n}$, applied n times as often, in the limit we obtain **continuously compounded interest** at a continuous growth rate of r.

Discrete and continuous population growth

The distinction between discrete and continuous growth also applies to populations. We have seen models of exponential population growth with constant continuous growth rates — usually unrealistic models.

However, in a non-technical context, the 'growth rate' of a population is sometimes taken to mean something like the percentage increase in population from the previous year. We can translate this *discrete* growth rate into a *continuous* growth rate, as in the following example.

Example

The Australian Bureau of Statistics estimated that in mid-2012 the resident population of Australia was 22 683 600, an increase of 1.6% from the previous year. Assuming (inaccurately) a constant population growth rate k, find k to three significant figures.

Solution

With an increase of 1.6% in a year, the discrete growth rate is $r = 0.016$. Thus, as discussed in this section, the corresponding continuous growth rate is

$$\begin{aligned} k &= \log_e(1+r) \\ &= \log_e 1.016 \\ &\approx 0.0159 = 1.59\% \quad \text{(to three significant figures).} \end{aligned}$$

References

- Data from `www.abs.gov.au/ausstats/abs@.nsf/mf/3101.0`
- Since 1973, the percentage increase in population per year in Australia has varied between 1.0% and 1.8% (Australian Bureau of Statistics, *Australian Historical Population Statistics, 2008*, 3105.0.65.001).

Note that in this example the discrete and continuous growth rates are very close. This is because $\log_e(1+x) \approx x$ for x close to 0. The two growth rates are close when they are small, but not when they are large.

Exercise 9

Return to our pheasants from exercise 4, which for a few years had a constant continuous growth rate of about 1. What was their discrete growth rate?

Logistic growth

We have seen many examples of exponential population growth based on an equation

$$\frac{dx}{dt} = kx,$$

saying intuitively that a larger population produces proportionately more offspring.

However, as we have discussed, exponential growth cannot continue forever in a finite ecosystem. There are always limits to growth. One way to model ecological constraints is by adding an extra factor corresponding to the **carrying capacity** R of the ecosystem:

$$\frac{dx}{dt} = kx\Big(1 - \frac{x}{R}\Big).$$

Here k (the continuous growth rate) and R are constants. This model is called a **logistic model** of growth. As x increases from 0 to R, the factor $1 - \frac{x}{R}$ decreases from 1 to 0, expressing the idea that as the population increases towards carrying capacity, growth becomes increasingly difficult; at carrying capacity, growth drops to zero.

Differential equations of this type can be solved explicitly, as we now illustrate.

Suppose there are initially 10 rabbits on an island with a carrying capacity of 1000. In the absence of ecological constraints, the rabbits reproduce with a growth rate of 2. Following the logistic model, we have

$$\frac{dx}{dt} = 2x\Big(1 - \frac{x}{1000}\Big) = \frac{x(1000-x)}{500}.$$

Using the fact that $\frac{dx}{dt} \cdot \frac{dt}{dx} = 1$ gives

$$\frac{dt}{dx} = \frac{500}{x(1000-x)}.$$

In order to integrate this expression we write it in terms of *partial fractions*, setting

$$\frac{500}{x(1000-x)} = \frac{A}{x} + \frac{B}{1000-x}$$

and solving for A and B. Cross-multiplying gives $500 = A(1000-x) + Bx$; so we obtain $A = B = \frac{1}{2}$. Thus we can write

$$\frac{dt}{dx} = \frac{1}{2x} + \frac{1}{2(1000-x)}.$$

We may antidifferentiate term-by-term to obtain

$$\begin{aligned} t &= \frac{1}{2}\log_e x - \frac{1}{2}\log_e(1000-x) + c \\ &= \frac{1}{2}\log_e\Big(\frac{x}{1000-x}\Big) + c, \end{aligned}$$

where c is a constant. Rearranging this equation we obtain

$$e^{2(t-c)} = \frac{x}{1000-x}.$$

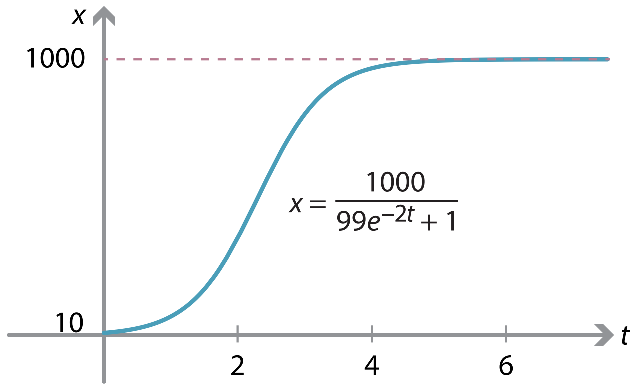
Substituting the initial condition $t = 0$, $x = 10$ gives $e^{-2c} = \frac{1}{99}$, so we have

$$\frac{1}{99}e^{2t} = \frac{x}{1000 - x}.$$

Solving now for x gives

$$x = \frac{1000}{99e^{-2t} + 1}.$$

This is the desired result. As t increases towards infinity, e^{-2t} approaches 0, so the denominator approaches 1, and therefore x approaches the carrying capacity of 1000.



Graph of rabbit population according to logistic model.

Using the same technique, it can be shown that the general logistic model

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{R}\right),$$

with k, R constants, has general solution

$$x = \frac{R}{e^{-k(t-c)} + 1},$$

where c is a constant.

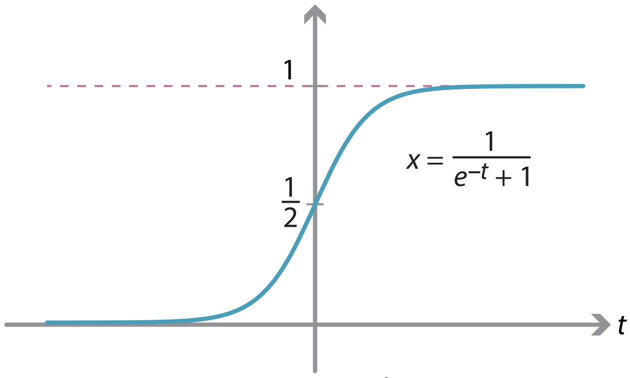
Exercise 10

Prove this.

Functions of this type are often called **logistic functions**. The simplest such function is

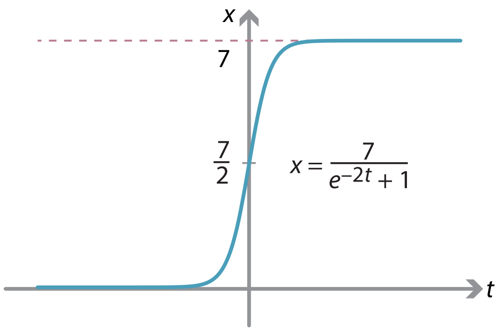
$$f(t) = \frac{1}{e^{-t} + 1},$$

which has the following graph.

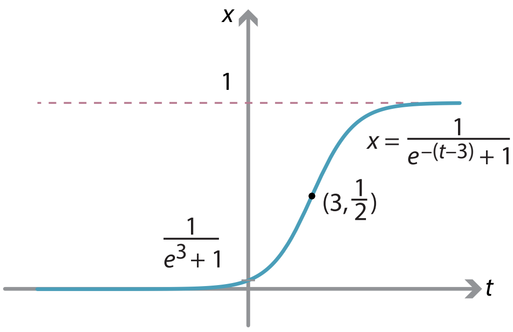


Graph of $x = \dfrac{1}{e^{-t}+1}$.

The next two graphs give further examples of logistic functions, with different values of k, R and c.



$c = 0, k = 2, R = 7.$



$c = 3, k = 1, R = 1.$

History and applications

World population

In 2010 the UN Department of Economic and Social Affairs released a series of reports on world population (see the *References* section). The UN estimated the continuous growth rate of world population then at 1.162%, and the total population at approximately 6 895 889 000.

The UN projected that the population growth rate would change significantly over time; assuming a constant growth rate is unrealistic. But for present purposes, let us assume a constant growth rate and see what happens — and how the result compares with the UN's more accurate modelling.

Let P be the world population at time t years after 2010. With continuous growth rate 1.162%, we have

$$\frac{dP}{dt} = 0.01162\,P,$$

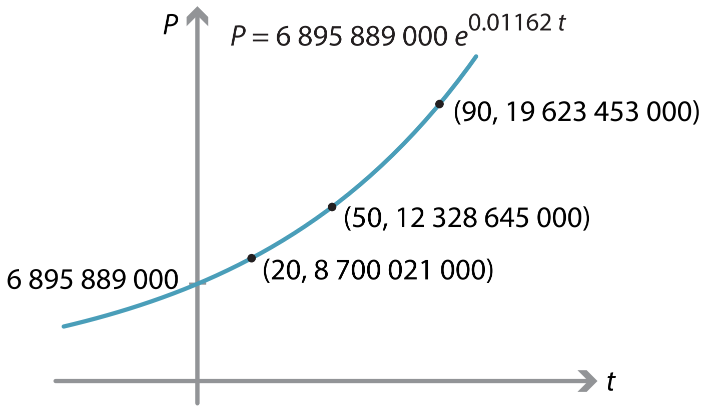
which has general solution $P(t) = Ce^{0.01162\,t}$. Since $P(0) = 6\,895\,889\,000$, we have

$$P(t) = 6\,895\,889\,000\,e^{0.01162\,t}.$$

We can now compute and graph world population.

World population at constant growth rate

Year	t (years)	P (nearest thousand)
2010	0	6 895 889 000
2020	10	7 745 604 000
2030	20	8 700 021 000
2060	50	12 328 645 000
2100	90	19 623 453 000



Graph of world population at constant growth rate 0.01162.

Assuming exponential growth, the world's population almost doubles in 50 years, and approaches 20 billion by 2100. However, all of the UN's projections were lower than this. For instance, the UN's 'medium fertility' projection gives a total world population of about half of our estimate, 10.9 billion, by 2100. A model of exponential growth is unrealistic.

HIV virus population growth and decay

In a series of papers in the 1990s, scientists used mathematical models to study HIV viruses in the human body (see the *References* section).

When a person contracts HIV, viruses infect white blood cells by attaching to certain proteins, and the infected cells are reprogrammed to produce viruses. After an initial acute phase, but before the onset of AIDS, the HIV viruses circulate within the body at low levels. It was previously thought that not much is happening during this phase, but a mathematical model helped to show that this is not the case.

The number n of HIV viruses (per millilitre of extracellular fluid) after t days can be modelled by the differential equation

$$\frac{dn}{dt} = kn + m,$$

where k is the rate at which viruses are cleared from the bloodstream by the immune system (that is, the continuous decay rate for viruses) and m is a constant rate of production of viruses from host cells (per day per millilitre of fluid).

We look at a simplified version of an experiment conducted by Perelson and others (see the *References* section). The experiment started by assuming that the level of HIV viruses had reached equilibrium. A typical value for the number of viruses in the bloodstream was 100 000 per millilitre.

Exercise 11

Explain why $-\dfrac{m}{k} \approx 100\,000$.

During the experiment, an antiviral agent was administered which effectively reduced the production of new viruses to zero, that is, $m = 0$. It was then found that the number of viruses decreased exponentially with a half-life of 0.24 days.

Exercise 12

Find the decay rate k.

From this information, several calculations can be made.

Exercise 13

a Based on the previous exercises, compute m, the production rate of viruses per day per millilitre of extracellular fluid.

b The average human body contains about 15 litres of extracellular fluid. Approximately how many viruses are produced per day?

The answer found by the experimenters (using a more sophisticated model than ours) was that about 10^{10} viruses are produced per day — far more than previously thought. Even when the HIV infection is in an apparently quiet phase, a huge number of viruses are being produced and destroyed every day!

Newton's law of cooling

The temperature of many objects can be modelled using a differential equation. **Newton's law of cooling** (or heating) states that the temperature of a body changes at a rate proportional to the difference in temperature between the body and its surroundings. It is a reasonably accurate approximation in some circumstances.

More precisely, let T denote the temperature of an object and T_0 the ambient temperature. If t denotes time, then Newton's law states that

$$\frac{dT}{dt} = -k(T - T_0),$$

where k is a positive constant. Thus, if the object is much hotter than its surroundings, then $T - T_0$ is large and positive, so $\frac{dT}{dt}$ is large and negative, so the object cools quickly. If the object is only slightly hotter than its surroundings, then $T - T_0$ is small positive, and the object cools slowly. So a cup of hot coffee will cool more quickly if you put it in the refrigerator!

This differential equation is of the same type as ones seen previously in this module.

Example

You take an ice-cream out of the freezer, kept at -18 °C. Outside it is 32 °C. After one minute, the ice-cream has warmed to -8 °C. What is the temperature of the ice-cream after five minutes?

Solution

Let T be the temperature of the ice-cream (in °C) after t minutes out of the freezer. Then Newton's law gives

$$\frac{dT}{dt} = -k(T - 32) = -kT + 32k.$$

Remembering that k is a constant, solving this differential equation gives a general solution of

$$T = Ce^{-kt} + 32.$$

Since $T = -18$ when $t = 0$, we obtain $-18 = C + 32$, so $C = -50$. Since $T = -8$ when $t = 1$, we have $-8 = -50\,e^{-k} + 32$, which gives $k = \log_e \frac{5}{4}$. We obtain

$$\begin{aligned} T &= -50\,e^{-(\log_e \frac{5}{4})\,t} + 32 \\ &= -50\left(\frac{4}{5}\right)^t + 32. \end{aligned}$$

(Here we used some index laws.) Hence, when $t = 5$, we have

$$T = -50\left(\frac{4}{5}\right)^5 + 32 = \frac{1952}{125} \approx 15.6\,^\circ\mathrm{C}.$$

Your ice-cream has well and truly melted! Note that we have effectively assumed that the ice-cream is a block of a single temperature — not very realistic.

Exercise 14

A cup of coffee is made with boiling water at a temperature of 100 °C, in a room at temperature 20 °C. After two minutes it has cooled to 80 °C. What is its temperature after five minutes? When will the coffee drop below 40 °C and taste cold?

Exercise 15

You are given a very hot sample of metal, and wish to know its temperature. You have a thermometer, but it only measures up to 200 °C and the metal is hotter than that!

You leave the metal in a room kept at 20 °C. After six minutes it has cooled sufficiently that you can measure its temperature; it is 80 °C. After another two minutes it is 50 °C. What was the initial temperature of the metal?

Logarithmic plots

Suppose that you have a set of data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ from an experiment or some observations, and you see that y grows or decays with respect to x. You wish to find out whether the growth or decay is exponential, or obeys a power law, or otherwise. A useful way to determine an exponential or power-law relationship is by using **logarithmic plots**.

For instance, if y is exponentially related to x, that is, if $y = Ce^{kx}$ for some constants C and k, then

$$\log_e y = kx + \log_e C,$$

so the graph of $\log_e y$ against x will be a *straight line* whose gradient is the continuous growth/decay rate k. Similarly, if y is related to x by a power law, $y = Cx^{\alpha}$, then

$$\log_e y = \alpha \log_e x + \log_e C,$$

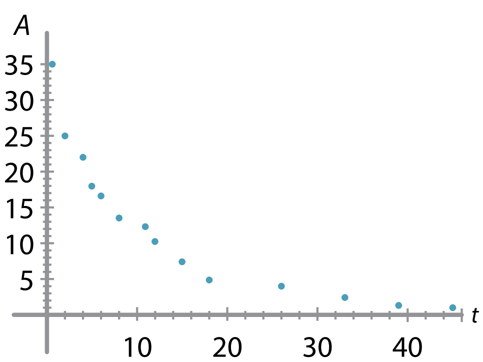
so the graph of $\log_e y$ against $\log_e x$ will be a straight line of gradient α.

We consider, as an example, some data taken from a 1905 experiment on radioactive decay (see the *References* section). The physicists Meyer and von Schweidler took measurements of radioactive activity A from a substance at various times t, measured in days. (The unit for radioactive activity is not of interest here.)

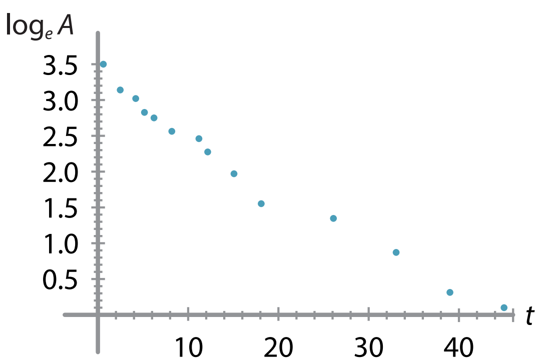
Measurements of radioactivity

t (time in days)	A (activity)
0.2	35.0
2.2	25.0
4.0	22.1
5.0	17.9
6.0	16.8
8.0	13.7
11.0	12.4
12.0	10.3
15.0	7.5
18.0	4.9
26.0	4.0
33.0	2.4
39.0	1.4
45.0	1.1

We can plot the graph of A against t and the graph of $\log_e A$ against t, as follows.



Graph of A against t.



Graph of $\log_e A$ against t.

We see that the graph of A against t looks rather close to exponential decay, although with a few bumps, and the graph of $\log_e A$ against t is quite close to linear, again with some bumps. These bumps might indicate some contamination, experimental error, or multiple types of radioactive substance in the sample. In practice, establishing simple mathematical relationships from empirical data can be difficult. However, exponential decay of radioactive substances is by now a well-established scientific fact.

Appendix

Solving a differential equation using an equilibrium value

In the section *Another differential equation for growth and decay*, we gave a method for solving differential equations of the form

$$\frac{dx}{dt} = kx + m,$$

where k, m are real constants and $k \neq 0$. We now give an alternative solution method.

We have seen that $-\frac{m}{k}$ is the *equilibrium* value of x. We define a new variable y as

$$y = x - \left(-\frac{m}{k}\right) = x + \frac{m}{k}.$$

So y is just a translation of x, and the equilibrium value of $x = -\frac{m}{k}$ corresponds to $y = 0$.

We may now express the differential equation in terms of y. Since $x = y - \frac{m}{k}$, we have

$$\begin{aligned}\frac{dy}{dt} &= \frac{dx}{dt}\\ &= kx + m\\ &= k\left(y - \frac{m}{k}\right) + m\\ &= ky.\end{aligned}$$

By changing variables from x to y, we have obtained a simpler equation. The general solution is

$$y = Ce^{kt},$$

where C is a real constant. Substituting back for x, we obtain the general solution

$$x = Ce^{kt} - \frac{m}{k}.$$

Answers to exercises

Exercise 1

Since (as calculated in the example)

$$e^{-0.000121\times 5730} \approx \frac{1}{2},$$

we may rewrite the equation $m = 100\,e^{-0.000121\,t}$ as

$$\begin{aligned} m &= 100\,e^{-0.000121\,t} \\ &= 100\left(e^{-0.000121\times 5730}\right)^{\frac{t}{5730}} \\ &\approx 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}. \end{aligned}$$

From this equation we can see that, when t increases by 5730, the exponent $\frac{t}{5730}$ increases by 1 and so m is multiplied by $\frac{1}{2}$.

Exercise 2

Since the population is given by $x(t) = 1000\,e^{0.1\,t}$, the population is double its initial value when

$$2000 = 1000\,e^{0.1\,t} \quad\Longrightarrow\quad t = \frac{\log_e 2}{0.1} \approx 6.93 \text{ years.}$$

Exercise 3

If the growth rate is k, then the population P will be given by $P(t) = Ce^{kt}$ after t years. The population will double over a time period of length T if

$$\begin{aligned} P(t+T) = 2P(t) \quad &\Longleftrightarrow \quad Ce^{k(t+T)} = 2Ce^{kt} \\ &\Longleftrightarrow \quad e^{kT} = 2 \\ &\Longleftrightarrow \quad T = \frac{1}{k}\log_e 2. \end{aligned}$$

Exercise 4

a Let P be the number of pheasants t years after introduction. With continuous growth rate $k = 1$, we have $P(t) = Ce^{t}$ for a constant C. Since $P(0) = 8$, we have $C = 8$ and so $P(t) = 8e^{t}$. After 60 years, the pheasant population is $P(60) = 8e^{60} \approx 9.14\times 10^{26}$.

b At 1.5 kg per pheasant, we have $9.14\times 10^{26}\times 1.5 \approx 1.37\times 10^{27}$ kg of pheasants. This is much greater than the mass of the earth.

c A continuous growth rate as rapid as 100% cannot persist for long. The pheasants will rapidly run out of food, land and other resources. The population's growth will slow down as these limits are reached. A more realistic model is discussed in the section *Links forward* (*Logistic growth*).

Exercise 5

Let $P(t)$ denote the population after t years.

a In this case we effectively have a constant continuous growth rate of $k = 0.1$, starting from a population of 1700. We obtain $P(t) = 1700\,e^{0.1\,t}$. After seven years, the number of birds is $P(7) = 1700\,e^{0.7} \approx 3420$ (to three significant figures).

b This is the situation in the example, which gives 3030 birds after seven years.

c This is effectively the situation in the earlier example, but with an extra 700 birds added at the end. Thus, to three significant figures, there are $2010 + 700 = 2710$ birds after seven years.

We see that the earlier the birds migrate, the larger the final population. The earlier they arrive, the more time they have to reproduce.

Exercise 6

a With the patch on, we have $\frac{dA}{dt} = 1 - 0.12A$, which has general solution

$$A(t) = Ce^{-0.12\,t} + \frac{1}{0.12}.$$

Since $A(0) = 0$, we obtain $C = -\frac{1}{0.12}$. Thus

$$\begin{aligned} A(t) &= -\frac{1}{0.12}e^{-0.12\,t} + \frac{1}{0.12} \\ &= \frac{1}{0.12}\big(1 - e^{-0.12\,t}\big), \end{aligned}$$

so

$$\begin{aligned} A(16) &= \frac{1}{0.12}\big(1 - e^{-0.12\times 16}\big) \\ &\approx 7.11\ \mu\text{g/L} \quad \text{(to three significant figures).} \end{aligned}$$

b With the patch off, $\frac{dA}{dt} = -0.12A$, which has general solution $A(t) = Ce^{-0.12\,t}$. Since $A(16) \approx 7.11$, we obtain $C \approx 48.5$, so

$$\begin{aligned} A(24) &\approx 48.5\,e^{-0.12\times 24} \\ &\approx 2.72\ \mu\text{g/L} \quad \text{(to three significant figures).} \end{aligned}$$

Exercise 7

a The population reaches 2 million when

$$2\,000\,000 = 500\,000\,e^{0.01\,t} + 500\,000.$$

Rearranging this gives

$$e^{0.01\,t} = 3 \quad \Longleftrightarrow \quad t = 100\log_e 3.$$

So $t \approx 109.86$ hours (to two decimal places).

b If there were no extraction, then we would have the differential equation

$$\frac{dp}{dt} = 0.015\,p - 0.005\,p = 0.01\,p.$$

Thus the growth rate is $k = 0.01$ and hence the doubling time is

$$\begin{aligned} T &= \frac{1}{k}\log_e 2 \\ &= 100\log_e 2 \\ &\approx 69.31 \text{ hours} \quad \text{(to two decimal places).} \end{aligned}$$

Exercise 8

a $M(t) = 100\,(1+r)^t$

b The amount in the account doubles when $(1+r)^t = 2$. Taking the natural logarithm of both sides gives $t\log_e(1+r) = \log_e 2$, so

$$t = \frac{\log_e 2}{\log_e(1+r)}.$$

c Rewriting $(1+r)^t$ as $e^{t\log_e(1+r)}$ gives $M = 100\,e^{t\log_e(1+r)}$. Therefore

$$\begin{aligned} \frac{dM}{dt} &= 100\log_e(1+r)\,e^{t\log_e(1+r)} \\ &= M\log_e(1+r), \end{aligned}$$

and hence the continuous growth rate is $\log_e(1+r)$.

Exercise 9

The discrete growth rate r and the continuous growth rate k are related by $k = \log_e(1+r)$. So $k = 1$ gives $1 = \log_e(1+r)$, and therefore

$$r = e - 1 \approx 1.72 \quad \text{(to two decimal places).}$$

This discrete growth rate of 1.72 per year is analogous to an interest rate of 172% per annum; the number of pheasants was multiplied by 2.72 each year.

Exercise 10

From

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{R}\right) = \frac{kx(R-x)}{R},$$

we obtain

$$\frac{dt}{dx} = \frac{R}{kx(R-x)}.$$

Writing this expression in terms of partial fractions we obtain

$$\frac{dt}{dx} = \frac{1}{kx} + \frac{1}{k(R-x)}.$$

Integrating gives

$$\begin{aligned} t &= \frac{1}{k}\log_e x - \frac{1}{k}\log_e(R-x) + c \\ &= \frac{1}{k}\log_e\left(\frac{x}{R-x}\right) + c, \end{aligned}$$

where c is a constant of integration. Rearranging, we obtain

$$x = \frac{R}{e^{-k(t-c)} + 1},$$

as desired.

Exercise 11

The equilibrium value of n is given by $-\frac{m}{k}$, which is assumed to be the existing number of HIV viruses before the experiment.

Exercise 12

After administering the agent so that $m = 0$, the number of viruses n obeys the differential equation $\frac{dn}{dt} = kn$, giving exponential decay. The half-life T is given by $T = -\frac{1}{k}\log_e 2$. Since $T = 0.24$ days, we have

$$0.24 = -\frac{1}{k}\log_e 2 \quad \iff \quad k = -\frac{1}{0.24}\log_e 2 \approx -2.88811.$$

Exercise 13

a Since $-\frac{m}{k} \approx 100\,000$ and $k \approx -2.88811$, we have

$$m \approx 100\,000 \times 2.88811 = 288\,811.$$

That is, approximately 288 811 viruses are produced per day per millilitre of extracellular fluid.

b Multiplying m by the 15 000 millilitres of extracellular fluid gives a production rate of 4.3×10^9 viruses per day. That's a lot!

Exercise 14

Let T be the temperature of the coffee (in °C) after t minutes. By Newton's law we have

$$\frac{dT}{dt} = -k(T - 20) = -kT + 20k,$$

so the general solution is $T(t) = Ce^{-kt} + 20$, where C is a constant. We have $T(0) = 100$, so $100 = C + 20$, and therefore $C = 80$. From $T(2) = 80$, we have $80 = 80\,e^{-2k} + 20$, and hence $k = -\frac{1}{2}\log_e \frac{3}{4}$. Thus

$$T(t) = 80\,e^{\frac{1}{2}(\log_e \frac{3}{4})\,t} + 20 = 80\left(\frac{3}{4}\right)^{\frac{t}{2}} + 20.$$

After five minutes, the temperature is

$$T = 80\left(\frac{3}{4}\right)^{\frac{5}{2}} + 20 \approx 58.97\,°\text{C}.$$

The temperature is 40 °C when

$$\begin{aligned} 40 &= 80\left(\frac{3}{4}\right)^{\frac{t}{2}} + 20 \\ \left(\frac{3}{4}\right)^{\frac{t}{2}} &= \frac{1}{4} \\ \frac{t}{2}\log_e \frac{3}{4} &= \log_e \frac{1}{4} \\ t &= \frac{2\log_e \frac{1}{4}}{\log_e \frac{3}{4}} \approx 9.64 \text{ minutes.} \end{aligned}$$

Exercise 15

Let T be the temperature of the metal (in °C) after t minutes. From Newton's law of cooling we have

$$\frac{dT}{dt} = -k(T - 20) = -kT + 20k,$$

which has general solution $T(t) = Ce^{-kt} + 20$. From $T(6) = 80$ and $T(8) = 50$ we obtain

$$Ce^{-6k} + 20 = 80 \qquad \text{and} \qquad Ce^{-8k} + 20 = 50,$$

giving $Ce^{-6k} = 60$ and $Ce^{-8k} = 30$. Dividing these two equations gives $e^{2k} = 2$, so that $k = \frac{1}{2}\log_e 2$. From $Ce^{-6k} = 60$ we then have

$$60 = Ce^{-\frac{6}{2}\log_e 2} = C \times 2^{-3} = \frac{C}{8},$$

so that $C = 480$. We have thus found the temperature $T(t)$ to be

$$T(t) = 480\, e^{-\frac{1}{2}(\log_e 2)\, t} + 20$$
$$= 480\left(\frac{1}{\sqrt{2}}\right)^t + 20,$$

and so the initial temperature was $T(0) = 500\,°\mathrm{C}$.

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