Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Functions: Module 6

Functions II

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Functions II – A guide for teachers (Years 11–12)

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Functions II

Assumed knowledge

This module builds on the module *Functions I*. It assumes knowledge of that module and also the modules required for it:

- *Algebra review*
- *Coordinate geometry*
- *Trigonometric functions and circular measure*
- *Exponential and logarithmic functions*.

Motivation

The importance of the concept of a function to modern mathematics was explained in the module *Functions I*. In this module, we shall develop five aspects of the theory.

1. **Arithmetic of functions.** We shall define how to add, subtract, multiply and divide functions.
2. **Odd and even functions.**
3. **Composition of functions.** In particular, we shall see that the order of composition is important, so that $\sin(x^2)$ and $(\sin x)^2$ are completely different functions.
4. **Geometrical transformations.** We shall see that many functions can be understood in terms of a geometrical transformation, such as a translation or a reflection.
5. **Functions and their inverses.** We shall investigate which functions have inverses. For example, the functions in the following pairs are inverses of each other:
   - $f(x) = x + 2$ and $g(x) = x - 2$
   - $f(x) = 2x$ and $g(x) = \dfrac{x}{2}$.

   We shall develop a sufficiently general concept of inverses to cover the example of x^2 and $\sqrt{x}$, where we have $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$, but with some restrictions on x.

All of these ideas are important in differential calculus and in curve sketching.

In *Functions I*, we covered:

- the concept of a function
- the difference between functions and relations
- the vertical-line test
- domains and ranges
- interval notation
- standard function notation.

In this module, we shall build on these ideas.

Content

The arithmetic of functions

Just as numbers can be added, subtracted, multiplied and divided to form other numbers, so functions can be added, subtracted, multiplied and divided to form other functions.

We have already seen in the TIMES module *Polynomials* (Year 10) that polynomials can be added, subtracted, multiplied and divided. The definitions given there are consistent with the definitions for functions in general.

In later sections of this module, we will look at two other ways of forming new functions: composition of functions and finding the inverse of a function.

Throughout this module, we consider **real functions**, that is, functions $f\colon D \to \mathbb{R}$, where the domain D is a subset of $\mathbb{R}$.

Addition and subtraction of functions

If f and g are functions with domains D_f and D_g, then the **sum of the functions** $f+g$ is defined by

$$(f+g)(x) = f(x) + g(x).$$

The domain of $f+g$ is $D_f \cap D_g$. That is, the function $f+g$ is defined only where both f and g are defined.

For example, let $f(x) = e^x$ and $g(x) = \frac{1}{x}$, where the domain of f is $\mathbb{R}$ and the domain of g is $\mathbb{R} \setminus \{0\}$. Then the function $f + g$ has domain $\mathbb{R} \cap (\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\}$, with

$$(f+g)(x) = e^x + \frac{1}{x}.$$

Similarly, if f and g are functions with domains D_f and D_g, then the **difference of the functions** $f - g$ is defined by

$$(f-g)(x) = f(x) - g(x).$$

The domain of $f - g$ is $D_f \cap D_g$.

Products and quotients of functions

If f and g are functions with domains D_f and D_g, then the **product of the functions** $f \cdot g$ is defined by

$$(f \cdot g)(x) = f(x) \cdot g(x).$$

The domain of $f \cdot g$ is $D_f \cap D_g$.

For example, using the functions $f(x) = e^x$ and $g(x) = \frac{1}{x}$ from above, we have

$$(f \cdot g)(x) = e^x \cdot \frac{1}{x} = \frac{e^x}{x}.$$

If f and g are functions with domains D_f and D_g, then the **quotient of the functions** $\dfrac{f}{g}$ is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}.$$

The domain of $\dfrac{f}{g}$ is $\{x \in D_f \cap D_g : g(x) \neq 0\}$.

Odd and even functions

For most functions $f(x)$, replacing x with $-x$ changes the function dramatically. For some functions, however, there is either no change or just a change in sign. For example, if $f(x) = x^6$, then $f(-x) = x^6$. On the other hand, if $f(x) = x^7$, then $f(-x) = -x^7$. The notion of odd and even functions generalises these two examples.

Definitions

- A function f is **even** if $f(-x) = f(x)$, for all x in the domain of f.
- A function f is **odd** if $f(-x) = -f(x)$, for all x in the domain of f.

Example

1 The polynomial function $f(x) = x^2 + x^4 + x^6$ is even.

2 The polynomial function $f(x) = x + x^3 + x^5$ is odd.

3 The polynomial function $f(x) = 1 + x + x^2$ is neither odd nor even.

4 We observe that

$$\sin(-x) = -\sin x \qquad \text{and} \qquad \cos(-x) = \cos x,$$

for all x. Thus $\sin x$ is an odd function and $\cos x$ is an even function. The function $\tan x$ is also an odd function, but on a slightly restricted domain: all reals except the odd multiples of $\frac{\pi}{2}$.

5 The functions $f(x) = e^x$ and $g(x) = \log_e x$ are neither odd nor even functions.

Note. It follows from the definition that, if a function f is odd or even, then its domain must be symmetric about the origin. That is, it follows that x is in the domain of f if and only if $-x$ is in the domain of f.

Exercise 1

a Show that the only function $f\colon \mathbb{R} \to \mathbb{R}$ which is both odd and even is the constant function $f(x) = 0$.

b Show that every polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ can be written as the sum of an odd function and an even function.

c Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function in a unique way.

Exercise 2

Prove the following:

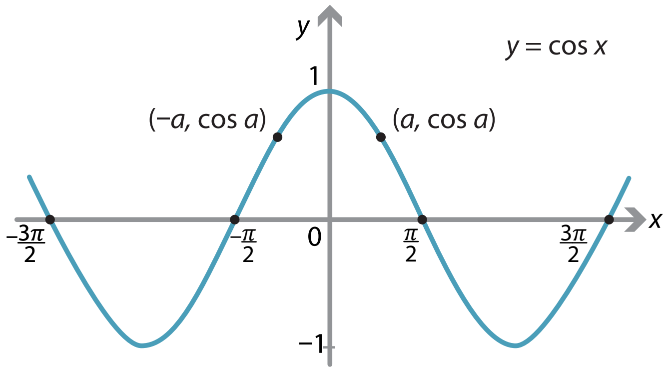
a The sum of two odd functions is odd, and the sum of two even functions is even.

b The product of two even functions is even, the product of two odd functions is even, and the product of an odd function and an even function is odd.

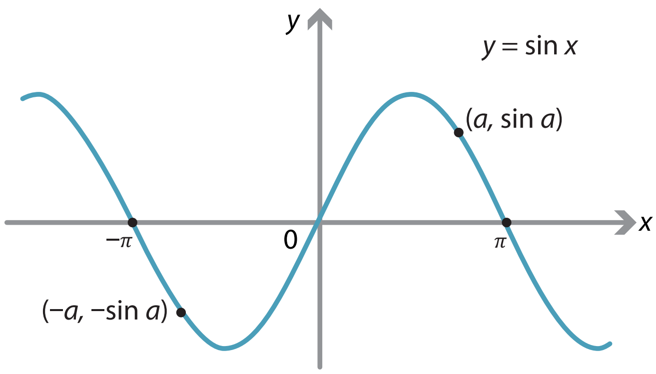
c Let f and g be functions on the same domain, and assume that each function takes at least one non-zero value. If f is odd and g is even, then the sum $f + g$ is neither odd nor even.

Symmetries of odd and even functions

We have observed that $\cos x$ is an even function. From the following figure, we can see that its graph $y = \cos x$ is symmetric about the y-axis. That is, it has reflection symmetry about the y-axis. Every even function has this property.

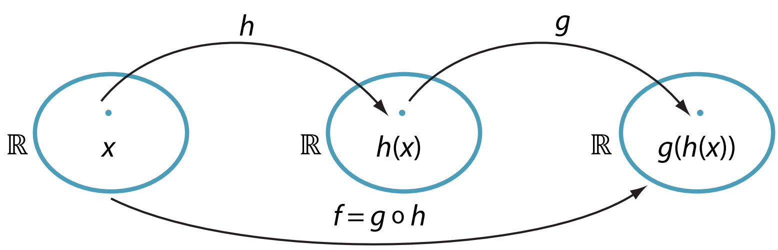


We have also observed that $\sin x$ is an odd function. Its graph $y = \sin x$ has rotational symmetry through 180° about the origin. Every odd function has this property.



Composites of functions

In this section, for convenience and simplicity, we will focus our attention on functions with domain and codomain the reals.



Without worrying about technicalities such as domains and ranges, we can imagine applying one function h to a real number and then applying another function g to the image. The result of this is called the **composite** or the **composition** of the two functions h and g. In the above diagram, we can write

$$f(x) = g(h(x)).$$

A common notation for this situation is $f = g \circ h$.

For example, consider the two functions $h(x) = x+3$ and $g(x) = \sin x$, and let $f = g \circ h$. Then

$$f(x) = g(h(x)) = g(x+3) = \sin(x+3).$$

What if we swap the order and look at $j = h \circ g$? We have

$$j(x) = h(g(x)) = h(\sin x) = \sin(x) + 3.$$

Clearly, $\sin(x+3) \neq \sin(x) + 3$, for any value of x.

Thus the order in which we compose two functions is important.

Notes.

1 Throughout this module, we shall assume that if we have an expression involving a trigonometric function, then the variables are measured in radians. Check that your calculator is in radian mode by verifying that $\sin(2) + 3 \approx 3.9093$.

2 The rule $a + b = b + a$, for all real numbers a, b, is the commutative law for addition. The fact that we can have $g \circ h \neq h \circ g$, for some functions g, h, says that composition of functions is *not commutative*.

3 Composition of functions is not the same as multiplication of functions:

$$f = h \circ g \qquad \text{means} \qquad f(x) = h(g(x))$$

$$j = h \cdot g \qquad \text{means} \qquad j(x) = h(x)\, g(x).$$

So, for example, if $g(x) = x^2$ and $h(x) = \sin x$, then $f(x) = \sin(x^2)$ and $j(x) = x^2 \sin x$. Clearly, $h \cdot g = g \cdot h$, since $h(x)\, g(x) = g(x)\, h(x)$ for all x.

4 It is possible to compose more than two functions. In particular, if $f(x) = k(h(g(x)))$, then $f = k \circ h \circ g$.

5 For all functions k, h, g, we have $(k \circ h) \circ g = k \circ (h \circ g)$. So composition of functions is *associative*. This allows us to write $k \circ h \circ g$ without any ambiguity.

Example

Let $g(x) = x^2$, $h(x) = x + 2$ and $j(x) = \sin x$. Write down the formula for $k = j \circ h \circ g$.

Solution

We have

$$\begin{aligned} g(x) &= x^2 \\ h(g(x)) &= x^2 + 2 \\ j(h(g(x))) &= \sin(x^2 + 2). \end{aligned}$$

So $k(x) = \sin(x^2 + 2)$.

Example

Write

$$f(x) = \frac{1}{\cos(7\log_2(x^2 + 3))}$$

as the composite of six functions.

Solution

Let

$$\begin{aligned} f_1(x) &= x^2 \\ f_2(x) &= x + 3 \\ f_3(x) &= \log_2 x \\ f_4(x) &= 7x \\ f_5(x) &= \cos x \\ f_6(x) &= \frac{1}{x}. \end{aligned}$$

Then

$$f = f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1.$$

We can compose functions whose domains are subsets of the real numbers. Consider functions $g\colon A \to \mathbb{R}$ and $h\colon B \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Then $g(h(x))$ is defined for each x in the domain of h such that $h(x)$ is in the domain of g. So we obtain a composite function $g \circ h\colon C \to \mathbb{R}$ with domain $C = \{x \in B : h(x) \in A\}$.

For example, consider the functions $g\colon \mathbb{R}^+ \to \mathbb{R}$ and $h\colon [0,\pi] \to \mathbb{R}$ given by $g(x) = \log_e x$ and $h(x) = \cos x$. Then the composite function $g \circ h$ is given by $(g \circ h)(x) = \log_e(\cos x)$, with domain $\{x \in [0,\pi] : \cos x \in \mathbb{R}^+\} = [0, \frac{\pi}{2})$.

Geometric transformations of graphs of functions

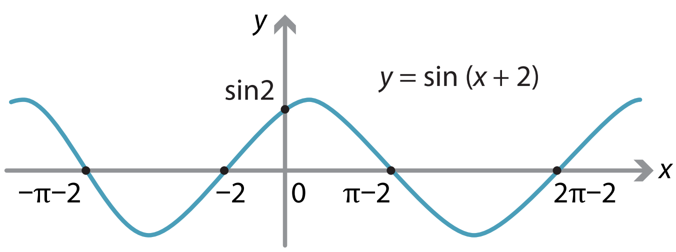
The graph of a function can be changed to produce the graph of a new function using:

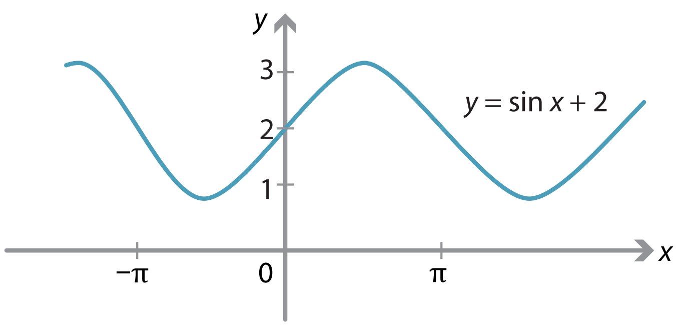
- vertical or horizontal translations
- reflections in either axis
- stretching from either axis.

For example, if $f(x) = \sin x$ and $g(x) = x + 2$, then

$$(f \circ g)(x) = \sin(x+2) \qquad \text{and} \qquad (g \circ f)(x) = \sin(x) + 2.$$

Both of these can be thought of as translations of $f(x) = \sin x$, so clearly we cannot talk about *the* translation of a function by 2.



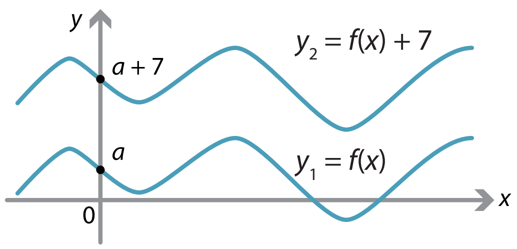


On the other hand, the image of the graph of a function under a rotation is usually not the graph of a function. Rotations of graphs of quadratic functions are discussed briefly in the module *Quadratics*.

Translations

Vertical translations in general

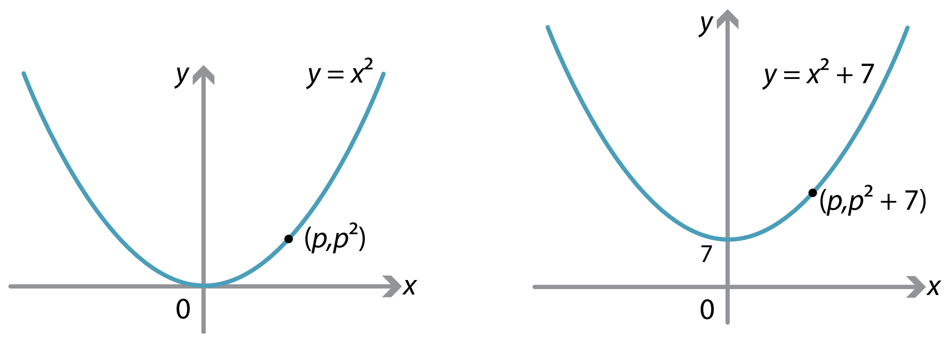
Consider the graph of $y_1 = f(x)$ and the graph of $y_2 = f(x)+7$, as shown in the following diagram.



We call $y_2 = f(x)+7$ the vertical translation 7 units up of $y_1 = f(x)$. Similarly, $y_3 = f(x)-7$ is the vertical translation 7 units down of $y_1 = f(x)$.

Vertical translations of parabolas

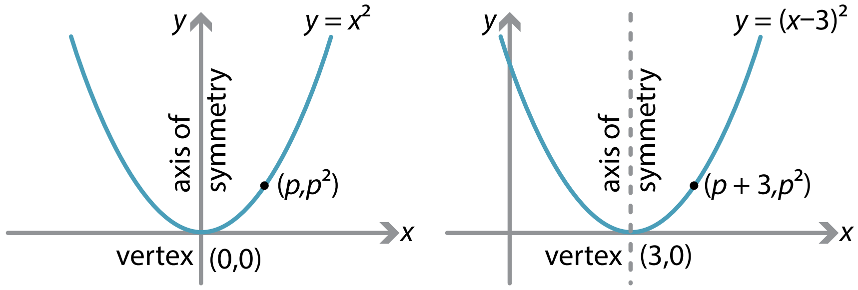
The standard parabola $y = x^2$ in shown on the left in the following diagram. We can describe each point on the parabola in terms of a **parameter** p; the corresponding point is (p, p^2).



If $y = x^2$ is translated 7 units up, we obtain $y = x^2 + 7$, which is also parametrised by p; the corresponding point is $(p, p^2 + 7)$.

Horizontal translations of parabolas

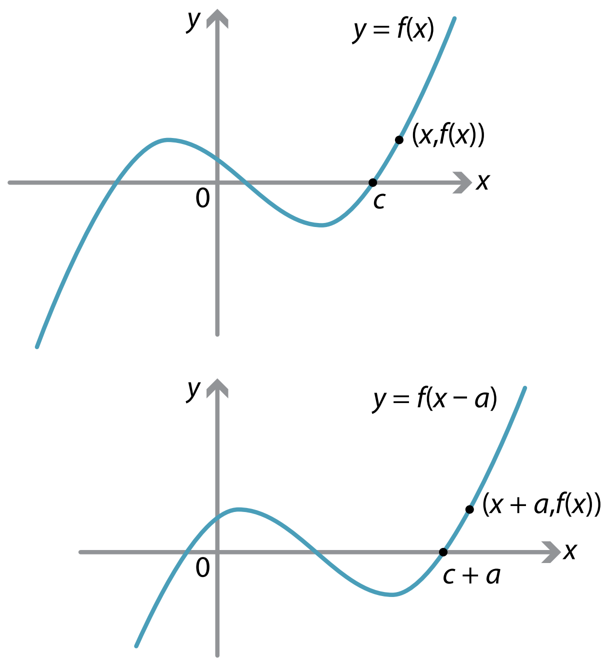
Suppose that the standard parabola $y = x^2$ is translated 3 units to the right. The vertex $(0,0)$ is moved to $(3,0)$, which is the vertex of the parabola $y = (x-3)^2$. Their graphs are as follows.



Clearly, the point (p, p^2) is moved to the point $(p+3, p^2)$. If $x = p+3$ and $y = p^2$, then $y = (x-3)^2$. In the same way, a translation 5 units to the left maps $y = x^2$ to $y = (x+5)^2$, with the point (p, p^2) moved to the point $(p-5, p^2)$.

Horizontal translations in general

Now we consider what happens to the graph of an arbitrary function $y = f(x)$ when it is translated a units to the right.



The translation moves the point (x, y) to the point $(x+a, y)$. We can view this translation as a mapping from the coordinate plane to itself, given by

$$(x, y) \mapsto (x', y'), \qquad \text{where} \quad x' = x + a \quad \text{and} \quad y' = y.$$

Here we have $x = x' - a$ and $y = y'$. So the graph $y = f(x)$ maps to $y' = f(x' - a)$. That is, $y = f(x)$ maps to $y = f(x - a)$.

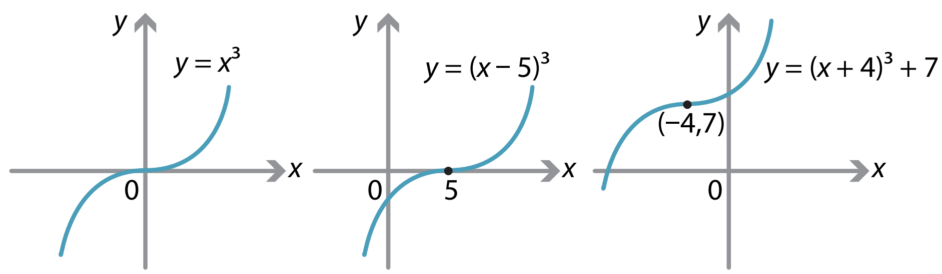
Example

Consider the graph of $y = x^3$.

1. If the graph is translated 5 units to the right, then what are the coordinates of the inflexion point and what is the equation of the image?
2. If the graph is translated 4 units to the left and 7 units up, then what are the coordinates of the inflexion point and what is the equation of the image?

Solution

1. The point of inflexion $(0,0)$ is translated to $(5,0)$. The equation of the image of the graph is $y = (x-5)^3$.
2. The point of inflexion $(0,0)$ is translated to $(-4,7)$. Translating the graph $y = x^3$ to the left by 4 units gives $y = (x+4)^3$. Then translating 7 units up yields $y = (x+4)^3 + 7$.



General translations

A general translation T of the plane is given by

$$T(x, y) = (x + a, y + b),$$

for some real numbers a, b. In particular, we have $T(0,0) = (a, b)$. Clearly, this mapping is the composite of a horizontal translation through a and a vertical translation through b. Thus, for example, it maps the graph of $y = x^2$ to the graph of $y = (x - a)^2 + b$.

Exercise 3

Find the translation which maps the graph of $y = x^2$ to $y = x^2 + 4x + 10$. Sketch the graph of $y = x^2 + 4x + 10$.

Summary for translations

The following table gives the image of the graph of $y = f(x)$ under various translations, with a and b positive constants.

Translations of the graph $y = f(x)$

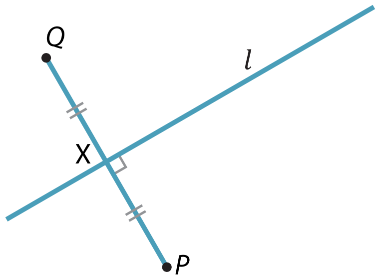
Translation	Image
a units up	$y = f(x) + a$
a units down	$y = f(x) - a$
b units to the right	$y = f(x - b)$
b units to the left	$y = f(x + b)$
a units up and b units to the right	$y = f(x - b) + a$
a units down and b units to the left	$y = f(x + b) - a$

We note that sometimes:

- 'up' is expressed as 'in the positive direction of the y-axis'
- 'down' is expressed as 'in the negative direction of the y-axis'
- 'to the right' is expressed as 'in the positive direction of the x-axis'
- 'to the left' is expressed as 'in the negative direction of the x-axis'.

Reflections

Assume we are given a line l in the plane. We can reflect any point P in the line l to obtain a point $Q = R_l(P)$, as shown in the following diagram.



The point Q is found by dropping a perpendicular from P to l, which meets l at X. We take Q on this perpendicular such that $PX = XQ$, with $Q \neq P$ unless P lies on l. This describes a mapping R_l from the plane $\mathbb{R}^2$ to itself. It is an **involution**, meaning that if you perform the mapping twice you get the identity.

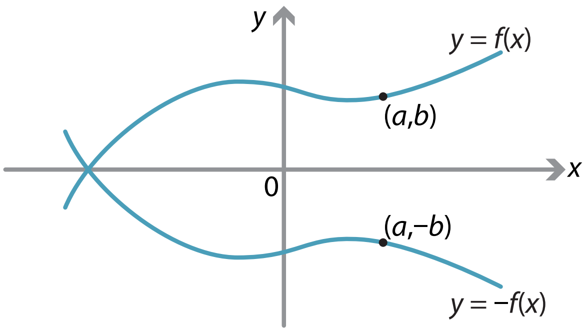
There are three important special cases:

- when l is the x-axis, $y = 0$
- when l is the y-axis, $x = 0$
- when l is the line $y = x$.

The third case is important when determining inverses of functions, and will be discussed in the next section of this module.

Reflection in the x-axis

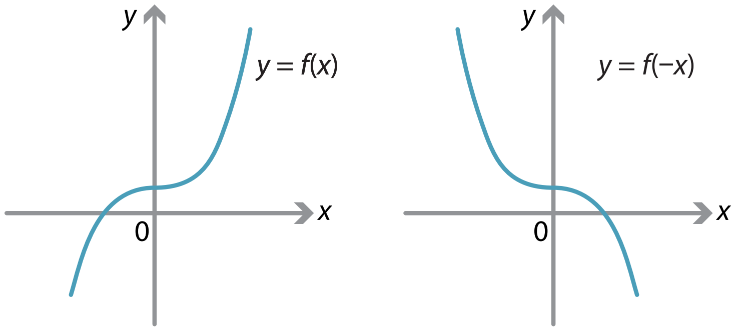
Reflection in the x-axis maps the point (a, b) to the point $(a, -b)$. Thus, if we consider the graph of $y = f(x)$ and reflect it in the x-axis, we obtain the graph of $y = -f(x)$.



For example, the standard parabola $y = x^2$ is reflected to the parabola $y = -x^2$. By applying suitable horizontal and vertical translations to $y = -x^2$, we can obtain any parabola of the form $y = -x^2 + ax + b$.

Reflection in the y-axis

Reflection in the y-axis maps the point (a, b) to the point $(-a, b)$. Thus, if we consider the graph of $y = f(x)$ and reflect it in the y-axis, we obtain the graph of $y = f(-x)$.



For example, the graph $y = x^2$ is mapped to itself, and the graph $y = x^3$ is mapped to $y = -x^3$.

Summary for reflections in an axis

We summarise reflections of a function $y = f(x)$ in the axes in the following table.

Reflections of the graph $y = f(x)$

Reflection	Image
Reflection in the x-axis	$y = -f(x)$
Reflection in the y-axis	$y = f(-x)$

Exercise 4

a Show that a function $f(x)$ is even if and only if the graph of $y = f(x)$ is mapped to itself by reflection in the y-axis.

b Show that a function $y = f(x)$ is odd if and only if the graph of $y = f(x)$ is mapped to itself by the composition of the reflections in the two coordinate axes.

Isometries

An **isometry** of the plane is a transformation which preserves distances between points. Isometries include rotations, translations, reflections, glide reflections and the identity map. In fact, it can be proved that every isometry can be obtained as a composition of at most three reflections.

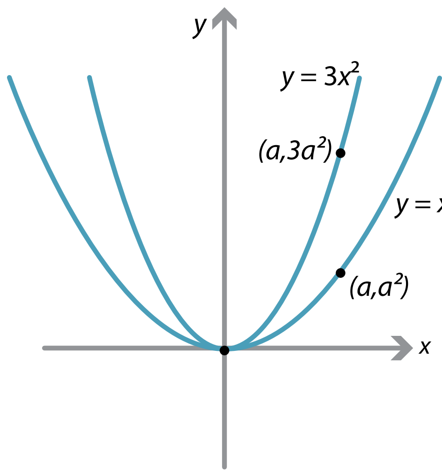
Isometries are sometimes also called **congruence transformations**. Two figures that can be transformed into each other by an isometry are said to be **congruent**.

So far in this section, we have looked at translations and reflections in the axes. Now we consider a family of transformations which are not isometries: the transformations which are 'stretches' from one of the axes.

Some transformations which are not isometries

Stretches from the x-axis

The mapping $(x, y) \mapsto (x, 3y)$ is called a **stretch from the x-axis**. Under this mapping, the point (a, a^2) on the graph of $y = x^2$ is sent to the point $(a, 3a^2)$ on the graph of $y = 3x^2$. This is shown in the following diagram.



We can describe this mapping as

$$(x, y) \mapsto (x', y'), \qquad \text{where} \quad x' = x \quad \text{and} \quad y' = 3y.$$

Thus $x = x'$ and $y = \dfrac{y'}{3}$, and so the graph of $y = x^2$ maps to $\dfrac{y'}{3} = (x')^2$. That is, $y = x^2$ maps to $y = 3x^2$.

Stretches from the y-axis

The mapping $(x, y) \mapsto (3x, y)$ is called a **stretch from the y-axis**. We can describe this mapping as

$$(x, y) \mapsto (x', y'), \qquad \text{where} \quad x' = 3x \quad \text{and} \quad y' = y.$$

Thus $x = \dfrac{x'}{3}$ and $y = y'$, and so the graph of $y = x^2$ maps to $y' = \left(\dfrac{x'}{3}\right)^2$. That is, $y = x^2$ maps to $y = \dfrac{x^2}{9}$.

Summary for stretches from the axes

We have seen that $(x, y) \mapsto (x, 3y)$ under a stretch from the x-axis by the factor 3. In general, the mapping $(x, y) \mapsto (x, ky)$ is the stretch from the x-axis by the factor k, where $k > 0$. Similarly, the mapping $(x, y) \mapsto (kx, y)$ is the stretch from the y-axis by the factor k, where $k > 0$. We summarise this in the following table.

Stretches of the graph $y = f(x)$

Stretch	Transformation	Image
Stretch from x-axis by the factor k, for $k > 0$	$(x, y) \mapsto (x, ky)$	$y = kf(x)$
Stretch from y-axis by the factor k, for $k > 0$	$(x, y) \mapsto (kx, y)$	$y = f\left(\frac{x}{k}\right)$

Note. The composition of these two stretches gives a transformation $(x, y) \mapsto (kx, ky)$. This is called a **dilation**. A dilation is an example of a similarity transformation.

Functions and their inverses

We begin with a simple example.

Example

Let $f(x) = 2x$ and $g(x) = \frac{x}{2}$.

Apply the function g to the number 3, and then apply f to the result:

$$g(3) = \frac{3}{2} \qquad \text{and} \qquad f\left(\frac{3}{2}\right) = 3.$$

A similar thing happens if we first apply f and then apply g:

$$f(3) = 6 \qquad \text{and} \qquad g(6) = 3.$$

It is clear that this will happen with any starting number. This is expressed as

$$f(g(x)) = x, \qquad \text{for all } x$$

$$g(f(x)) = x, \qquad \text{for all } x.$$

The function f reverses the effect of g, and the function g reverses the effect of f. We say that f and g are inverses of each other.

As another example, we have

$$(\sqrt[3]{x})^3 = x \qquad \text{and} \qquad \sqrt[3]{x^3} = x,$$

for all real x. So the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of each other.

If $x \geq 0$, then $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$. If $x < 0$, then $\sqrt{x}$ is not defined. So the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverses of each other, but we need to be careful about domains. We will look at this more carefully later in this section.

Basics

In an earlier section of this module, we defined the composite of two functions h and g by $(g \circ h)(x) = g(h(x))$.

Definitions

- The **zero function** $\underline{0}\colon \mathbb{R} \to \mathbb{R}$ is defined by $\underline{0}(x) = 0$, for all x.
- The **identity function** $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ is defined by $\mathrm{id}(x) = x$, for all x.

Example

Consider a function $f\colon \mathbb{R} \to \mathbb{R}$.

1 Prove that

  a $\underline{0} \circ f = \underline{0}$

  b $f \circ \mathrm{id} = f$

  c $\mathrm{id} \circ f = f$.

2 Show that $f \circ \underline{0}$ does not necessarily equal $\underline{0}$.

Solution

1 a We have $(\underline{0} \circ f)(x) = \underline{0}(f(x)) = 0$, for all x, and so $\underline{0} \circ f = \underline{0}$.

  b We have $(f \circ \mathrm{id})(x) = f(\mathrm{id}(x)) = f(x)$, for all x, and so $f \circ \mathrm{id} = f$.

  c We have $(\mathrm{id} \circ f)(x) = \mathrm{id}(f(x)) = f(x)$, for all x, and so $\mathrm{id} \circ f = f$.

2 Consider the function given by $f(x) = 2$, for all x. Then $f \circ \underline{0}(x) = f(\underline{0}(x)) = f(0) = 2$, and so $f \circ \underline{0} \neq \underline{0}$.

Definition

Let f be a function with both domain and range all real numbers. Then the function g is the **inverse** of f if

$$f(g(x)) = x, \quad \text{for all } x, \quad \text{and}$$

$$g(f(x)) = x, \quad \text{for all } x.$$

That is, $f \circ g = \text{id}$ and $g \circ f = \text{id}$.

Notes.

- Clearly, if g is the inverse of f, then f is the inverse of g.
- We denote the inverse of f by f^{-1}. We read f^{-1} as 'f inverse'. Note that f inverse has nothing to do with the function $\frac{1}{f}$.

Example

Let $f(x) = x + 2$ and let $g(x) = x - 2$. Show that f and g are inverses of each other.

Solution

We have

$$f(g(x)) = f(x-2) = x - 2 + 2 = x, \quad \text{for all } x \qquad (f \circ g = \text{id})$$

and

$$g(f(x)) = g(x+2) = x + 2 - 2 = x, \quad \text{for all } x \qquad (g \circ f = \text{id}).$$

Hence, the functions f and g are inverses of each other.

Exercise 5

Find the inverse of

a $f(x) = x + 7$

b $f(x) = 4x + 5$.

Example

Let $f(x) = ax + b$ with $a \neq 0$. Find the inverse of f.

Solution

We have $x = \dfrac{f(x) - b}{a}$, for all x. So let $g(x) = \dfrac{x-b}{a}$. Then

$$f(g(x)) = f\Big(\frac{x-b}{a}\Big) = a\Big(\frac{x-b}{a}\Big) + b = x$$

$$g(f(x)) = g(ax+b) = \frac{(ax+b)-b}{a} = x,$$

for all x. Hence, g is the inverse of f.

Exercise 6

a Show that $f(x) = x^5$ and $g(x) = x^{\frac{1}{5}}$ are inverses of each other.

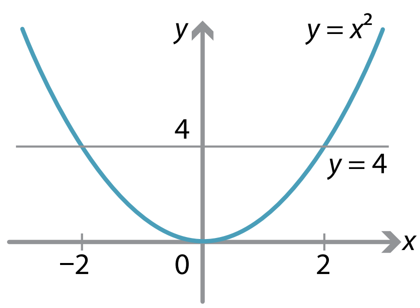
b Find the inverse of $f(x) = x^3 + 2$.

We do not yet have a general enough concept of inverses, since x^2 and $\sqrt{x}$ do not fit into this framework, nor do e^x and $\log_e x$. We will give a definition that covers these functions later in this section.

The horizontal-line test

Consider the function $f(x) = x^2$, which has domain the reals and range $A = \{x : x \geq 0\}$. Does f have an inverse?

The following graph shows that it does not. We have $f(-2) = f(2) = 4$, and so $f^{-1}(4)$ would have to take two values, -2 and 2! Hence, f does not have an inverse.



This idea can be formulated as a test.

Horizontal-line test

Let f be a function. If there is a horizontal line $y = c$ that meets the graph $y = f(x)$ at more than one point, then f does not have an inverse.

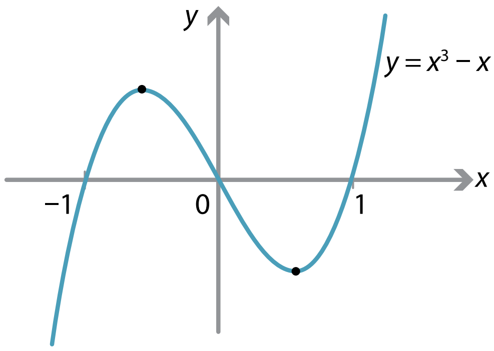
Note. Remember that the vertical-line test determines whether a relation is a function.

Example

Consider the function

$$f(x) = x^3 - x = (x+1)x(x-1).$$

Its graph is shown in the following diagram.



Does f have an inverse?

Solution

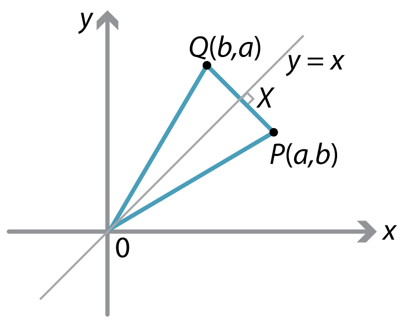
The line $y = 0$ meets the graph at three points. By the horizontal-line test, the function f does not have an inverse.

Finding inverses

Suppose that $f(x)$ and $g(x)$ are inverse functions. Then $f(a) = b$ if and only if $g(b) = a$. So (a, b) is a point on the graph $y = f(x)$ if and only if (b, a) is a point on the graph $y = g(x)$. The two points (a, b) and (b, a) are closely related, as the next theorem shows.

Theorem

The points $P(a, b)$ and $Q(b, a)$ are reflections of each other in the line $y = x$.



Proof

Let X be the point where PQ meets the line $y = x$. Then

$$\text{Gradient of } PQ = \frac{b-a}{a-b} = -1, \qquad \text{Gradient of } OX = 1,$$

and so PQ is perpendicular to OX. We also have $OP^2 = a^2 + b^2 = OQ^2$, giving $OP = OQ$.

It now follows that $\triangle OXP \equiv \triangle OXQ$ (RHS), since OX is a common side, $OP = OQ$ and $OXP = 90° = OXQ$. Hence, $XP = XQ$.

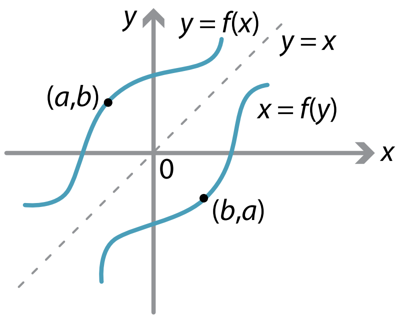
We have shown that $PQ \perp OX$ and $XP = XQ$. So Q is the reflection of P in the line $y = x$, as required. □

Note. An alternative way to show that $PX = QX$ in this proof is by using the formula for the distance of a point from a line (see the module *Coordinate geometry*). The formula gives $PX = \dfrac{|b-a|}{\sqrt{2}} = QX$.

Corollary

Assume that f has an inverse. Then the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$.

We have seen that reflection in the line $y = x$ interchanges the points (a, b) and (b, a). So a function $y = f(x)$ is reflected to the relation $x = f(y)$. This is illustrated in the following diagram. A point (a, b) on the graph $y = f(x)$ is mapped to the point (b, a), which lies on $x = f(y)$.



This gives us a method for finding the inverse of $y = f(x)$ in some circumstances: Write down $x = f(y)$ and then, if possible, make y the subject of this new equation.

For example, suppose $y = 4x + 7$. Swapping x and y yields

$$\begin{aligned} x &= 4y + 7 \\ 4y &= x - 7 \\ y &= \frac{x-7}{4}. \end{aligned}$$

Check that the functions $f(x) = 4x + 7$ and $g(x) = \dfrac{x-7}{4}$ are inverses of each other.

As a second example, suppose $y = x^3 + 2$. Swapping x and y yields

$$\begin{aligned} x &= y^3 + 2 \\ y^3 &= x - 2 \\ y &= \sqrt[3]{x-2}. \end{aligned}$$

Exercise 7

Consider the function $f(x) = 3x + 2$.

a Find the inverse of f.

b Show that the graph $y = f(x)$ meets the graph $y = f^{-1}(x)$ on the line $y = x$.

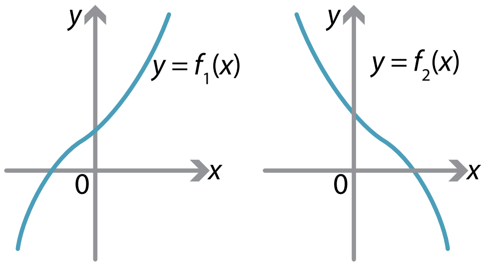
c Draw a sketch showing $y = f(x)$ and $y = f^{-1}(x)$.

Increasing and decreasing functions

Consider two functions f_1 and f_2 with domain $\mathbb{R}$ and range $\mathbb{R}$. Assume that:

- f_1 is **strictly increasing**, that is, $a < b \implies f_1(a) < f_1(b)$
- f_2 is **strictly decreasing**, that is, $a < b \implies f_2(a) > f_2(b)$.

Then both f_1 and f_2 satisfy the horizontal-line test, and so they both have inverses. For example, the functions f_1 and f_2 could be those shown in the following diagram.



Notes.

1 In the module *Applications of differentiation*, we see that:

- If $f'(x) > 0$, for all x, then the function f is strictly increasing.
- If $f'(x) < 0$, for all x, then the function f is strictly decreasing.

All such functions will satisfy the horizontal-line test.

2 It is easy to write down examples where it is not possible to give a formula for the inverse. For example, suppose

$$f(x) = x^5 + 3x^3 + 7x - 8.$$

Then

$$f'(x) = 5x^4 + 9x^2 + 7 \geq 7, \qquad \text{for all } x.$$

Thus f is a strictly increasing function, and hence f has an inverse function. The graph of f is $y = x^5 + 3x^3 + 7x - 8$. If we want to find the inverse of f, we first interchange x and y. This gives

$$x = y^5 + 3y^3 + 7y - 8.$$

From the look of this equation, it does not seem possible to make y the subject, and indeed this the case. (There is a discussion of the solution of polynomial equations in the module *Polynomials*.) So we see that it is not always possible to explicitly write down a formula for the inverse, even when it exists.

Restricted domains and ranges

To deal with some of the simplest and most important examples, we have to generalise the idea of an inverse function by restricting the domain or the range or both.

For any set S, we shall use id_S to denote the identity function on S. That is, the function $\mathrm{id}_S\colon S \to S$ is given by $\mathrm{id}_S(x) = x$, for all $x \in S$.

Example

Define the set

$$A = \{x \in \mathbb{R} : x \geq 0\}.$$

Then A consists of all the positive reals together with 0. Now define the functions:

$$f\colon A \to A, \quad f(x) = x^2 \qquad \text{and} \qquad g\colon A \to A, \quad g(x) = \sqrt{x}.$$

Note that $\mathrm{domain}(f) = \mathrm{range}(f) = A$ and $\mathrm{domain}(g) = \mathrm{range}(g) = A$. We have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, \qquad \text{for all } x \in A,$$

and so $f \circ g = \mathrm{id}_A$. Similarly, we can check that $g \circ f = \mathrm{id}_A$. It is natural to say

$$g = f^{-1} \qquad \text{and} \qquad f = g^{-1}.$$

Building on the previous example, we can also define functions $h\colon A \to A$ and $j\colon A \to A$ by $h(x) = x^4$ and $j(x) = x^{\frac{1}{4}}$, and we can show that $h \circ j = j \circ h = \mathrm{id}_A$. We say that $j = h^{-1}$ and $h = j^{-1}$.

There is no reason why this idea cannot be generalised to functions whose domain and range are not necessarily the same. We first give a definition which allows us to work in this situation.

Definition

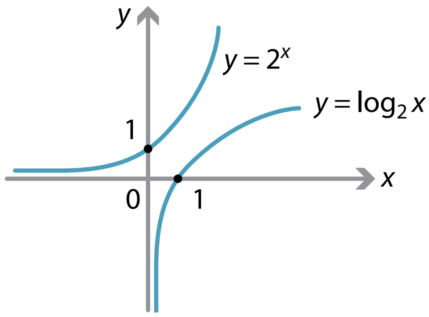
Let f be a function with domain X and range Y. Then a function g with domain Y and range X such that

$$f \circ g = \mathrm{id}_Y \qquad \text{and} \qquad g \circ f = \mathrm{id}_X$$

is the **inverse** of f. Clearly, if g is the inverse of f, then f is the inverse of g. We write $g = f^{-1}$ and $f = g^{-1}$.

Example

Let $f\colon \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = 2^x$, and let $g\colon \mathbb{R}^+ \to \mathbb{R}$ be defined by $g(x) = \log_2 x$.



Then

$$(f \circ g)(x) = f(g(x)) = f(\log_2 x) = 2^{\log_2 x} = x, \qquad \text{for all } x > 0.$$

That is, $f \circ g = i_{\mathbb{R}^+}$. Also,

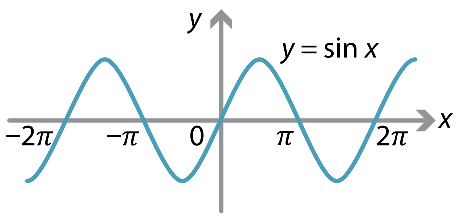
$$(g \circ f)(x) = g(f(x)) = g(2^x) = \log_2(2^x) = x, \qquad \text{for all } x \in \mathbb{R}.$$

That is, $g \circ f = i_{\mathbb{R}}$. So $f = g^{-1}$ and $g = f^{-1}$.

Trigonometric functions

A scientific calculator reports that $\sin^{-1}(0.5) = \frac{\pi}{6}$. However, $\sin(\frac{5\pi}{6}) = 0.5$ as well. So why does $\sin^{-1}(0.5)$ not equal $\frac{5\pi}{6}$?

We start by considering the graph of $y = \sin x$.

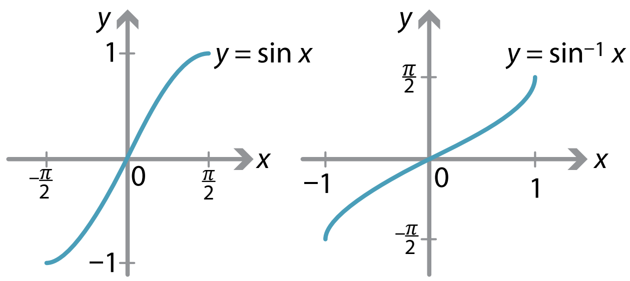


This function spectacularly fails the horizontal-line test: whenever $-1 \leq c \leq 1$, the line $y = c$ meets the graph $y = \sin x$ infinitely often!

To obtain an inverse function, we must restrict the domain of the function $\sin x$ so that, for each c with $-1 \leq c \leq 1$, there is exactly one value of x with $\sin x = c$. Choice is involved, but clearly one would want to include the first quadrant $[0, \frac{\pi}{2}]$ in the domain, and to choose an interval if possible.

The natural choice is to define $y = \sin x$, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. This function has an inverse, called $\sin^{-1} x$, with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We have:

- $\sin(\sin^{-1} x) = x$, for all x with $-1 \le x \le 1$
- $\sin^{-1}(\sin x) = x$, for all x with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.



In summary: $\sin^{-1} a = b$ if and only if $\sin b = a$, for $a \in [-1, 1]$ and $b \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise 8

a Work out how to restrict the domain of $\cos x$ and how to define $\cos^{-1} x$.

b Repeat for $\tan x$.

Links forward

Differentiation

In this module, we have seen various ways of forming new functions. How to differentiate sums, differences, products, quotients, compositions and inverses is discussed in the module *Introduction to differential calculus*. We repeat them here for convenience.

Theorem

Let f, g be differentiable functions. Then the derivative of $f(x) + g(x)$ is $f'(x) + g'(x)$, and the derivative of $f(x) - g(x)$ is $f'(x) - g'(x)$. That is,

$$\frac{d}{dx}\big[f(x) + g(x)\big] = f'(x) + g'(x), \qquad \frac{d}{dx}\big[f(x) - g(x)\big] = f'(x) - g'(x).$$

Theorem (Product rule)

Let f, g be differentiable functions. Then the derivative of their product is

$$\frac{d}{dx}\big[f(x)\,g(x)\big] = f(x)\,g'(x) + g(x)\,f'(x).$$

Theorem (Quotient rule)

Let f, g be differentiable functions. Then the derivative of their quotient is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\big(g(x)\big)^2}.$$

Theorem (Chain rule)

Let f, g be differentiable functions. Then the derivative of their composition is

$$\frac{d}{dx}\big[f \circ g(x)\big] = \frac{d}{dx}\big[f(g(x))\big] = f'(g(x))\,g'(x).$$

Inverses

The derivative of an inverse function in terms of the original function is presented in the module *Introduction to differential calculus* through the use of the chain rule. We give a direct proof here and an example.

Theorem

Let $y = f(x)$ be a strictly increasing, differentiable function on the interval (a,b). Then the inverse function $x = g(y)$ exists, and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}.$$

Proof

As f is strictly increasing, the inverse function $x = g(y)$ exists. Consider

$$\frac{g(y+k) - g(y)}{k}.$$

Define

$$h = g(y+k) - g(y).$$

We are assuming f is differentiable, and so it is continuous. Thus its inverse g is continuous. This implies that, as $k \to 0$, $g(y+k) \to g(y)$ and so $h \to 0$.

Since $g(y+k) = x + h$, we obtain $y + k = f(x+h)$ and therefore $k = f(x+h) - f(x)$. We now have

$$\frac{g(y+k) - g(y)}{k} = \frac{h}{f(x+h) - f(x)},$$

and so it follows that

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}.$$ □

The result also holds if the function f is strictly decreasing.

Example

Let $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be given by $f(x) = \sin x$. Find the derivative of f^{-1}.

(This result is also established in the module *The calculus of trigonometric functions*. Here we derive the result using the notation of the previous theorem.)

Solution

We first note that f is a strictly increasing function on this interval, and therefore the inverse exists. Let $y = f(x)$ and let g be the inverse function of f. Then $x = g(y)$. By the previous theorem, we have

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)} \\ &= \frac{1}{\cos x} \\ &= \frac{1}{\cos(g(y))} \\ &= \frac{1}{\cos(\sin^{-1} y)}. \end{aligned}$$

Since $\sin^{-1} y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we must have $\cos(\sin^{-1} y) > 0$. So we can use the Pythagorean identity to obtain

$$\begin{aligned} g'(y) &= \frac{1}{\sqrt{\cos^2(\sin^{-1} y)}} \\ &= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} y)}} \\ &= \frac{1}{\sqrt{1 - y^2}}. \end{aligned}$$

Definite integrals

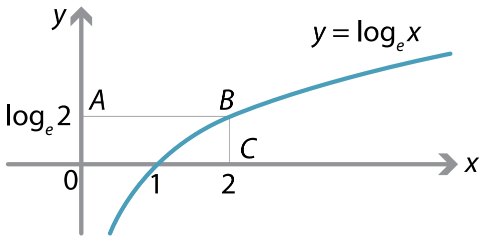
We illustrate the use of inverses in determining definite integrals through an example.

Example

Find

$$\int_1^2 \log_e x \, dx.$$

Solution



We note that e^x and $\log_e x$ are inverses of each other. From the diagram, we have

$$\begin{aligned}\int_1^2 \log_e x \, dx &= \text{Area of rectangle } OABC - \int_0^{\log_e 2} e^y \, dy \\ &= 2\log_e 2 - \left[e^y\right]_0^{\log_e 2} \\ &= 2\log_e 2 - 1.\end{aligned}$$

It is clear that this technique can be used profitably in many similar situations.

Transformations of the plane

Composition of transformations

The geometrical transformations of the plane we have discussed in this module are functions. For any of these transformations T, we have

$$T : \mathbb{R}^2 \to \mathbb{R}^2.$$

For example, the translation 3 units to the right and 2 units down is defined by

$$T_1(x, y) = (x+3, y-2).$$

The stretch from the y-axis by a factor of 3 is the transformation defined by

$$T_2(x, y) = (3x, y).$$

The reflection in the x-axis is given by

$$R_x(x, y) = (x, -y).$$

We now consider compositions of these transformations. For example, we have

$$T_1 \circ T_2(x, y) = T_1(T_2(x, y)) = T_1(3x, y) = (3x+3, y-2).$$

Taking them in the opposite order, we obtain

$$T_2 \circ T_1(x, y) = T_2(T_1(x, y)) = T_2(x+3, y-2) = (3x+9, y-2).$$

As expected, we did not get the same result. In general, composition of transformations of the plane is not commutative.

On the other hand, composition of *translations* is commutative. This is easily shown, as follows. Let S and T be two translations, defined by

$$S(x, y) = (x+a, y+b) \qquad \text{and} \qquad T(x, y) = (x+c, y+d).$$

Then

$$S \circ T(x, y) = S(T(x, y)) = S(x+c, y+d) = (x+c+a, y+d+b)$$

and

$$T \circ S(x, y) = T(S(x, y)) = T(x+a, y+b) = (x+a+c, y+b+d).$$

Hence, $S \circ T = T \circ S$. (Note that two transformations of the plane T_1 and T_2 are equal if $T_1(x, y) = T_2(x, y)$, for all $(x, y) \in \mathbb{R}^2$.)

Inverses of transformations

Again, consider the translation 3 units to the right and 2 units down, which is defined by

$$T_1(x, y) = (x+3, y-2).$$

We will find the inverse transformation. To do this, assume that there is a transformation T_4, with $T_4(x, y) = (x', y')$, such that $T_1 \circ T_4(x, y) = (x, y)$. We have

$$T_1(T_4(x, y)) = T_1(x', y') = (x'+3, y'-2),$$

and therefore $x'+3 = x$ and $y'-2 = y$. This gives

$$T_4(x, y) = (x', y') = (x-3, y+2).$$

We can now check that $T_4 \circ T_1(x, y) = (x, y)$ and $T_1 \circ T_4(x, y) = (x, y)$, for all $(x, y) \in \mathbb{R}^2$. So T_4 is the inverse of the transformation T_1. That is, $T_4 = T_1^{-1}$.

Next consider the reflection in the x-axis, given by

$$R_x(x, y) = (x, -y).$$

We note that $R_x \circ R_x(x, y) = R_x(x, -y) = (x, -(-y)) = (x, y)$. This means that R_x is the inverse of itself. Earlier, we called this an *involution*.

Groups of isometries

Recall that an isometry is a distance-preserving map from the plane $\mathbb{R}^2$ to itself.

Just as we have the identity function id on $\mathbb{R}$, we have the identity transformation I on $\mathbb{R}^2$, given by

$$I(x, y) = (x, y).$$

This transformation is clearly an isometry.

Now let E be the set of all isometries of the plane. We make the following observations:

- **Closure.** For all $S, T \in E$, we have $S \circ T \in E$.
- **Associativity.** For all $R, S, T \in E$, we have $(R \circ S) \circ T = R \circ (S \circ T)$.
- **Identity.** There is $I \in E$ such that, for all $T \in E$, we have $T \circ I = I \circ T = T$.
- **Inverses.** For each $T \in E$, there is an element $T^{-1} \in E$ such that $T \circ T^{-1} = T^{-1} \circ T = I$.

These are the four defining properties of a **group**. Note again that the commutative property does not necessarily hold.

Groups were first studied in depth by Galois when considering solutions of equations. Group theory provides a powerful tool for studying symmetries, and so groups are used in areas such as physics, chemistry and encryption.

Answers to exercises

Exercise 1

a Assume that $f : \mathbb{R} \to \mathbb{R}$ is both even and odd. Let $x \in \mathbb{R}$. Since f is even, we have $f(-x) = f(x)$, and since f is odd, we have $f(-x) = -f(x)$. Therefore $f(x) = -f(x)$, which implies that $f(x) = 0$. So f is constant zero.

b Consider a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. We will assume that n is even. (The case that n is odd is similar.) Take

$$g(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_n x^n$$

$$h(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{n-1} x^{n-1}.$$

Then $f(x) = g(x) + h(x)$, with $g(x)$ an even function and $h(x)$ an odd function.

c Suppose

$$f(x) = g(x) + h(x) \qquad (1)$$

such that $g(x)$ is odd and $h(x)$ is even. Then

$$\begin{aligned} f(-x) &= g(-x) + h(-x) \\ &= -g(x) + h(x). \qquad (2) \end{aligned}$$

Adding equations (1) and (2) gives $f(x) + f(-x) = 2h(x)$, and we obtain

$$h(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad g(x) = \frac{f(x) - f(-x)}{2}.$$

This gives us the uniqueness of $g(x)$ and $h(x)$. We can easily check that the function $g(x)$ defined this way actually is an odd function, and similarly that $h(x)$ actually is an even function.

Exercise 2

a If f_1 and f_2 are odd functions, then

$$(f_1 + f_2)(-x) = f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) = -(f_1 + f_2)(x),$$

so $f_1 + f_2$ is odd. If g_1 and g_2 are even functions, then

$$(g_1 + g_2)(-x) = g_1(-x) + g_2(-x) = g_1(x) + g_2(x) = (g_1 + g_2)(x),$$

so $g_1 + g_2$ is even.

b Suppose f_1 and f_2 are even, and g_1 and g_2 are odd. If $h = f_1 f_2$, then

$$h(-x) = f_1(-x)\, f_2(-x) = f_1(x)\, f_2(x) = h(x),$$

so h is even. If $j = g_1 g_2$, then

$$j(-x) = g_1(-x)\, g_2(-x) = -g_1(x) \times -g_2(x) = g_1(x)\, g_2(x) = j(x),$$

so j is even. If $k = f_1 g_1$, then

$$k(-x) = f_1(-x)\, g_1(-x) = f_1(x) \times -g_1(x) = -f_1(x)\, g_1(x) = -k(x),$$

so k is odd.

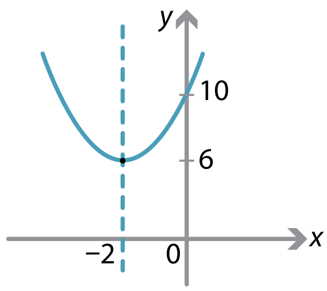
c Let $f\colon D \to \mathbb{R}$ with $f(a) \neq 0$, for some $a \in D$, and let $g\colon D \to \mathbb{R}$ with $g(b) \neq 0$, for some $b \in D$. Assume that f is odd and g is even. Then

$$\begin{aligned} (f+g)(-a) &= f(-a) + g(-a) \\ &= -f(a) + g(a) && \text{since } f \text{ is odd and } g \text{ is even} \\ &\neq f(a) + g(a) && \text{since } f(a) \neq 0 \\ &= (f+g)(a). \end{aligned}$$

So $f + g$ is not even. Similarly, we can use b to show that $f + g$ is not odd.

Exercise 3

Let $y = x^2 + 4x + 10 = (x+2)^2 + 6$. To map $y = x^2$ to $y = (x+2)^2 + 6$, translate it 2 units to the left and 6 units up.



Exercise 4

a First assume that f is even. Then $f(-x) = f(x)$, for all x in the domain. So reflection in the y-axis maps $y = f(x)$ to $y = f(-x) = f(x)$.

For the converse, assume that $y = f(x)$ is mapped to itself under reflection in the y-axis. Then $y = f(-x)$ is $y = f(x)$. Thus $f(-x) = f(x)$, for all x in the domain of f, and so f is even.

b Note that $y = f(x)$ is mapped to $y = -f(x)$ by reflection in the x-axis, which is then mapped to $y = -f(-x)$ by reflection in the y-axis. If the resulting graph is $y = f(x)$, then $f(x) = -f(-x)$, for all x in the domain of f, and so f is odd. Check the converse.

Exercise 5

a Let $g(x) = x - 7$. Then

$$f(g(x)) = f(x-7) = x - 7 + 7 = x$$

$$g(f(x)) = g(x+7) = x + 7 - 7 = x,$$

for all x. Hence, f and g are inverses of each other.

b Let $g(x) = \dfrac{x-5}{4}$. Then

$$f(g(x)) = f\Big(\frac{x-5}{4}\Big) = 4 \times \Big(\frac{x-5}{4}\Big) + 5 = x$$

$$g(f(x)) = g(4x+5) = \frac{4x+5-5}{4} = x.$$

Thus f and g are inverses of each other.

Exercise 6

a Note that domain(f) = range$(f) = \mathbb{R}$ and domain(g) = range$(g) = \mathbb{R}$. We have

$$(f \circ g)(x) = f(g(x)) = f(x^{\frac{1}{5}}) = (x^{\frac{1}{5}})^5 = x$$

$$(g \circ f)(x) = g(f(x)) = g(x^5) = (x^5)^{\frac{1}{5}} = x,$$

for all $x \in \mathbb{R}$. Thus $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, as required.

b Let $f(x) = x^3 + 2$. We have $x^3 = f(x) - 2$ and so $x = \sqrt[3]{f(x) - 2}$. Take $g(x) = \sqrt[3]{x - 2}$. This gives $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, as required.

Exercise 7

The function $f(x) = 3x + 2$ has graph $y = 3x + 2$. To find the inverse, we swap x and y. This gives

$$x = 3y + 2$$

$$y = \frac{x - 2}{3}.$$

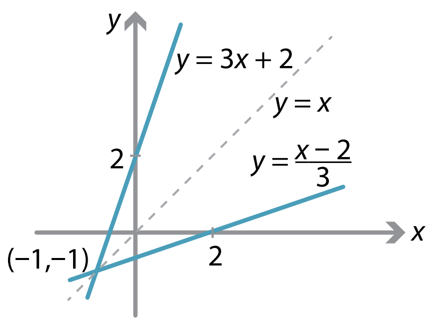
Thus the inverse is given by $f^{-1}(x) = \dfrac{x - 2}{3}$. To find where the two lines meet, we solve $f(x) = f^{-1}(x)$. This gives

$$3x + 2 = \frac{x - 2}{3}$$

$$9x + 6 = x - 2$$

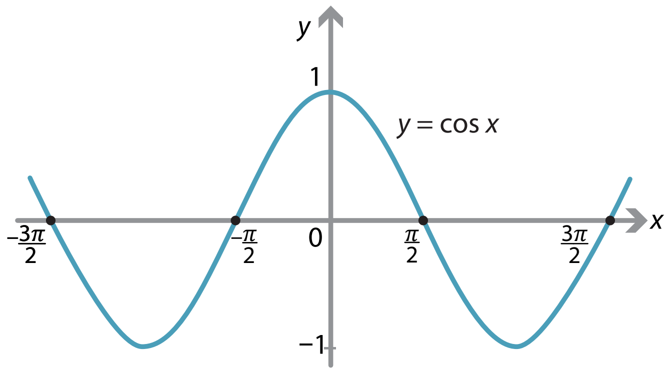
$$x = -1.$$

Substituting $x = -1$ into $y = 3x + 2$ yields $y = -1$. Hence, they meet on the line $y = x$.



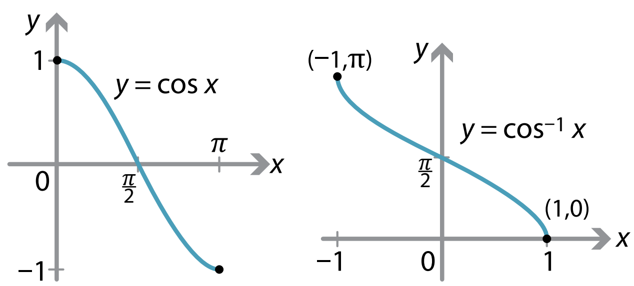
Exercise 8

a Consider the graph of $y = \cos x$.

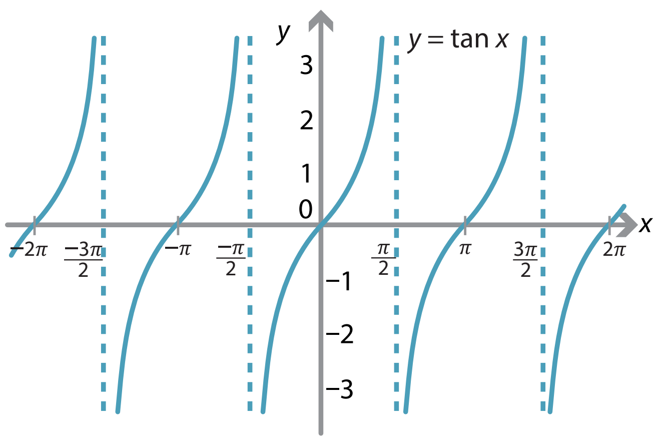


We must restrict the domain of $\cos x$ so that, for each c with $-1 \leq c \leq 1$, there is exactly one value of x such that $\cos x = c$. We restrict the domain of $\cos x$ to the interval $[0, \pi]$. This gives us a strictly decreasing function with domain $[0, \pi]$ and range $[-1, 1]$. The inverse function $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$.

So $\cos^{-1} a = b$ if and only if $\cos b = a$, for $a \in [-1, 1]$, $b \in [0, \pi]$.

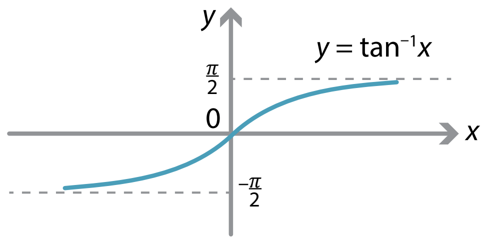


b Consider the graph of $y = \tan x$.



We restrict the domain of $\tan x$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. This gives us a strictly increasing function with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and range $\mathbb{R}$. The inverse function $\tan^{-1} x$ has domain $\mathbb{R}$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

So $\tan^{-1} a = b$ if and only if $\tan b = a$, for $a \in \mathbb{R}$, $b \in (-\frac{\pi}{2}, \frac{\pi}{2})$.



0 1 2 3 4 5 6 7 8 9 10 11 12