Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Functions: Module 4

# Quadratics

Education Services Australia

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

*Quadratics – A guide for teachers (Years 11–12)*

Principal author: Peter Brown, University of NSW

Dr Michael Evans, AMSI
Associate Professor David Hunt, University of NSW
Dr Daniel Mathews, Monash University

Editor: Dr Jane Pitkethly, La Trobe University

Illustrations and web design: Catherine Tan, Michael Shaw

Full bibliographic details are available from Education Services Australia.

Published by Education Services Australia
PO Box 177
Carlton South Vic 3053
Australia

Tel: (03) 9207 9600
Fax: (03) 9910 9800
Email: info@esa.edu.au
Website: www.esa.edu.au



This publication is funded by the Australian Government Department of Education, Employment and Workplace Relations.

Supporting Australian Mathematics Project

Australian Mathematical Sciences Institute
Building 161
The University of Melbourne
VIC 3010
Email: enquiries@amsi.org.au
Website: www.amsi.org.au

# Quadratics

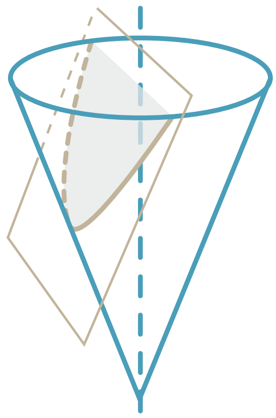
## Assumed knowledge

- The content of the module *Quadratic equations* (Years 9–10).
- The content of the module *Algebra review*, which includes factoring quadratic expressions, completing the square and solving quadratic equations using various techniques including the quadratic formula.
- The content of the module *Coordinate geometry*.

## Motivation

Quadratic equations arose in antiquity in the context of problem solving. These problems were generally expressed verbally, but their solution involved some method for solving a quadratic equation — often also expressed verbally.

The Greeks studied the parabola and its geometry. This curve occurs naturally as the edge of the cross-section of a cone sliced at an angle to its main axis.



The parabola can also be thought of as the graph of the quadratic function. It models the path taken by a ball thrown under gravity at an angle to the horizontal, assuming we ignore air resistance.

Since the Greeks did not have the notion of a coordinate system, their studies were heavily geometric and often hard to understand. With the advent of coordinate geometry, the study of the parabola became much easier and a full and complete analysis became possible.

In this module, we will summarise all of the main features and properties of quadratic equations and functions.

We will also look at a geometric definition of the parabola and its representation using parametric equations.

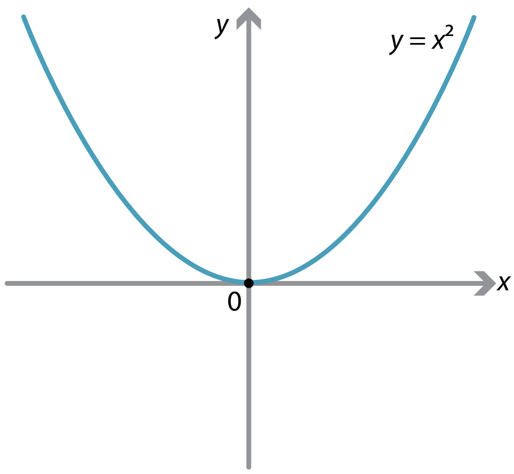
One of the many interesting characteristics of the parabola is its reflective property, used in collecting light from a distant source.

It is not intended that all the material in this module be taught in one topic in the classroom. Our aim here is to present the teacher with a comprehensive overview of quadratic functions and the geometry of the parabola.

# Content

## The basic parabola

The graph of the quadratic function $y = x^2$ is called a **parabola**. The basic shape can be seen by plotting a few points with $x = -3, -2, -1, 0, 1, 2$ and $3$.




We will refer to the above graph as the **basic parabola**. As we will see, all parabolas can be obtained from this one by translations, rotations, reflections and stretching.

The basic parabola has the following properties:

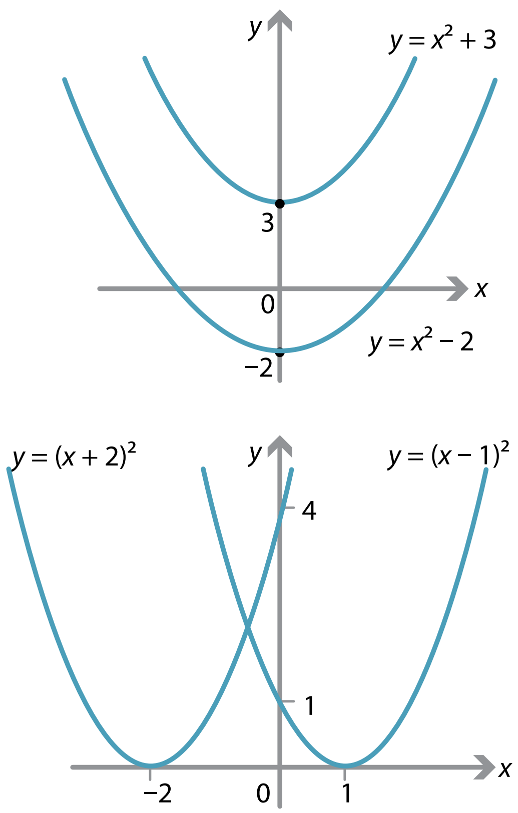
- It is symmetric about the $y$-axis, which is an **axis of symmetry**.
- The minimum value of $y$ occurs at the origin, which is a minimum turning point. It is also known as the **vertex** of the parabola.
- The arms of the parabola continue indefinitely.

## Transformations of the parabola

### Translations

We can translate the parabola vertically to produce a new parabola that is similar to the basic parabola. The function $y = x^2 + b$ has a graph which simply looks like the standard parabola with the vertex shifted $b$ units along the $y$-axis. Thus the vertex is located at $(0, b)$. If $b$ is positive, then the parabola moves upwards and, if $b$ is negative, it moves downwards.

Similarly, we can translate the parabola horizontally. The function $y = (x - a)^2$ has a graph which looks like the standard parabola with the vertex shifted $a$ units along the $x$-axis. The vertex is then located at $(a, 0)$. Notice that, if $a$ is positive, we shift to the right and, if $a$ is negative, we shift to the left.




These two transformations can be combined to produce a parabola which is congruent to the basic parabola, but with vertex at $(a, b)$.

For example, the parabola $y = (x-3)^2 + 4$ has its vertex at $(3, 4)$ and its axis of symmetry has the equation $x = 3$.

In the module *Algebra review*, we revised the very important technique of completing the square. This method can now be applied to quadratics of the form $y = x^2 + qx + r$, which are congruent to the basic parabola, in order to find their vertex and sketch them quickly.

## Example

Find the vertex of $y = x^2 - 4x + 8$ and sketch its graph.

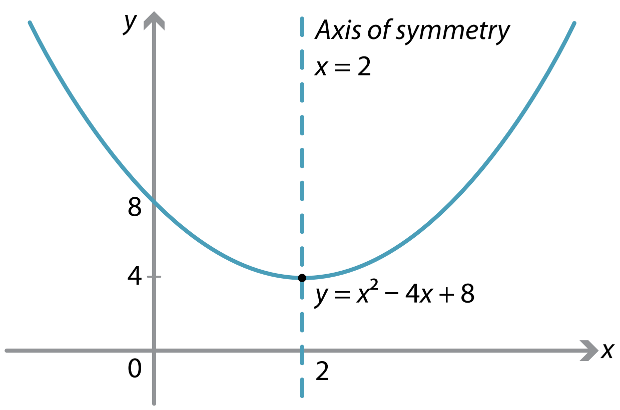
### Solution

Completing the square, we have

$$\begin{aligned} y &= x^2 - 4x + 8 \\ &= (x-2)^2 + 4. \end{aligned}$$

Hence the vertex is $(2, 4)$ and the axis of symmetry has equation $x = 2$.

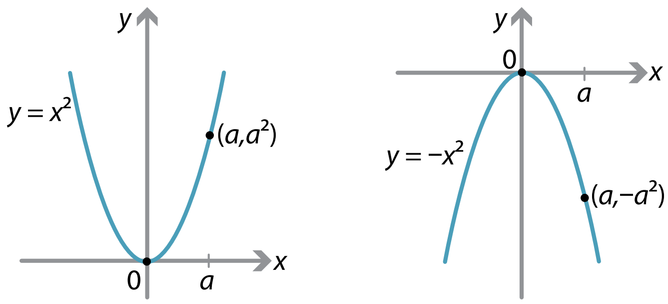
The graph is shown below.




We can, of course, find the vertex of a parabola using calculus, since the derivative will be zero at the $x$-coordinate of the vertex. The $y$-coordinate must still be found, and so completing the square is generally much quicker.

## Reflections

Parabolas can also be reflected in the $x$-axis. Thus the parabola $y = -x^2$ is a reflection of the basic parabola in the $x$-axis.




### Example

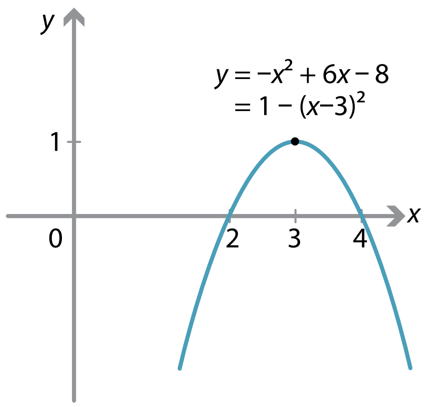
Find the vertex of $y = -x^2 + 6x - 8$ and sketch its graph.

#### Solution

Completing the square, we have

$$\begin{aligned} y &= -\big(x^2 - 6x + 8\big) \\ &= -\big((x-3)^2 - 1\big) = 1 - (x-3)^2. \end{aligned}$$

Hence the vertex is $(3, 1)$ and the parabola is 'upside down'. The equation of the axis of symmetry is $x = 3$. The graph is shown below.



### Exercise 1

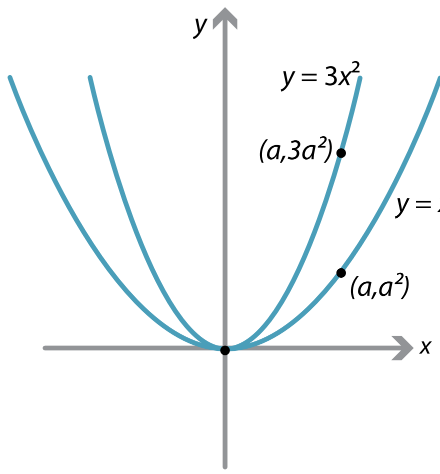
Find the vertex of each parabola and sketch it.

**a** $y = x^2 + x + 1$

**b** $y = -x^2 - 6x - 13$

## Stretching

Not all parabolas are congruent to the basic parabola. For example, the arms of the parabola $y = 3x^2$ are steeper than those of the basic parabola. The $y$-value of each point on this parabola is three times the $y$-value of the point on the basic parabola with the same $x$-value, as you can see in the following diagram. Hence the graph has been stretched in the $y$-direction by a factor of 3.




In fact, there is a similarity transformation that takes the graph of $y = x^2$ to the graph of $y = 3x^2$. (Map the point $(x, y)$ to the point $(\frac{1}{3}x, \frac{1}{3}y)$.) Thus, the parabola $y = 3x^2$ is similar to the basic parabola.

In general, the parabola $y = ax^2$ is obtained from the basic parabola $y = x^2$ by stretching it in the $y$-direction, away from the $x$-axis, by a factor of $a$.

### Exercise 2

Sketch the graphs of $y = \frac{1}{2}x^2$ and $y = x^2$ on the same diagram and describe the relationship between them.

This transformation of stretching can now be combined with the other transformations discussed above. Once again, the basic algebraic technique is completing the square.

## Example

Find the vertex of the parabola $y = 2x^2 + 4x + 9$ and sketch its graph.

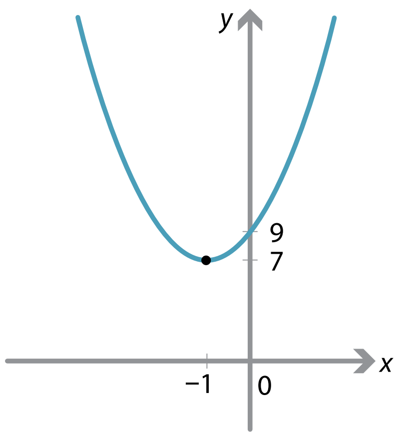
### Solution

By completing the square, we obtain

$$\begin{aligned} y &= 2x^2 + 4x + 9 \\ &= 2\Big(x^2 + 2x + \frac{9}{2}\Big) \\ &= 2\Big((x+1)^2 + \frac{7}{2}\Big) \\ &= 2(x+1)^2 + 7. \end{aligned}$$

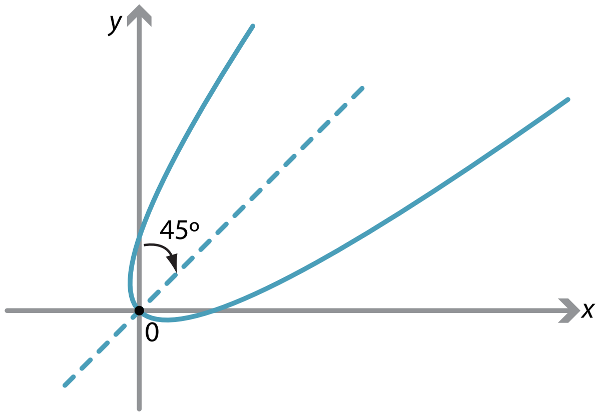
Hence the vertex is at $(-1, 7)$.

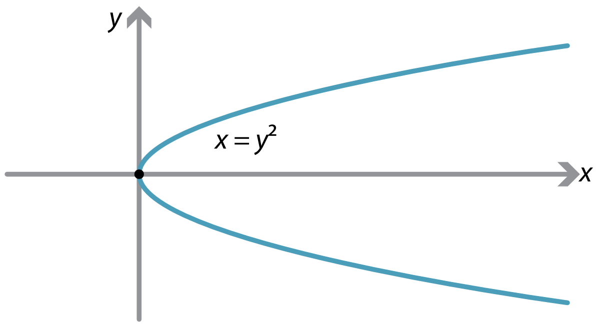
The basic parabola is stretched in the $y$-direction by a factor of 2 (and hence made steeper) and translated.




## Rotations

The basic parabola can also be rotated. For example, we can rotate the basic parabola clockwise about the origin through 45° or through 90°, as shown in the following two diagrams.




Algebraically, the equation of the first parabola is complicated and generally not studied in secondary school mathematics.

The second parabola can be obtained from $y = x^2$ by interchanging $y$ and $x$. It can also be thought of as a reflection of the basic parabola in the line $y = x$. Its equation is $x = y^2$ and it is an example of a relation, rather than a function (however, it can be thought of as a function of $y$).

## Summary

All parabolas can be obtained from the basic parabola by a combination of:

- translation
- reflection
- stretching
- rotation.

Thus, all parabolas are similar.

This is an important point, since in the module *Polynomials*, it is seen that the graphs of higher degree equations (such as cubics and quartics) are *not*, in general, obtainable from the basic forms of these graphs by simple transformations. The parabola, and also the straight line, are special in this regard.

For the remainder of the *Content* section of this module, we will restrict our attention to parabolas whose axis is *parallel* to the $y$-axis. It will be to these types that we refer when we use the term 'parabola'.

## Intercepts

Thus far we have concentrated on the transformational aspects of the graphs of quadratics and the coordinates of the vertex. Of course one may also be interested in the points at which the graph crosses the $x$- and $y$-axes. These are called the **intercepts**. It is easy to find the $y$-intercept of a parabola — it always has one — by putting $x = 0$. Not all parabolas, however, have $x$-intercepts. When $x$-intercepts exist, they are obtained by putting $y = 0$ and solving the resulting quadratic equation.

As we know, there are many ways to solve a quadratic equation, and there are obvious choices of strategy depending on the given problem. If we have already completed the square to find the vertex, then we can use that to quickly find the $x$-intercepts.

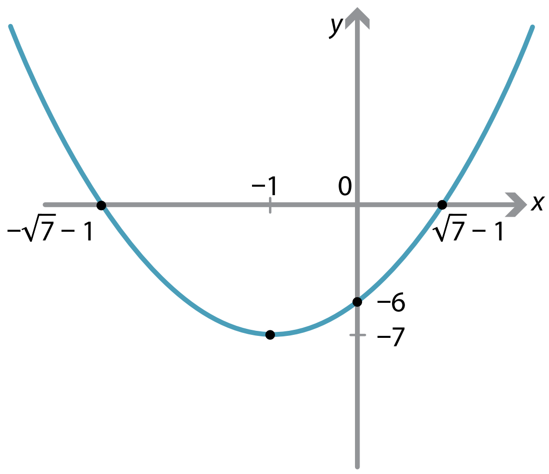
### Example

Find the vertex and intercepts of $y = x^2 + 2x - 6$ and sketch the parabola.

#### Solution

Completing the square gives $y = (x+1)^2 - 7$, so the vertex is at $(-1, -7)$.

Putting $x = 0$, we find that the $y$-intercept is $-6$. Putting $y = 0$ to find the $x$-intercepts, we have $(x+1)^2 - 7 = 0$ and so $(x+1)^2 = 7$, which gives $x = -1+\sqrt{7}$, $x = -1-\sqrt{7}$.



Completing the square will also tell us when there are no $x$-intercepts. For example, given the quadratic $y = x^2 + 6x + 13$, completing the square yields $y = (x+3)^2 + 4$. We can see that $y$ is never zero, since the right-hand side is at least 4 for any value of $x$.

In some instances — especially when completing the square is harder — we may prefer to use the factor method instead. For example, the quadratic $y = x^2 - 5x + 6$ factors to $y = (x-3)(x-2)$, and so we can easily see that the $x$-intercepts are at $x = 2$ and $x = 3$.

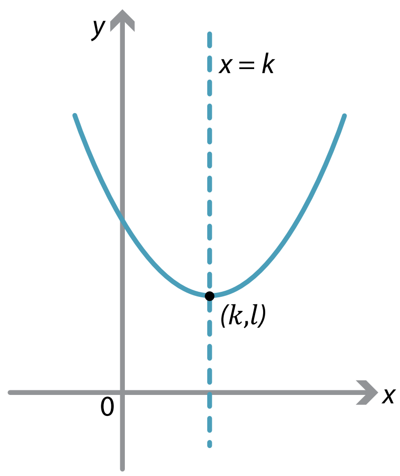
There are a number of strategies to find the vertex. You will have seen in the examples that, since the graph of the quadratic function is symmetric about the vertex, the $x$-intercepts are symmetric about the vertex. Hence the *average* of the $x$-intercepts gives the $x$-coordinate of the vertex. This is very useful in those situations in which we have already found the $x$-intercepts.

For example, the quadratic function $y = x^2 - 5x + 6$ has its $x$-intercepts at $x = 2$ and $x = 3$. Hence the $x$-coordinate of the vertex is the average of these numbers, which is $2\frac{1}{2}$. From this we can find the $y$-coordinate of the vertex, which is $-\frac{1}{4}$.

Of course, not all parabolas have $x$-intercepts. In the next section, we give a formula for the $x$-coordinate of the vertex of $y = ax^2 + bx + c$.

## The axis of symmetry

All parabolas have exactly one axis of symmetry (unlike a circle, which has infinitely many axes of symmetry). If the vertex of a parabola is $(k, l)$, then its axis of symmetry has equation $x = k$.



We can find a simple formula for the value of $k$ in terms of the coefficients of the quadratic. As usual, we complete the square:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\Big[x^2 + \frac{b}{a}x + \frac{c}{a}\Big] \\ &= a\Big[\Big(x + \frac{b}{2a}\Big)^2 + \frac{c}{a} - \Big(\frac{b}{2a}\Big)^2\Big]. \end{aligned}$$

We can now see that the $x$-coordinate of the vertex is $-\dfrac{b}{2a}$. Thus the equation of the axis of symmetry is

$$x = -\frac{b}{2a}.$$

We could also find a formula for the $y$-coordinate of the vertex, but it is easier simply to substitute the $x$-coordinate of the vertex into the original equation $y = ax^2 + bx + c$.

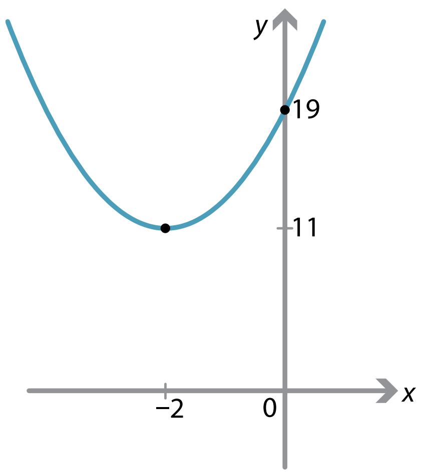
## Example

Sketch the parabola $y = 2x^2 + 8x + 19$ by finding the vertex and the $y$-intercept. Also state the equation of the axis of symmetry. Does the parabola have any $x$-intercepts?

### Solution

Here $a = 2$, $b = 8$ and $c = 19$. So the axis of symmetry has equation $x = -\frac{b}{2a} = -\frac{8}{4} = -2$. We substitute $x = -2$ into the equation to find $y = 2 \times (-2)^2 + 8 \times (-2) + 19 = 11$, and so the vertex is at $(-2, 11)$. Finally, putting $x = 0$ we see that the $y$-intercept is 19.

There are no $x$-intercepts.



### Exercise 3

A parabola has vertex at $(1,3)$ and passes through the point $(3,11)$. Find its equation.

## The quadratic formula and the discriminant

The quadratic formula was covered in the module *Algebra review*. This formula gives solutions to the general quadratic equation $ax^2+bx+c=0$, when they exist, in terms of the coefficients $a,b,c$. The solutions are

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad x=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

provided that $b^2-4ac \geq 0$.

The quantity $b^2-4ac$ is called the **discriminant** of the quadratic, often denoted by $\Delta$, and should be found first whenever the formula is being applied. It *discriminates* between the types of solutions of the equation:

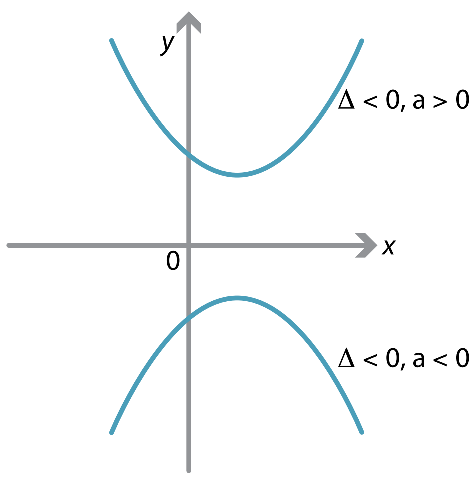
- $\Delta > 0$ tells us the equation has two distinct real roots
- $\Delta = 0$ tells us the equation has one (repeated) real root
- $\Delta < 0$ tells us the equation has no real roots.

### Exercise 4

For what values of $k$ does the equation $(4k+1)x^2-6kx+4=0$ have one real solution?

A quadratic expression which always takes positive values is called **positive definite**, while one which always takes negative values is called **negative definite**.

Quadratics of either type never take the value 0, and so their discriminant is negative. Furthermore, such a quadratic is positive definite if $a>0$, and negative definite if $a<0$.



## Example

Show that the quadratic expression $4x^2 - 8x + 7$ always takes positive values for any value of $x$.

### Solution

In this case, $a = 4$, $b = -8$ and $c = 7$. So

$$\Delta = (-8)^2 - 4 \times 4 \times 7 = -48 < 0$$

and $a = 4 > 0$. Hence the quadratic is positive definite.

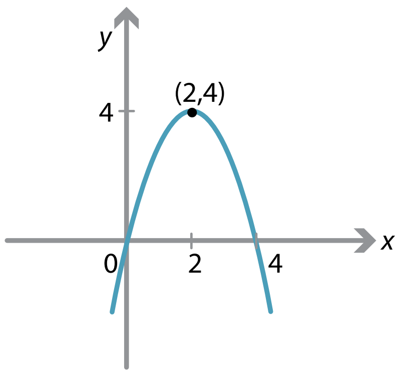
## Exercise 5

For what values of $k$ does the equation $(4k+1)x^2 - 2(k+1)x + (1-2k) = 0$ have one real solution? For what values of $k$, if any, is the quadratic negative definite?

## Applications to maxima and minima

Since every parabola has a vertex, the $y$-coordinate of the vertex gives the maximum or minimum value of $y$ for all possible values of $x$.

For example, the vertex of the parabola $y = 4x - x^2$ is $(2,4)$ and the greatest $y$-value is 4.

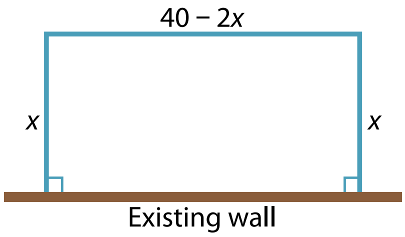



There are a number of geometric problems and word problems in which the variables are connected via a quadratic function. It is often of interest to maximise or minimise the dependent variable.

## Example

A farmer wishes to build a large rectangular enclosure using an existing wall. She has 40 m of fencing wire and wishes to maximise the area of the enclosure. What dimensions should the enclosure have?

### Solution

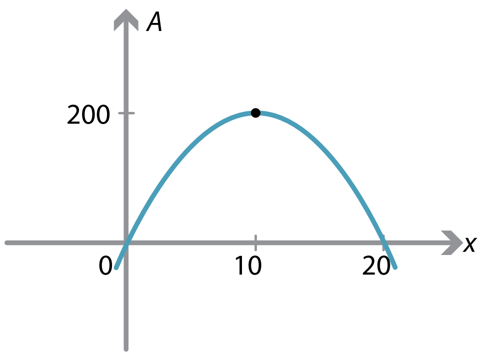



With all dimensions in metres, let the width of the enclosure be $x$. Then the three sides to be made from the fencing wire have lengths $x$, $x$ and $40-2x$, as shown in the diagram.

Thus the area $A$ is given by

$$A = x(40-2x) = 40x - 2x^2.$$

This gives a quadratic relationship between the area and the width, which can be plotted.




Clearly the area will be a maximum at the vertex of the parabola. Using $x = -\dfrac{b}{2a}$, we see that the $x$-value of the vertex is $x = 10$.

Hence the dimensions of the rectangle with maximum area are 10 m by 20 m, and the maximum area of the enclosure is 200 $m^2$.

*Note.* Differential calculus can be used to solve such problems. While this is quite acceptable, it is easier to use the procedure outlined above. You may use calculus, however, to solve the following exercise.

### Exercise 6

A piece of wire with length 20 cm is divided into two parts. One part is bent to form a circle while the other is bent to form a square. How should the wire be divided in order that the sum of the areas of the circle and square is minimal?

## Sum and product of the roots

We have seen that, in the case when a parabola crosses the $x$-axis, the $x$-coordinate of the vertex lies at the average of the intercepts. Thus, if a quadratic has two real roots $\alpha, \beta$, then the $x$-coordinate of the vertex is $\frac{1}{2}(\alpha+\beta)$. Now we also know that this quantity is equal to $-\frac{b}{2a}$. Thus we can express the sum of the roots in terms of the coefficients $a, b, c$ of the quadratic as $\alpha+\beta = -\frac{b}{a}$.

In the case when the quadratic does not cross the $x$-axis, the corresponding quadratic equation $ax^2+bx+c=0$ has no real roots, but it will have complex roots (involving the square root of negative numbers). The formula above, and other similar formulas shown below, still work in this case.

We can find simple formulas for the sum and product of the roots simply by expanding out. Thus, if $\alpha, \beta$ are the roots of $ax^2+bx+c=0$, then dividing by $a$ we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta.$$

Comparing the first and last expressions we conclude that

$$\alpha+\beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a}.$$

From these formulas, we can also find the value of the sum of the squares of the roots of a quadratic without actually solving the quadratic.

### Example

Suppose $\alpha, \beta$ are the roots of $2x^2 - 4x + 7 = 0$. Find the value of $\alpha^2+\beta^2$ and explain why the roots of the quadratic cannot be real.

#### Solution

Using the formulas above, we have $\alpha+\beta = 2$ and $\alpha\beta = \frac{7}{2}$. Now $(\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$, so $\alpha^2+\beta^2 = 2^2 - 2\times\frac{7}{2} = -3$. If the roots were real, then the sum of their squares would be positive. Since the sum of their squares is $-3$, the roots cannot be real.

### Exercise 7

Find the monic quadratic with roots $2-3\sqrt{5}$, $2+3\sqrt{5}$.

### Exercise 8

Suppose $\alpha, \beta$ are the roots of $2x^2 - 4x + 7 = 0$.

Find the value of

**a** $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **b** $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$.

Note that, in the previous exercise, the desired expressions are *symmetric*. That is, interchanging $\alpha$ and $\beta$ does not change the value of the expression. Such expressions are called **symmetric functions of the roots**.

### Exercise 9

Suppose $\alpha, \beta$ are the roots of $ax^2 + bx + c = 0$. Find a formula for $(\alpha - \beta)^2$ in terms of $a$, $b$ and $c$.

You may recall that the arithmetic mean of two positive numbers $\alpha$ and $\beta$ is $\frac{1}{2}(\alpha + \beta)$, while their geometric mean is $\sqrt{\alpha\beta}$. Thus, if a quadratic has two positive real solutions, we can express their arithmetic and geometric mean using the above formulas.

### Exercise 10

Without solving the equation, find the arithmetic and geometric mean of the roots of the equation $3x^2 - 17x + 12 = 0$.

## Constructing quadratics

It is a fundamental fact that two points uniquely determine a line. That is, given two distinct points, there is one and only one line that passes through these points.

How many (distinct) non-collinear points in the plane are required to determine a parabola? The answer is three.

This idea may be expressed by the following theorem:

### Theorem

*If two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = Ax^2 + Bx + C$ take the same value for three different values of $x$, then $f(x) = g(x)$ for all values of $x$.*

**Proof**

We start by observing that a quadratic equation can have at most two solutions. Hence, if the quadratic equation $dx^2 + ex + f = 0$ has more than two solutions, then $d = e = f = 0$.

Now, if $f(x)$ and $g(x)$ agree for $x = \alpha$, $x = \beta$ and $x = \gamma$, then

$$f(x) - g(x) = (a - A)x^2 + (b - B)x + (c - C) = 0$$

when $x = \alpha$, $x = \beta$ and $x = \gamma$. In that case, the quadratic equation above has three different solutions and so the coefficients are 0. Thus, $a = A$, $b = B$ and $c = C$, and so $f(x) = g(x)$ as claimed. $\square$

### Example

Find the equation of the quadratic function whose graph passes through $(-1,2)$, $(0,3)$ and $(1,6)$.

#### Solution

Suppose the equation is $y = ax^2 + bx + c$. Substituting the coordinates of the three points, we have

$$2 = a - b + c, \qquad 3 = c, \qquad 6 = a + b + c.$$

Thus, $c = 3$ and so $a - b = -1$ and $a + b = 3$. Therefore $a = 1$ and $b = 2$.

Hence the function is $y = x^2 + 2x + 3$.

### Exercise 11

Express $x^2$ in the form $a(x-1)^2 + b(x-2)^2 + c(x-3)^2$.

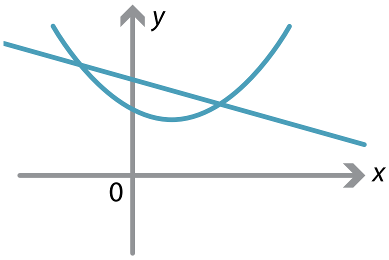
## Intersection of lines and parabolas

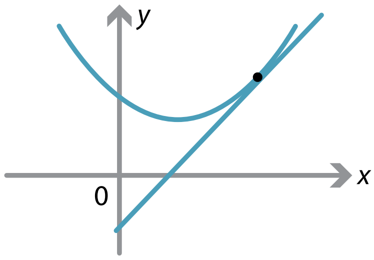
In this section we investigate under what conditions a line and a parabola might meet in the plane. It is easy to see that there are three main cases to consider.

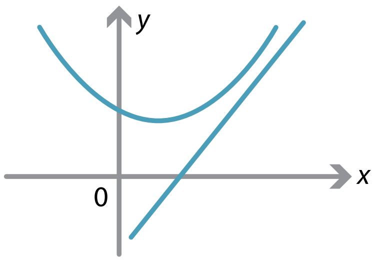
A line may:

- intersect a parabola at two points
- touch a parabola at one point (in this case we say the line is **tangent** to the parabola)
- not intersect the parabola at all.

The three cases are shown in the diagrams below.




There is also a special case when the line is parallel to the $y$-axis. In this case the line will cut the parabola once. Having noted this case, we will not need to consider it further, since the intersection point is easily found.

Suppose the line has the form $y = mx + d$ and the parabola is $y = ax^2 + bx + c$. Equating, we have

$$ax^2 + bx + c = mx + d \implies ax^2 + (b-m)x + (c-d) = 0.$$

This gives a quadratic equation. Hence we can conclude:

- A line can meet a parabola in *at most two* points.
- If the discriminant of the resulting quadratic is negative, the line does not meet the parabola at all.
- If the discriminant of the resulting quadratic is positive, the line meets the parabola in two distinct points.
- If the discriminant of the resulting quadratic is zero, then the line is tangent to the parabola.

### Example

Discuss the intersection of the line $y = 3x + 2$ and the parabola $y = 4x^2 - 9x + 11$.

#### Solution

Equating the two equations, we have

$$4x^2 - 9x + 11 = 3x + 2 \implies 4x^2 - 12x + 9 = 0.$$

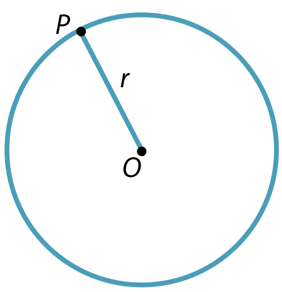
Now the discriminant of this quadratic is $b^2 - 4ac = 144 - 4 \times 4 \times 9 = 0$ and so the equation has only one solution. Hence the line is tangent to the parabola.

### Exercise 12

Use the ideas above to show that the line $y = -2x + 5$ is tangent to the circle $x^2 + y^2 = 5$.

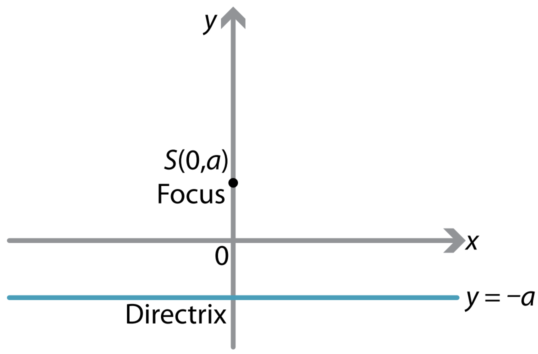
## Focus-directrix definition of the parabola

The Greeks defined the parabola using the notion of a **locus**. A locus is a set of points satisfying a given condition. These points will generally lie on some curve. For example, the circle with centre $O$ and radius $r$ is the locus of a point $P$ moving so that its distance from the point $O$ is always equal to $r$.

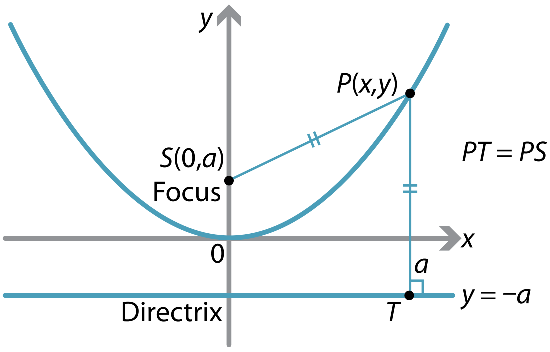



The locus definition of the parabola is only slightly more complicated. It is very important and gives us a new way of viewing the parabola. One of the many applications of this is the refection principle, which we will look at in a later section.

We fix a point in the plane, which we will call the **focus**, and we fix a line (not through the focus), which we will call the **directrix**. It is easiest to take the directrix parallel to the $x$-axis and choose the origin so that it is equidistant from the focus and directrix. Thus, we will take the focus at $S(0, a)$ and the directrix with equation $y = -a$, where $a > 0$. This is as shown in the following diagram.




We now look at the locus of a point moving so that its distance from the focus is equal to its perpendicular distance to the directrix. Let $P(x, y)$ be a point on the locus. Our aim is to find the equation governing the coordinates $x$ and $y$ of $P$.




The distance $PS$ is equal to $\sqrt{x^2 + (y-a)^2}$, and we can see from the diagram that the perpendicular distance $PT$ of $P$ to the directrix is simply $y + a$.

Since $PS = PT$, we can square each length and equate, giving

$$x^2 + (y-a)^2 = (y+a)^2 \implies x^2 = 4ay.$$

The last equation may also be written as

$$y = \frac{x^2}{4a}$$

and so we see that it is the equation of a parabola with vertex at the origin.

The positive number $a$ is called the **focal length** of the parabola.

Any parabola of the form $y = Ax^2 + Bx + C$ can be put into the standard form

$$(x-p)^2 = \pm 4a(y-q),$$

with $a > 0$, where $(p, q)$ is the vertex and $a$ is the focal length. When a parabola is in this standard form, we can easily read off its vertex, focus and directrix.

## Example

Find the vertex, focal length, focus and directrix for the parabola $y = x^2 - 4x + 3$.

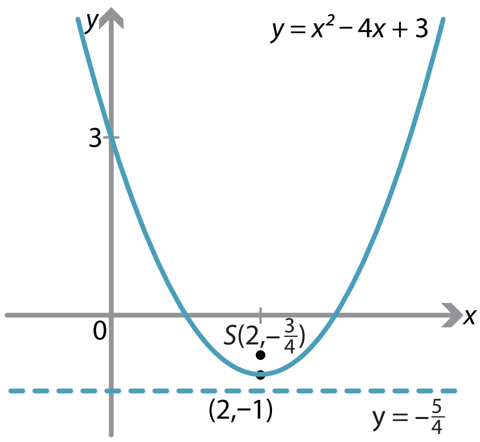
### Solution

We complete the square and rearrange as follows:

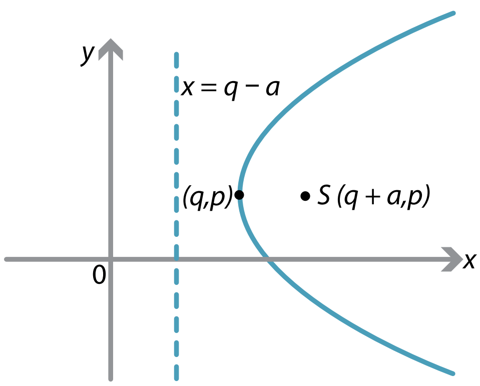
$$\begin{aligned} y = x^2 - 4x + 3 &\implies y = (x-2)^2 - 1 \\ &\implies (x-2)^2 = 4 \times \frac{1}{4}(y+1). \end{aligned}$$

Hence the vertex is $(2, -1)$ and the focal length is $\frac{1}{4}$.

The focus is then $(2, -\frac{3}{4})$ and the equation of the directrix is $y = -\frac{5}{4}$.




A parabola may also have its directrix parallel to the $y$-axis and then the standard form is $(y-p)^2 = \pm 4a(x-q)$, with $a > 0$.



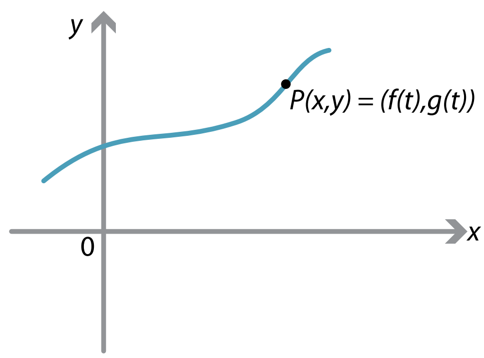
### Exercise 13

Put the parabola $y^2 - 2y = 4x - 5$ into the above standard form (for a directrix parallel to the $y$-axis) and thus find its vertex, focus and directrix, and sketch its graph.

## Parametric equations of a parabola

Since every parabola is congruent to $x^2 = 4ay$, for some choice of $a > 0$, we can study the geometry of every parabola by confining ourselves to this one! This shows the power of both coordinate geometry and transformations.

Imagine a particle moving in the plane along a curve $C$ as shown.




The $x$- and $y$-coordinates of the particle can be thought of as functions of a new variable $t$, and so we can write $x = f(t)$, $y = g(t)$, where $f, g$ are functions of $t$. In some physical problems $t$ is thought of as time.

The equations

$$x = f(t)$$

$$y = g(t)$$

are called the **parametric equations** of the point $P(x, y)$ and the variable $t$ is called a **parameter**.

It is often very useful to take a cartesian equation $y = F(x)$ and introduce a parameter $t$ so that the $x$- and $y$-coordinates of any point on the curve can be expressed in terms of this parameter.

Here is a very simple example of a parametrisation. Suppose we begin with the line whose equation is $y = 3x - 4$. We can introduce a new variable $t$ and write $x = t$. Then we have $y = 3t - 4$. Thus we can rewrite the equation $y = 3x - 4$ in parametric form as

$$x = t$$

$$y = 3t - 4.$$

## Exercise 14

Eliminate the parameter $t$ from the equations

$$x = 2t - 3$$

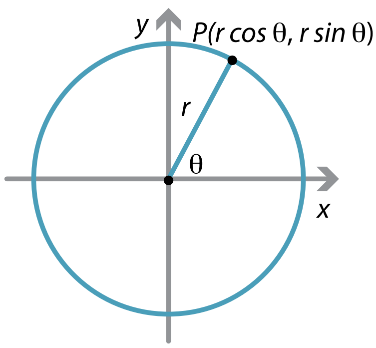
$$y = 5 - 3t$$

and describe the resulting function.

Another common parametrisation is that used for a circle. The cartesian equation of a circle of radius $r$ centred at the origin is $x^2 + y^2 = r^2$. We can put $x = r\cos\theta$, $y = r\sin\theta$, and we see that

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2,$$

so these satisfy the original equation. The angle $\theta$ here is the parameter and measures the angle between the positive $x$-axis and a radial line to the circle.




Thus we can describe the circle parametrically by

$$x = r\cos\theta$$

$$y = r\sin\theta,$$

where $0 \le \theta < 2\pi$.

The parabola $x^2 = 4ay$, where $a > 0$, may also be parametrised and there are many ways to do this. We will choose the parametrisation

$$x = 2at$$
$$y = at^2.$$

The parameter in this case is $t$.

This is the standard parametrisation for the parabola $x^2 = 4ay$ and the reason for this parametrisation will become apparent later in this module.

You should check that this parametrisation actually satisfies the original equation.

### Example

Express the parabola $x^2 = 12y$ parametrically.

#### Solution

In this case we see that $a = 3$ and so the general point on the parabola can be expressed parametrically as

$$x = 6t$$
$$y = 3t^2.$$

## Chords, tangents and normals

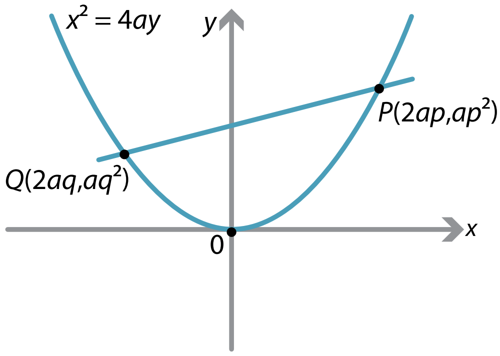
### Chords

Again, we consider the parabola $x^2 = 4ay$, for some $a > 0$, and the parametrisation

$$x = 2at$$
$$y = at^2.$$

When we take two distinct points $P, Q$ on the parabola, we can join them to form a line segment called a **chord**. Sometimes we will refer to the line through $P$ and $Q$ as the chord.

Let the value of the parameter $t$ at the points $P$ and $Q$ be $p$ and $q$ respectively. Thus $P$ has coordinates $(2ap, ap^2)$ and $Q$ has coordinates $(2aq, aq^2)$.




A chord of the parabola.

The gradient of the line $PQ$ is given by

$$\frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p^2 - q^2)}{2a(p-q)} = \frac{p+q}{2}.$$

So the gradient of the chord is the average of the values of the parameter. (One of the many reasons this parametrisation was chosen.)

Hence the equation of the chord is

$$y - ap^2 = \frac{p+q}{2}(x - 2ap),$$

which rearranges to

$$y = \frac{p+q}{2}x - apq.$$

It is best to reconstruct this equation each time rather than memorise it.

A chord which passes through the focus of a parabola is called a **focal chord**. A given chord will be a focal chord if the point $(0, a)$ lies on it. Substituting these coordinates into the equation of the chord above we have

$$a = \frac{p+q}{2} \times 0 - apq,$$

which is equivalent to $pq = -1$. This gives us a simple condition to tell when the chord joining two points is a focal chord.

Thus $PQ$ is a focal chord if and only if $pq = -1$. The basic fact should be committed to memory.

## Example

Suppose $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points lying on the parabola $x^2 = 4ay$.

Prove that, if $OP$ and $OQ$ are perpendicular, then $pq = -4$.

### Solution

The gradient of $OP$ is $\dfrac{ap^2}{2ap} = \dfrac{p}{2}$ and similarly the gradient of $OQ$ is $\dfrac{q}{2}$. Since these lines are perpendicular, the product of their gradients is $-1$ and so $\dfrac{p}{2} \times \dfrac{q}{2} = -1$, which implies that $pq = -4$.

Note that, in this case, $PQ$ can never be a focal chord.

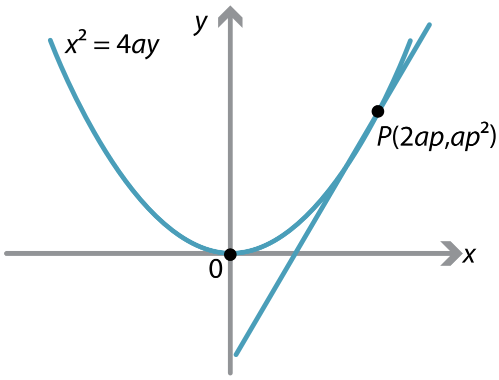
## Exercise 15

Suppose $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points lying on the parabola $x^2 = 4ay$. Prove that, if $PQ$ is a focal chord, then the length of the chord $PQ$ is given by $a\big(p + \frac{1}{p}\big)^2$.

## Tangents

We have seen that the gradient of the chord joining two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$ is $\frac{1}{2}(p + q)$.

If we now imagine the point $Q$ moving along the parabola to the point $P$, then the line through $P$ and $Q$ will become a tangent to the parabola at the point $P$. Putting $q = p$, we see that the gradient of this tangent is $\frac{1}{2}(p + p) = p$.




A tangent to the parabola.

We could also find the gradient of the tangent by differentiating

$$y = \frac{x^2}{4a}$$

and substituting $x = 2ap$. In either case, the gradient of the tangent to $x^2 = 4ay$ at the point $P(2ap, ap^2)$ is $p$. (This is the main reason for this choice of parametrisation.)

Hence the equation of the tangent to the parabola at $P$ is given by

$$y - ap^2 = p(x - 2ap) \implies y = px - ap^2.$$

This formula is best derived each time it is needed rather than memorised.

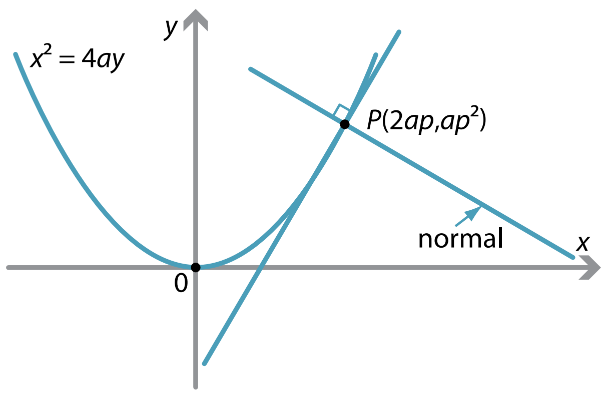
## Normals

The line through a point $P$ perpendicular to the tangent to the curve at $P$ is called the **normal** to the curve at $P$.

Since the normal and tangent are perpendicular, the product of their gradients is $-1$ and so the gradient of the normal at $P(2ap, ap^2)$ is $-\frac{1}{p}$. Hence the equation of the normal at $P$ is

$$y - ap^2 = -\frac{1}{p}(x - 2ap) \implies x + py = 2ap + ap^3.$$

Once again, this is best derived each time rather than memorised.




### Example

Find the point of intersection of the tangents to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. What can be said about this intersection point if the chord $PQ$ is a focal chord?

### Solution

The equations of the tangents at $P$ and $Q$ are respectively $y = px - ap^2$ and $y = qx - aq^2$. Equating these we find

$$px - ap^2 = qx - aq^2 \implies x = a(p+q).$$

Substituting back we find that the $y$-coordinate is $y = apq$ and so the point of intersection is $\big(a(p+q), apq\big)$.

Now, if $PQ$ is a focal chord, then $pq = -1$ and so the point of intersection becomes $\big(a(p - \frac{1}{p}), -a\big)$. While the $x$-coordinate of this point can vary, the $y$-coordinate is always $-a$ and so the point of intersection always lies on the directrix.

### Exercise 16

Find the point of intersection $R$ of the normals to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

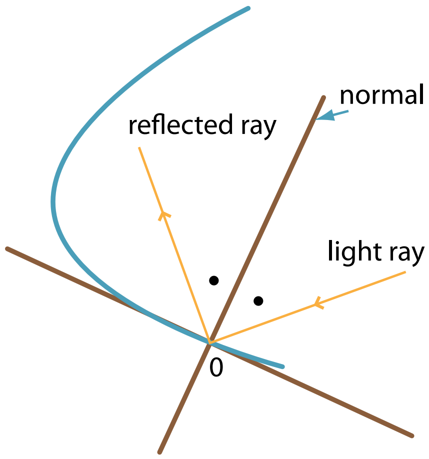
Show that, if $PQ$ is a focal chord, then the point $R$ always lies on a certain parabola, and find the equation of this parabola.

## The reflection property of the parabola

While the parabola has many beautiful geometric properties as we have seen, it also has a remarkable property known as the **reflection property**, which is used in such diverse places as car headlights and satellite dishes. A car headlight emits light from one source, but is able to focus that light into a beam so that the light does not move off in all directions — as lamplight does — but is focused directly ahead. Satellite dishes do the opposite. They collect electromagnetic rays as they arrive and concentrate them at one point, thereby obtaining a strong signal.

Headlight reflectors and satellite dishes are designed so their cross-sections are parabolic in shape and the point of collection or emission is the *focus* of the parabola.

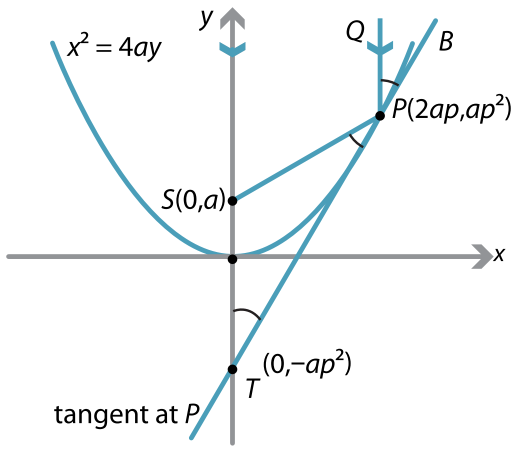
To show how this works we need a basic fact from physics: when light (or any electromagnetic radiation) is reflected off a surface, the angle of incidence equals the angle of reflection. That is, when a light ray bounces off the surface of a reflector, then the angle between the light ray and the normal to the reflector at the point of contact equals the angle between the normal and the reflected ray. This is shown in the following diagram.




Note that this implies that the angle between the ray and the tangent is also preserved after reflection, which is a more convenient idea for us here.

Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$ with focus at $S$ and let $T$ be the point where the tangent at $P$ cuts the $y$-axis.

Suppose $PQ$ is a ray parallel to the $y$-axis. Our aim is to show that the line $PS$ will satisfy the reflection property, that is, $\angle QPB$ is equal to $\angle SPT$.




Notice that, since $QP$ is parallel to $ST$, $\angle QPB$ is equal to $\angle STP$, so we will show that $\angle STP = \angle SPT$.

Now $S$ has coordinates $(0, a)$, and $T$ has coordinates $(0, -ap^2)$, obtained by putting $x = 0$ in the equation of the tangent at $P$. Hence

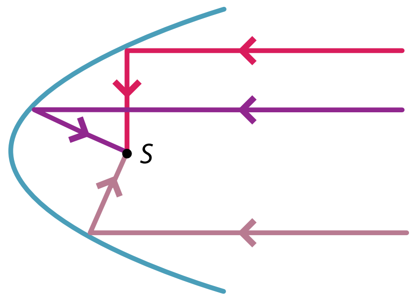
$$SP^2 = (2ap - 0)^2 + (ap^2 - a)^2 = a^2(p^2 + 1)^2$$

after a little algebra. Also,

$$ST^2 = (a + ap^2)^2 = a^2(1 + p^2)^2$$

and so $SP = ST$. Hence $\Delta STP$ is isosceles and so $\angle STP = \angle SPT$.

Thus the reflection property tells us that any ray parallel to the axis of the parabola will bounce off the parabola and pass through the focus. Conversely, any ray passing through the focus will reflect off the parabola in a line parallel to the axis of the parabola (so that light emanating from the focus will reflect in a straight line parallel to the axis).

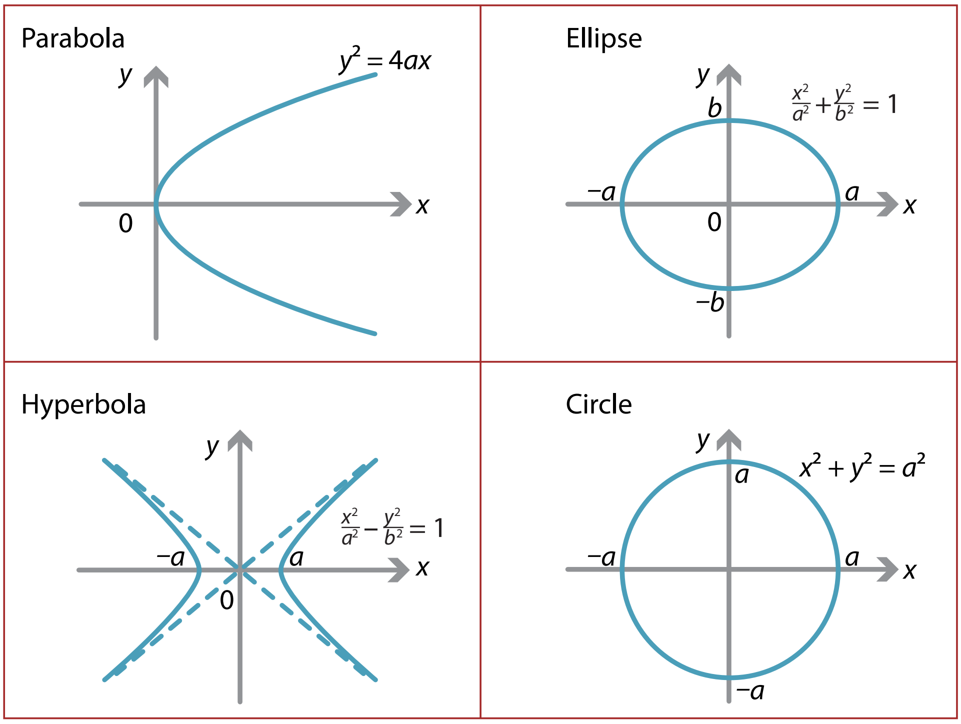



# Links forward

## Conic sections

The parabola is one of the curves known as the **conic sections**, which are obtained when a plane intersects with a double cone.

The non-degenerate conic sections are the parabola, ellipse, hyperbola and circle.

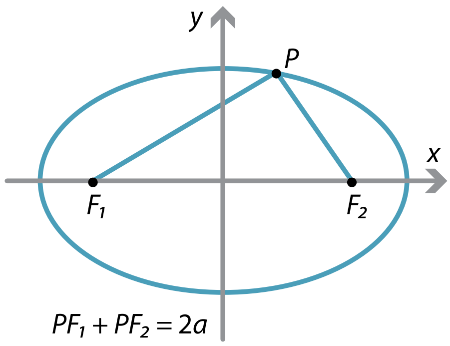


The degenerate case produces also a line, a pair of lines and a point.

## Locus definitions

The ellipse and hyperbola can be defined as a locus in various ways.

Suppose we fix two distinct points $F_1$ and $F_2$ in the plane. Then the ellipse can be defined as the locus of points $P$ such that the sum of the distances $PF_1 + PF_2$ is a fixed constant $2a$. (Imagine pinning down the ends of a piece of string at the points $F_1$ and $F_2$, then using a pencil to pull the string tight and allowing the pencil to move around the page. The sum of the distances is then the length of the string, which remains constant.)




Locus definition of the ellipse.

Thus, if we set the points to be $F_1(-c, 0)$ and $F_2(c, 0)$, with $0 < c < a$, then the sum of the distances is

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Taking the second square root onto the other side, squaring, simplifying and squaring again, we obtain the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we write $b^2 = a^2 - c^2$, then the equation becomes

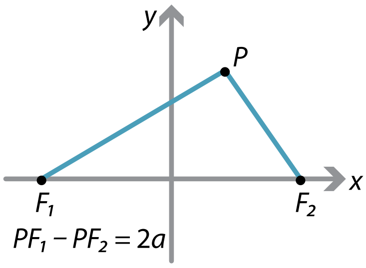
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

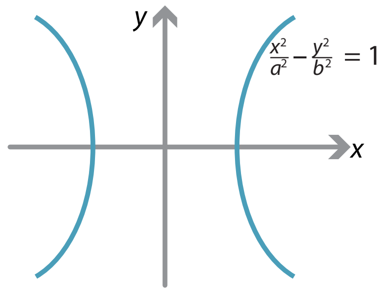
This is the cartesian equation of the ellipse.

## Exercise 17

Perform the algebra suggested above to derive the equation of the ellipse.

If, instead, we require the *difference* of the distances $PF_1 - PF_2$ to be a fixed constant $2a$, then we obtain part of a hyperbola. The equation has the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

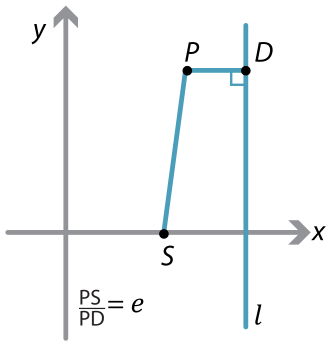



Locus definition of the hyperbola.

### Focus-directrix definitions

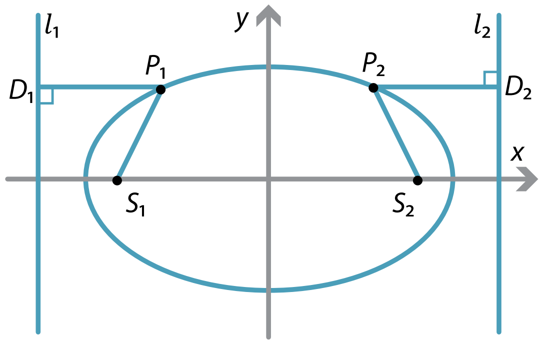
Recall that the parabola can be defined as the locus of the point $P$ moving so that the distance $PS$, from $P$ to the focus $S$, is equal to the perpendicular distance of $P$ to the directrix $l$. Thus, if $D$ is the nearest point on $l$ to $P$, we have $PS = PD$, which we can write as $\frac{PS}{PD} = 1$.




The ellipse and the hyperbola also have focus-directrix definitions.

We choose a fixed point $S$, which will be the focus, and a line $l$ (not containing $S$). Take the locus of all points $P$ such that, if $D$ is the nearest point on $l$ to $P$, then the ratio of the distances $\frac{PS}{PD}$ is equal to a fixed positive number $e$, called the **eccentricity**.

- If $e$ is equal to 1, then we have, of course, the parabola. The name *parabola* comes from the Greek meaning 'fall beside' since, in the case of the parabola, the distance $PS$ is equal to (i.e., falls beside) the distance $PD$.
- If $e$ is less than 1, then we obtain an *ellipse* (from the Greek meaning 'fall short of' — the original sense was that $PS$ falls short of $PD$).
- If $e$ is greater than 1, then we have a *hyperbola* (from the Greek meaning 'exceed' — the original sense was that $PS$ exceeds $PD$).




Focus-directrix definition of the ellipse.

As we saw above, the cartesian equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It can be shown that the eccentricity $e$ of the ellipse is connected to the quantities $a$ and $b$ by the equation $b^2 = a^2(1 - e^2)$. The foci can be written as $(\pm ae, 0)$, while the equations of the directrices are $x = \pm\frac{a}{e}$.

Similarly, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the eccentricity is connected to the quantities $a$ and $b$ by the equation $b^2 = a^2(e^2 - 1)$.

The ellipse also has a reflective property in that any light beam emanating from one of the two foci will reflect off the curve and pass through the other focus. This idea has been widely exploited in acoustics and optics.

## Second-degree equations

Algebraically, every second-degree equation of the form $Ax^2 + By^2 + Cxy + Dx + Ey = F$ can be shown, by means of linear changes of variables, to be similar to one of the conic sections (including the degenerate cases) we have listed above.

For example, the equation $x^2 + y^2 - 4x + 6y = 0$ can be written as $(x-2)^2 + (y+3)^2 = 13$, which is a circle. Similarly, the equation $x^2 - y^2 = 0$ can be written as $(x-y)(x+y) = 0$ and so as $y = x$ or $y = -x$, which is a pair of lines.

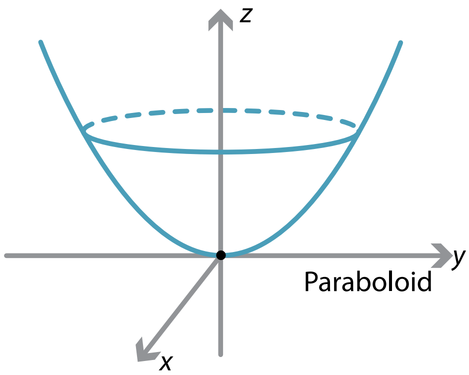
An equation such as $x^2 + y^2 + 6xy = 1$ in fact represents a hyperbola, but one which is rotated about the origin. The theory of matrices is generally used to analyse such equations, but this is not part of secondary school mathematics.

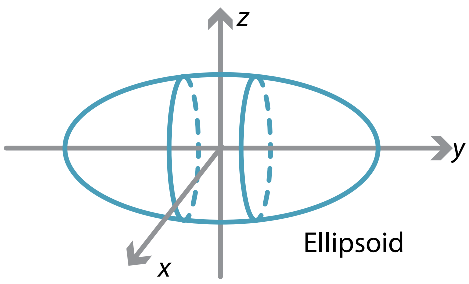
## Three-dimensional analogues

Each of these curves has a 3-dimensional analogue. For example, the parabolic reflector is an example of a **paraboloid**. The basic paraboloid has equation $z = x^2 + y^2$ and its cross-section in the $y$–$z$ plane is a parabola. Similarly, an **ellipsoid** has the basic shape of a football, with equation

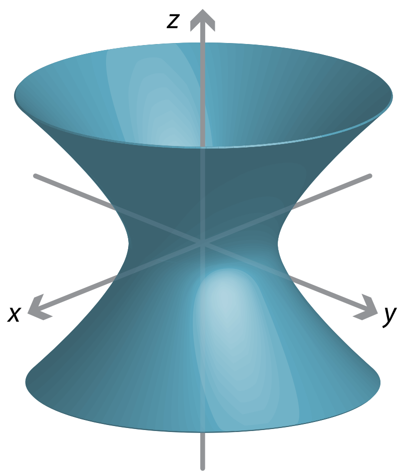
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

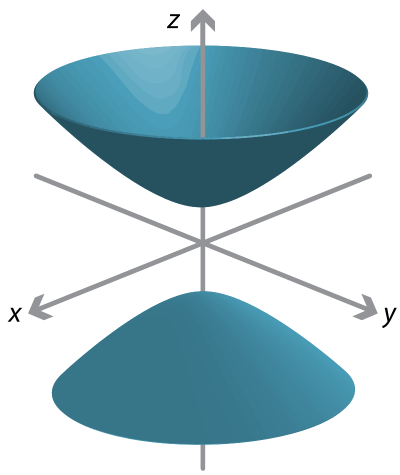
In the case when $a = b = c$, we have a sphere.




There are also 3-dimensional **hyperboloids** of one and two sheets.



## History and applications

The history of the algebraic aspects of quadratics was covered in the module *Quadratic equations* (Years 9–10).

Historically, the geometric properties of the parabola were studied by the ancient Greeks. Menaechmus (*c.* 380–320 BCE) appears to have been the first to study the properties of the parabola along with the hyperbola and ellipse. These curves arose through the study of the cone. Apollonius of Perga (262–190 BCE) wrote a major treatise on the conic sections, but his definition of the parabola was in terms of ratios and without any use of coordinates. The focus-directrix definition used in this module goes back to Pappus (290–350 CE). It was not until the advent of coordinate geometry at the time of Descartes that further real progress could be made, although the reflective property was known to the Greeks.

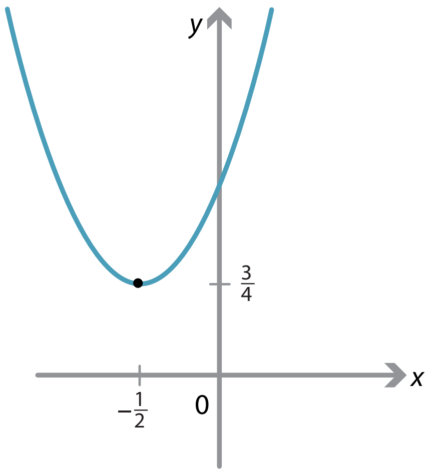
Galileo (1564–1642) realised that the motion of a projectile under gravity formed a parabolic path. Kepler (1571–1630) was the first to realise that the planets revolve around the sun in orbits that are very close to being elliptical. Newton proved this using his universal law of gravitation and the calculus.


The reflective properties of the parabola and the ellipse have been exploited in architecture to obtain remarkable acoustic properties in large church buildings, and later in the design of powerful telescopes and reflectors.

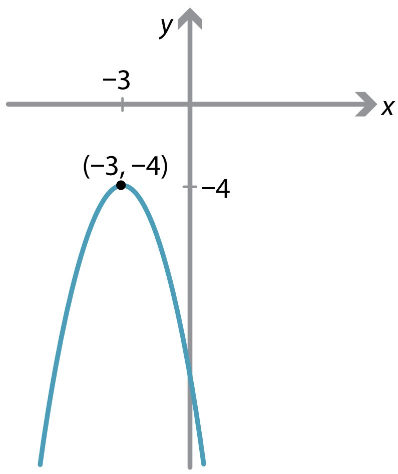
# Answers to exercises

## Exercise 1

**a**

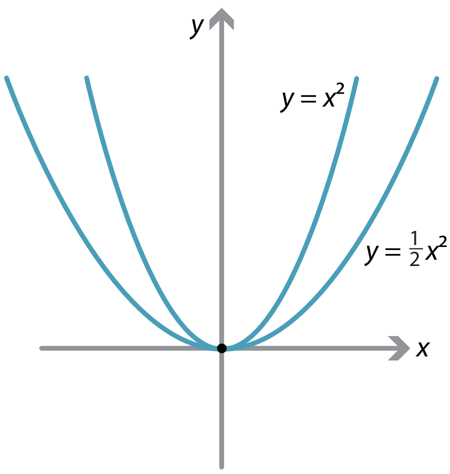



**b**




## Exercise 2

The arms of the parabola $y = x^2$ are steeper.




## Exercise 3

$y = 2(x-1)^2 + 3$

## Exercise 4

The discriminant is $36k^2 - 16(4k+1)$ and we set this equal to zero in order to obtain one real solution to the given equation. This gives $9k^2 - 16k - 4 = 0$, which has solutions $k = 2$ or $k = -\frac{2}{9}$.

## Exercise 5

The discriminant is $\Delta = 36k^2$. One real solution occurs when $\Delta = 0$, which is when $k = 0$. The quadratic is never negative definite, since $\Delta \geq 0$ for all values of $k$.

## Exercise 6

Divide the wire into $4x$ cm for the circle and $(20-4x)$ cm for the square. (The 4 is to make the algebra easier.) The total area is then

$$S = \frac{4x^2}{\pi} + (5-x)^2.$$

Since the leading coefficient is positive, the parabola has a minimum. We set $\frac{dS}{dx} = 0$ to find the stationary point, which occurs when $x = \frac{5\pi}{4+\pi}$. Hence, we use $\frac{20\pi}{4+\pi}$ cm for the circle and $\frac{80}{4+\pi}$ cm for the square.

## Exercise 7

$x^2 - 4x - 41$

## Exercise 8

**a** $\frac{4}{7}$ **b** $-\frac{12}{49}$

## Exercise 9

We have

$$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2-4ac}{a^2}.\end{aligned}$$

(Note that, if $a = 1$, then $(\alpha-\beta)^2$ is the discriminant of the quadratic. Similarly, we can define the discriminant of a monic cubic with roots $\alpha, \beta, \gamma$ to be $(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$, and so on for higher degree equations.)

## Exercise 10

The arithmetic mean is $\frac{17}{6}$ and the geometric mean is 2. (Note that the actual roots are $\frac{17\pm\sqrt{145}}{6}$ !)

## Exercise 11

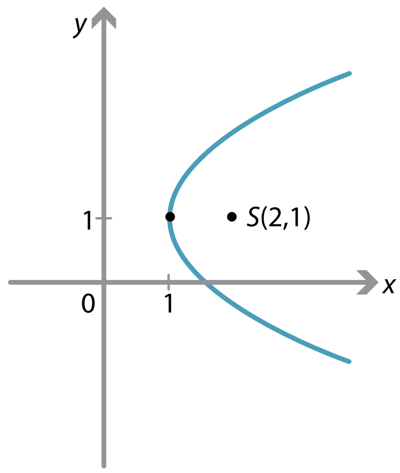
Successively substitute $x = 1$, $x = 2$, $x = 3$ and solve the resulting equations for $a, b, c$ to obtain $x^2 = 3(x-1)^2 - 3(x-2)^2 + (x-3)^2$.

## Exercise 12

Substituting the line into the circle we obtain $x^2-4x+4=0$. The discriminant of this quadratic is 0 and so the line is tangent to the circle at $(2,1)$.

## Exercise 13

The parabola can be written as $(y-1)^2=4(x-1)$. So the vertex is $(1,1)$, the focal length is $a=1$, the focus is $(2,1)$, and the $y$-axis is the directrix.




## Exercise 14

The line $3x+2y=1$.

## Exercise 15

If $PQ$ is a focal chord, then $pq=-1$. Hence we can write the coordinates of $Q$ as $\left(-\frac{2a}{p},\frac{a}{p^2}\right)$. Thus

$$\begin{aligned}
PQ^2 &= \left(2ap+\frac{2a}{p}\right)^2+\left(ap^2-\frac{a}{p^2}\right)^2\\
&= 4a^2p^2+8a^2+\frac{4a^2}{p^2}+a^2p^4-2a^2+\frac{a^2}{p^4}\\
&= a^2\left(p^4+4p^2+6+\frac{4}{p^2}+\frac{1}{p^4}\right)\\
&= a^2\left(p+\frac{1}{p}\right)^4,
\end{aligned}$$

where we recall the numbers $1,4,6,4,1$ from Pascal's triangle and the binomial theorem. So $PQ=a\left(p+\frac{1}{p}\right)^2$.

## Exercise 16

The equations of the normals at $P$ and $Q$ are respectively

$$x + py = 2ap + ap^3 \qquad \text{and} \qquad x + qy = 2aq + aq^3.$$

Subtracting we have

$$y(p-q) = a(p^3 - q^3) + 2a(p-q) = a(p-q)(p^2 + q^2 + pq + 2)$$

so $y = a(p^2 + q^2 + pq + 2)$. Substituting back we have $x = -apq(p+q)$ after simplifying. Hence $R$ is the point $\big(-apq(p+q), a(p^2 + q^2 + pq + 2)\big)$.

Now, if $PQ$ is a focal chord, then $pq = -1$. Thus the $x$- and $y$-coordinates of $R$ become $x = a(p+q)$ and $y = a(p^2+q^2+1) = a\big((p+q)^2+3\big)$, since $pq = -1$. We can then eliminate $p$ and $q$ to obtain

$$y = a\left(\left(\frac{x}{a}\right)^2 + 3\right) = \frac{x^2}{a} + 3a,$$

which is a parabola.

## Exercise 17

Squaring

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2},$$

we have

$$a\sqrt{(x+c)^2 + y^2} = a^2 + xc.$$

Squaring again and rearranging, we arrive at

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2).$$

Dividing by $a^2(a^2 - c^2)$ gives the desired equation.

0 1 2 3 4 5 6 7 8 9 10 11 12