Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Functions: Module 10

Polynomials

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Polynomials – A guide for teachers (Years 11–12)

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Polynomials

Assumed knowledge

- The content of the module *Coordinate geometry*.
- Familiarity with quadratic equations and functions and their graphs.
- The content of the module *Functions II*.
- The content of the module *Introduction to differential calculus*.

Motivation

> Il y a quelque chose à compléter dans cette démonstration. Je n'ai pas le temps … Après cela, il y aura, j'espère, des gens qui trouveront leur profit à déchiffrer tout ce gâchis.
>
> (There is something to complete in this proof. I do not have the time … Later, there will be, I hope, some people who will find it to their advantage to decipher all this mess.)
>
> — Évariste Galois on polynomials, the night before his death, 29 May 1832.

By now we are familiar with linear and quadratic functions, i.e.,

$$f(x) = ax + b \qquad \text{or} \qquad f(x) = ax^2 + bx + c,$$

where a, b, c are real constants. We know how to graph linear functions and how to solve linear equations $ax + b = 0$. We also know how to draw graphs of quadratic functions (which are parabolas) and how to solve quadratic equations $ax^2 + bx + c = 0$.

It's then natural to consider a slightly more complicated function, with just one more term, an x^3 term,

$$f(x) = ax^3 + bx^2 + cx + d.$$

Here a, b, c, d are real constants. The situation now becomes a bit more complicated. A cubic function's graph is not always of the same shape, and a cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

is more difficult to solve than a quadratic one. In secondary school mathematics such equations can only be solved in special cases. (We will give a sketch of a general method in the *Links forward* section.)

But why stop at cubic functions? Why not add an x^4 term? Why not an $x^{3095283}$ term? In general, if we allow any terms of the type ax^n, where a is a real constant and n a non-negative integer, then the functions we obtain are called **polynomial functions**. A general expression for a polynomial function is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0.$$

The highest power n occurring is called the **degree**, and $a_0, a_1, \dots, a_n$ are real numbers called the **coefficients**.

Some examples of polynomials are the functions

$$f(x) = 3x^3 - \frac{11}{3}x + 42, \qquad g(x) = \sqrt{7}x^{3982} + 5x^{13} - 8 \qquad \text{and} \qquad h(x) = 6.$$

Polynomials are a natural type of function to consider: a generalisation of linear, quadratic and cubic functions. They can sometimes be solved exactly. Their graphs can be sketched. Many quantities in the real world are related by polynomial functions.

Importantly, any smooth function can be *approximated* by polynomials. So proficiency in dealing with polynomials is also important.

The reader should be aware of the module *Polynomials* for Years 9–10, which provides useful revision of some concepts in polynomials, and covers some interesting related topics. This module is designed to complement the Years 9–10 module.

Content

Some jargon

We start by defining some words used to describe interesting bits and pieces of a polynomial. Take a general polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \dots, a_0$ are real numbers with $a_n \neq 0$. We have already defined the degree n and the coefficients $a_0, a_1, \dots, a_n$. The coefficient of x^k is a_k. The highest degree term $a_n x^n$ is called the **leading term** and its coefficient a_n is called the **leading coefficient**. If the leading coefficient is 1, the polynomial is called **monic**. The term a_0 is called the **constant term**.

Polynomials of degree 1, 2, 3, 4, 5 are respectively called **linear**, **quadratic**, **cubic**, **quartic** and **quintic**. A degree-zero polynomial is just a constant function, such as $f(x) = 3$.

A **root** or a **zero** of a polynomial $f(x)$ is a number r such that $f(r) = 0$.

Example

Let $f(x) = 18x^4 - 11x^3 - 7x + 12$. What is this polynomial's degree, leading term, leading coefficient, coefficient of x^3, coefficient of x^2, and constant term? Is $f(x)$ monic?

Solution

The degree of $f(x)$ is 4. The leading term is $18x^4$ and the leading coefficient is 18. The coefficient of x^3 is -11. The coefficient of x^2 is 0. The constant term is 12. As the leading coefficient is 18 (not 1), the polynomial $f(x)$ is not monic.

A note of caution. Some of the coefficients may be zero. In the above example, the coefficient of x^2 was zero; no x^2 term is written. However the leading coefficient is always non-zero, as it's the coefficient of the highest power of x which actually appears.

Exercise 1

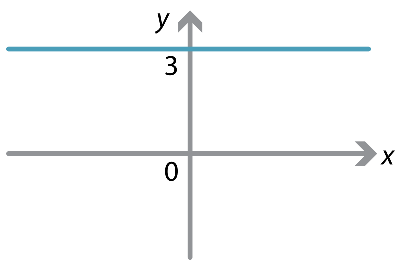
Show that 3 is a root of the polynomial $2x^3 - 8x^2 + 7x - 3$.

Polynomial function gallery

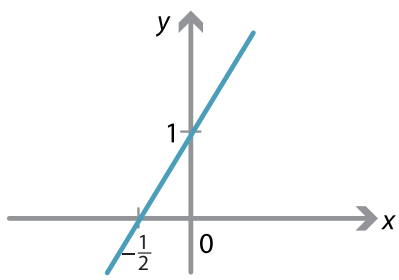
We can draw the graph of a polynomial function $f(x)$ by plotting all points (x, y) in the Cartesian plane with y-value given by $f(x)$. In other words, we draw the graph of the equation $y = f(x)$.

We will examine some graphs of polynomial functions. We'll start from simpler, low-degree polynomials, making some observations as we go.

We are very familiar with graphing polynomials of degree 0 or 1, i.e., constant or linear functions.



Graph of $f(x) = 3$.

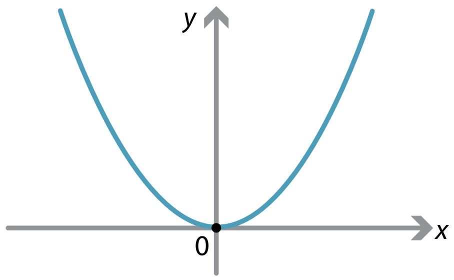


Graph of $f(x) = 2x + 1$.

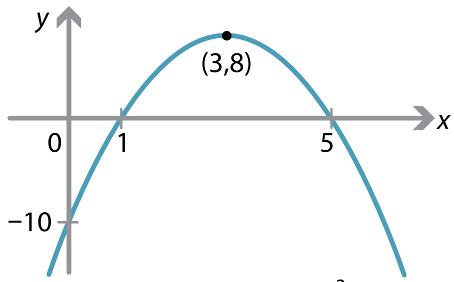
Quadratics

We should also be comfortable with graphing polynomials of degree 2, i.e., quadratic functions. We know that we can always complete the square, so we can rewrite a quadratic function $f(x) = ax^2 + bx + c$ as $a(x-h)^2 + k$ for some constants h and k. (Can you find a formula for h and k in terms of a, b, c?) As we know from the module *Functions II*, the graph of $y = a(x-h)^2 + k$ can be obtained from the standard parabola $y = x^2$ by reflections, dilations and translations. In other words, any quadratic graph is a standard parabola, suitably reflected, dilated and translated.

In particular, the turning point of $y = a(x-h)^2 + k$ is at (h, k).



Graph of $f(x) = x^2$.



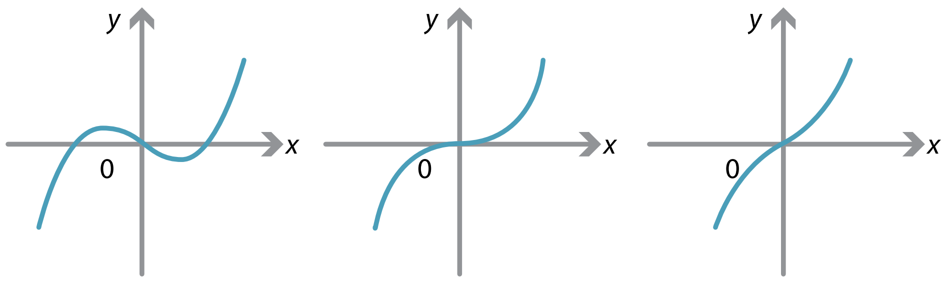
Graph of $f(x) = -2(x-3)^2 + 8$.

The sign of a (whether positive or negative) determines the general shape of the graph. When a is positive, then as $x \to \pm\infty$, we see that $y \to \infty$, so the graph is curved like a smile ('happy'). If you imagine the graph as a road, and driving along it in the positive x-direction, you would always be turning left. (In the language of calculus, the derivative is increasing.) We say the graph is **convex** (or 'concave up').

On the other hand, when a is negative, then as $x \to \pm\infty$, we have $y \to -\infty$, so the graph is curved like a frown ('sad'). Driving along it in the positive x-direction, you would always be turning right, and the graph is **concave** (or 'concave down').

Cubics

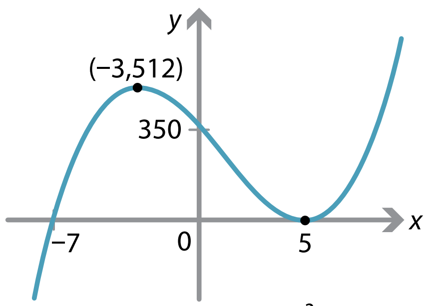
As we move from quadratic to cubic polynomials, things become more complicated. Unlike the situation with quadratics, not every cubic graph is obtained from the standard cubic graph $y = x^3$ by reflections, dilations and translations. Indeed, if we consider three very simple cubic polynomials $x^3 - x$, x^3 and $x^3 + x$, we get three distinct shapes.



Graph of $f(x) = x^3 - x$. Graph of $f(x) = x^3$. Graph of $f(x) = x^3 + x$.

Note in particular the graph of $y = x^3 + x$, which has no turning points and no stationary points of inflexion. This example shows that it's possible for a cubic graph to have no stationary points at all.

The graph of a cubic polynomial may have one, two or three x-intercepts. The examples above have one and three intercepts; below is an example with two of them.



Graph of $f(x) = 2(x-5)^2(x+7)$.

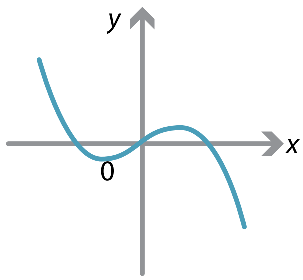
Note. On our graphs we mark turning points, stationary points of inflexion and intercepts where appropriate. We will have more to say about finding these points later on.

Exercise 2

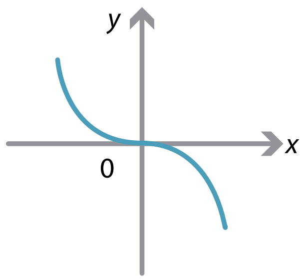
Using calculus, check that the turning points on the above graph are correct.

The behaviour of the above cubic graphs $y = f(x)$ is not so different from that of $y = x^3$, at least when x is large. If x is a large positive number, then $f(x)$ is also a large positive number; if x is a large negative number, then $f(x)$ is also a large negative number. Asymptotically, $f(x)$ behaves similarly to x^3.

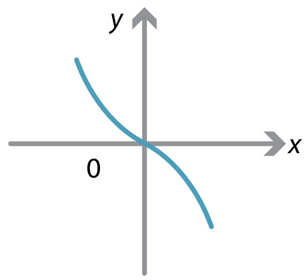
However, in all the examples seen so far, the leading coefficient (i.e., the coefficient of x^3) has been positive. In the case that the leading coefficient is negative, the behaviour for large x is reversed. If x is a large positive number, then $f(x)$ is a large *negative* number; if x is a large negative number, then $f(x)$ is a large *positive* number. Below we graph the cubic polynomials $-x^3+x$, $-x^3$ and $-x^3-x$, which all have a negative leading coefficient. Asymptotically, these cubic polynomials behave similarly to $-x^3$.



Graph of $f(x)=-x^3+x$.



Graph of $f(x)=-x^3$.



Graph of $f(x)=-x^3-x$.

It appears, then, that the leading term of a cubic polynomial determines its behaviour for large x. In particular, a cubic graph goes to $-\infty$ in one direction and $+\infty$ in the other. So it must cross the x-axis at least once.

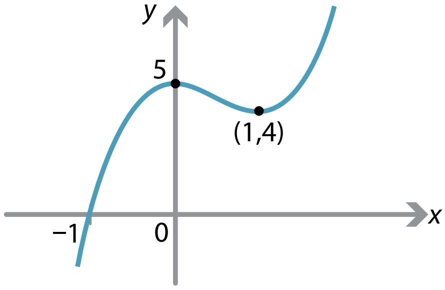
Furthermore, all the examples of cubic graphs have precisely zero or two turning points, an *even* number. (One way to see why: Think about moving along a cubic graph in the positive x-direction. Either you go up all the way, or you go down all the way, or you go up-down-up, or down-up-down. That's zero or two turns. Indeed, if you start going upwards, and you end going upwards, then you must have turned an even number of times.)

Exercise 3

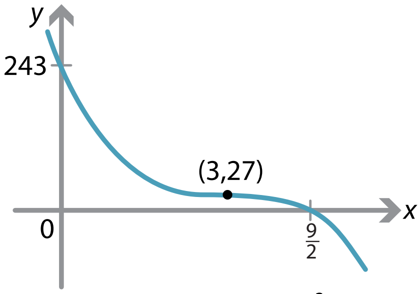
As mentioned earlier, not all cubic polynomial graphs are obtained by reflecting, dilating and translating the standard cubic $y=x^3$. Equivalently, not all cubic polynomials are of the form $f(x)=a(x-h)^3+k$. This exercise explains why, purely algebraically.

- **a** Explain how the graph of $y=a(x-h)^3+k$ is related to the standard cubic graph $y=x^3$.
- **b** (Harder.) Show that if a cubic polynomial $f(x)=ax^3+bx^2+cx+d$ can be rewritten in the form $f(x)=a(x-h)^3+k$, then $b^2-3ac=0$.
- **c** Find an example of a cubic polynomial $f(x)=ax^3+bx^2+cx+d$ where $b^2-3ac\neq 0$.

In fact, when looking at the graph of a cubic polynomial $f(x)=ax^3+bx^2+cx+d$, the number b^2-3ac tells us about the shape of the graph. Depending on whether this number is negative, zero or positive, the graph has zero, one or two stationary points. (We'll see why a little later on.)



Graph of $f(x) = 2x^3 - 3x^2 + 5$.
$b^2 - 3ac = 9$

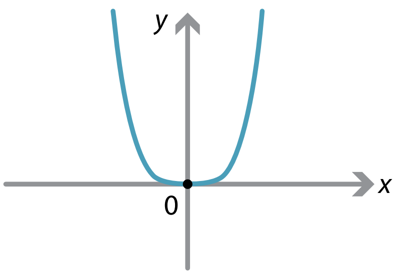


Graph of $f(x) = -8(x-3)^3 + 27$.
$b^2 - 3ac = 0$

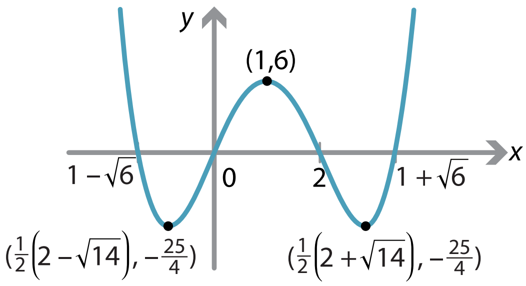
Notice that all the cubic graphs we have seen are quite symmetric. There is always a point on the graph about which the graph is symmetric under 180° rotation. This is true in general.

Quartics

We turn next to quartic graphs, where an even wider variety of shapes is possible. First, consider the graph of the standard quartic $f(x) = x^4$, which is something like a steeper and flatter version of a parabola, and the graph of $f(x) = x^4 - 4x^3 - x^2 + 10x$, which has four x-intercepts.

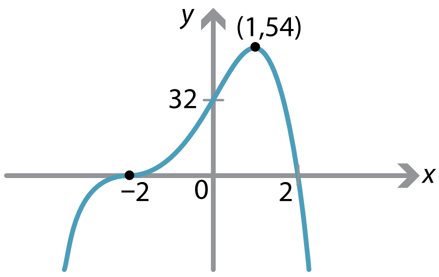


Graph of $f(x) = x^4$.

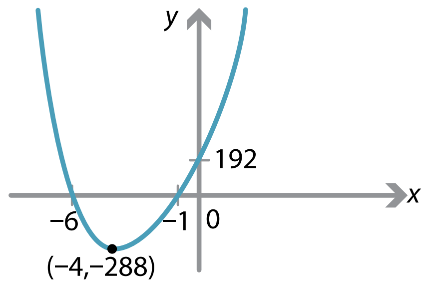


Graph of $f(x) = x^4 - 4x^3 - x^2 + 10x$.

Note that both the graphs above are rather symmetric: they have vertical axes of symmetry at $x = 0$ and $x = 1$, respectively. (Compare the 180° rotational symmetry of cubic graphs.) However, not all quartic graphs are so symmetric. For instance, consider the following graphs of $f(x) = -2(x+2)^3(x-2)$ and $f(x) = (x+1)(x+6)(x^2+32)$.

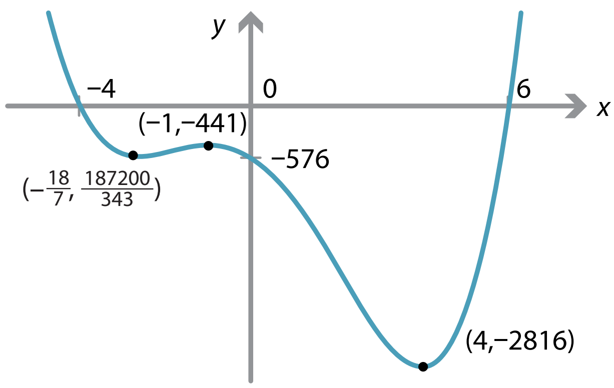


Graph of $f(x) = -2(x+2)^3(x-2)$.



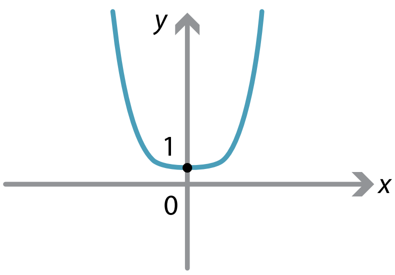
Graph of $f(x) = (x+1)(x+6)(x^2+32)$.

Both these graphs have two x-intercepts and only one turning point. The first has a stationary point of inflexion; the second does not. We can also obtain asymmetric W-shaped graphs, such as the following.

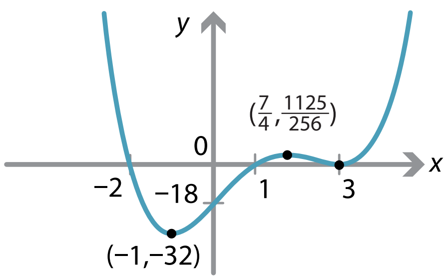


Graph of $f(x) = (x-6)(x+4)(7x^2+10x+24)$.

So far, we have seen quartic graphs with one, two or four x-intercepts. It's also possible to have zero or three x-intercepts, as shown below. We'll see that it's impossible to obtain five or more x-intercepts.



Graph of $f(x) = x^4 + 1$.



Graph of $f(x) = (x-1)(x+2)(x-3)^2$.

All these quartics, for large x, behave roughly like $y = x^4$ or $y = -x^4$ depending on the leading coefficient. When the leading coefficient is positive, then $f(x) \to \infty$ as x becomes large (positive or negative). When the leading coefficient is negative, then $f(x) \to -\infty$ as x becomes large. Again the leading term determines the asymptotic behaviour.

Higher degree

As we consider quintics and higher, we will see more turning points, more x-intercepts and more points of inflexion. Let us summarise some of the observations we have made.

Graphs of polynomials

Type of polynomial	Number of x-intercepts	Number of turning points
linear	1	0
quadratic	from 0 to 2	1
cubic	from 1 to 3	0 or 2
quartic	from 0 to 4	1 or 3

We might make several conjectures based on these observations. We'll see why these are all true as we proceed. You can formulate more conjectures as well.

Conjectures

1 The graph of a polynomial of degree n has at most n x-intercepts.

2 The graph of a polynomial of odd degree has at least one x-intercept.

3 The graph of a polynomial of degree n has at most $n-1$ turning points.

4 The graph of a polynomial of even degree has at least one turning point.

5 The graph of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ behaves asymptotically like its leading term $a_n x^n$. That is, when x is large (approaching $+\infty$ or $-\infty$), $f(x)$ has the same behaviour (approaching $+\infty$ or $-\infty$) as $a_n x^n$.

Exercise 4

Another possible conjecture is that the graph of a polynomial of even degree has an odd number of turning points, while the graph of a polynomial of odd degree has an even number of turning points. Assuming the above conjectures, explain why this is true.

Solving polynomial equations

Often we need to find the solutions of an equation $f(x) = 0$, where $f(x)$ is a polynomial, i.e., the roots of $f(x)$. This amounts to finding the x-intercepts of the graph $y = f(x)$.

When f is linear or quadratic, solving $f(x) = 0$ amounts to solving a linear or quadratic equation. We know how to solve these.

For cubics and higher, the question is more difficult. There is a (complicated!) formula for solving cubic equations, containing many $+$, $-$, $\times$, $\div$ and **radical signs** (i.e., $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc.). There is also a (monstrously complicated!) formula for quartic equations. However, *there is no such formula for a solution of a general quintic or higher degree equation.* This astonishing fact was proved by Niels Heinrik Abel in 1824, and independently by the extraordinary French mathematician Évariste Galois quoted at the beginning of this module. (For more on the brief, tumultuous and tragic life of Galois, see the *History* section of this module.) Think about it: How would you go about *proving* that there is no formula to solve an equation?

The insolvability of quintic polynomials is a university-level topic, although we shall say something about it in the *Links forward* section. In secondary school mathematics, the only higher degree polynomial equations encountered are ones with simple solutions.

One simple higher degree polynomial equation is $x^5 = 2$, which obviously has solution $x = \sqrt[5]{2}$. Another type of simple polynomial equation is found in the following exercise.

Exercise 5

Solve the equation $x^4 - 5x^2 + 6 = 0$.

For other simple polynomial equations, solutions can be found by educated trial and error. Our next task is to learn the tricks for educated trial and error.

Solving by factorising

Let's suppose we are asked to solve the cubic equation $x^3 - 4x^2 + 2x + 3 = 0$. So we let $f(x) = x^3 - 4x^2 + 2x + 3$, and we seek the roots of $f(x)$. We begin finding solutions by trial and error. Of course there are many guesses we could make!

The trick is to guess *factors of the constant term.* Here the constant term is 3, so we consider factors of 3. We must be careful to check both the *positive* and *negative* factors of 3, so we check 1, −1, 3, −3:

$$f(1) = 1^3 - 4\cdot 1^2 + 2\cdot 1 + 3 = 1 - 4 + 2 + 3 = 2,$$

$$f(-1) = (-1)^3 - 4(-1)^2 + 2(-1) + 3 = -1 - 4 - 2 + 3 = -4,$$

$$f(3) = 3^3 - 4\cdot 3^2 + 2\cdot 3 + 3 = 27 - 36 + 6 + 3 = 0,$$

$$f(-3) = (-3)^3 - 4(-3)^2 + 2(-3) + 3 = -27 - 36 - 6 + 3 = -66.$$

We have found a solution $x = 3$.

Now, *as soon as we have a solution, we can factorise* $f(x)$. Specifically, when we have a solution $x = a$, then $(x - a)$ is a factor of $f(x)$. In the present case, $x = 3$ is a solution, so $(x-3)$ is a factor. We can use polynomial division to do this factorisation.

$$
\begin{array}{rrrrrrrrrrr}
 & & & x^2 & - & x & - & 1 & & & \\
\cline{4-11}
x & - & 3 & \multicolumn{1}{|r}{x^3} & - & 4x^2 & + & 2x & + & 3 & \\
 & & & x^3 & - & 3x^2 & & & & & \\
\cline{4-6}
 & & & & & -x^2 & + & 2x & + & 3 & \\
 & & & & & -x^2 & + & 3x & & & \\
\cline{6-8}
 & & & & & & & -x & + & 3 & \\
 & & & & & & & -x & + & 3 & \\
\cline{8-10}
 & & & & & & & & & 0 &
\end{array}
$$

(Recall how polynomial long division works, following the above example. We start from the $x-3 \mid x^3 - 4x^2 + 2x + 3$. Dividing the x into the leading x^3 gives x^2. Multiplying down x^2 by $x-3$ gives $x^3 - 3x^2$, which we then subtract from $x^3 - 4x^2 + 2x + 3$ to obtain $-x^2 + 2x + 3$. Dividing x into $-x^2$ gives $-x$; then multiplying down $-x$ by $x-3$ gives $-x^2 + 3x$. We proceed until we arrive at the remainder of 0.)

Thus $f(x) = (x-3)(x^2 - x - 1)$ and we must solve $(x-3)(x^2 - x - 1) = 0$. If the product of two numbers is zero, then one of them must be zero. As $x - 3 = 0$ only has the solution $x = 3$, it remains to solve $x^2 - x - 1 = 0$, which is just a quadratic equation. Completing the square we have

$$\left(x - \frac{1}{2}\right)^2 - \frac{5}{4} = 0, \qquad \text{so} \quad x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

We now have all solutions to $f(x) = 0$. The complete set of solutions is

$$x = 3, \quad \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad \text{or} \quad \frac{1}{2} - \frac{\sqrt{5}}{2}.$$

To make clear how the ‘tricks’ used here work, we state them as theorems.

Theorem

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial where all the coefficients $a_0, \ldots, a_n$ are integers. Then any integer solution to the equation $f(x) = 0$ must be a factor (positive or negative) of the constant term a_0.

Proof

Suppose k is an integer solution, so $f(k) = 0$. Thus we have

$$a_n k^n + a_{n-1} k^{n-1} + \cdots + a_1 k + a_0 = 0,$$

and each a_i is an integer. Now every term with a k in it is a multiple of k — that's every term on the left-hand side other than the constant term. But the terms on the left-hand side have to add up to 0, so the final term a_0 must be a multiple of k as well. That is, k is a factor of a_0. □

Theorem (Factor theorem)

Let $f(x)$ be a polynomial with real coefficients. If a is a real number such that $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Proof

We can perform polynomial division, dividing $f(x)$ by $(x - a)$, to obtain a quotient $q(x)$ and a remainder r which is just a constant. (When you divide by a linear polynomial, you get a constant remainder.) This means that

$$f(x) = (x - a)q(x) + r.$$

Now substituting $x = a$ into the above gives $f(a) = 0 + r$. As $f(a) = 0$, this means the remainder r is 0. Thus $f(x) = (x - a)q(x)$, and $(x - a)$ is a factor of $f(x)$. □

Exercise 6

Solve the equation $x^3 + 2x^2 + 3x + 6 = 0$.

Exercise 7

Prove the following corollary of the factor theorem: Let $f(x)$ be a polynomial with real coefficients. If $r_1, r_2, \dots, r_k$ are distinct real numbers, each of which is a zero of $f(x)$, then the polynomial $(x - r_1)(x - r_2)\cdots(x - r_k)$ is a factor of $f(x)$.

Number of solutions

We know that a linear equation $ax + b = 0$ always has one solution $x = -\frac{b}{a}$. On the other hand, a quadratic equation $ax^2 + bx + c = 0$ may have zero, one or two real solutions.

For instance, the equation $x^2 + 1 = 0$ has no real solutions; while $x^2 - 2x + 1 = 0$ has only one real solution ('repeated twice') since it factorises to $(x - 1)^2 = 0$.

The quadratic formula tells us that the number of solutions of a quadratic equation $ax^2 + bx + c = 0$ is determined by the **discriminant** $b^2 - 4ac$. There are zero, one or two solutions depending on whether $b^2 - 4ac$ is negative, zero or positive.

For general polynomials, we can state the following.

Theorem

Let $f(x)$ be a polynomial of degree n. Then $f(x) = 0$ has at most n distinct real solutions.

Proof

Suppose instead that there were more than n distinct solutions. Let $n+1$ of these distinct solutions be $r_1, r_2, \dots, r_{n+1}$. It follows from the factor theorem that the polynomial $(x - r_1)(x - r_2)\cdots(x - r_{n+1})$ is a factor of $f(x)$. (See Exercise 7.) Thus

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_{n+1})q(x),$$

for some polynomial $q(x)$. But now the right-hand side has degree at least $n+1$, while $f(x)$ has degree n. This is a contradiction, and so $f(x)$ can have at most n distinct solutions. $\square$

The above theorem is equivalent to saying that the graph $y = f(x)$ of a polynomial $f(x)$ of degree n has at most n x-intercepts. So we have proved Conjecture 1 from the previous section.

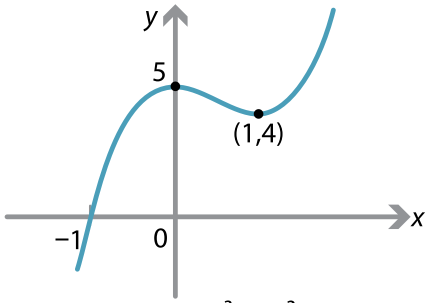
It turns out that if we allow square roots of negative numbers, leading to the *complex numbers*, then we can always find n solutions (counting multiple roots) to an equation of degree n. This is called the *fundamental theorem of algebra*. See the *Appendix* for details.

Behaviour of polynomials at infinity

Understanding the behaviour of a polynomial function $f(x)$ when x is large (that is, as $x \to \pm\infty$) helps us to sketch the graph of $y = f(x)$.

We've seen from examples that, for polynomial functions up to degree 4, the graph of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ behaves asymptotically like its leading term $a_n x^n$.

Let's consider how this behaviour arises. Take a cubic polynomial that we saw earlier, $f(x) = 2x^3 - 3x^2 + 5$.



Graph of $f(x) = 2x^3 - 3x^2 + 5$.

When x is large (positive or negative), x^2 is much larger again, and x^3 dwarfs them both. So the x^3 term will 'dominate' the others. One way to see this algebraically is to write

$$f(x) = 2x^3 - 3x^2 + 5 = 2x^3\Big(1 - \frac{3}{2x} + \frac{5}{2x^3}\Big).$$

When x is large, both $\frac{3}{2x}$ and $\frac{5}{2x^3}$ become very small, and so effectively the dominant term is $2x^3$. This explains why, as seen on the graph,

$$f(x) \to +\infty \text{ as } x \to +\infty, \qquad \text{and} \qquad f(x) \to -\infty \text{ as } x \to -\infty.$$

A similar argument applies to any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$: for large x, the $a_n x^n$ term dominates the others.

Looking at the asymptotic behaviour, we can say that the graph $y = x^n$ is roughly U-shaped when n is even, in the sense that $x^n \to +\infty$ as $x \to \pm\infty$. The actual graph may have many turning points, but if we imagine 'zooming out' and looking only at the large-scale picture when x is large, the graph is U-shaped. In a similar way, the graph of $y = x^n$ is roughly /-shaped when n is odd. The behaviour of a polynomial graph $y = f(x)$ when x is large will depend on whether the degree of $f(x)$ is even or odd, and on the sign of the leading coefficient a_n. We can summarise the outcomes, along with rough shapes of graphs when x is large, in a table.

Graph of a polynomial $f(x)$ with leading term $a_n x^n$

If n is	and a_n is	then as $x \to +\infty$	and as $x \to -\infty$	so shape is roughly
even	positive	$f(x) \to +\infty$	$f(x) \to +\infty$	$\cup$
even	negative	$f(x) \to -\infty$	$f(x) \to -\infty$	$\cap$
odd	positive	$f(x) \to +\infty$	$f(x) \to -\infty$	/
odd	negative	$f(x) \to -\infty$	$f(x) \to +\infty$	\

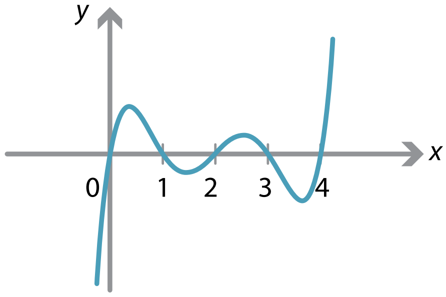
This confirms our Conjecture 5.

Example

Sketch the graph of $y = x(x-1)(x-2)(x-3)(x-4)$, marking all intercepts (but not turning points).

Solution

It's clear that the polynomial $f(x) = x(x-1)(x-2)(x-3)(x-4)$ has roots at $x = 0, 1, 2, 3, 4$, and so these are the x-intercepts. The y-intercept is 0. Also, $f(x)$ is a quintic polynomial with leading term x^5. (This can be seen without fully expanding out the brackets.) Hence $f(x)$ behaves like x^5, for large x: as $x \to +\infty$, $f(x) \to +\infty$, and as $x \to -\infty$, $f(x) \to -\infty$. This is enough information to sketch the graph.



Graph of $f(x) = x(x-1)(x-2)(x-3)(x-4)$.

Finally, we can use the asymptotic behaviour of a polynomial $f(x)$ to obtain information about its x-intercepts. When we have an odd-degree polynomial, the graph of $y = f(x)$ must go from $-\infty$ to $+\infty$, or vice versa. Therefore the graph of $y = f(x)$ must cross the x-axis, giving at least one x-intercept. This proves that the graph of a polynomial of odd degree has at least one x-intercept, confirming Conjecture 2.

Stationary points

The **stationary points** of a graph $y = f(x)$ are those points (x, y) on the graph where $f'(x) = 0$. A stationary point can be a turning point or a stationary point of inflexion.

Differentiating the term $a_k x^k$ in a polynomial gives $k a_k x^{k-1}$. So if a polynomial $f(x)$ has degree n, then its derivative $f'(x)$ has degree $n-1$. To find stationary points of $y = f(x)$, we must solve the polynomial equation $f'(x) = 0$ of degree $n-1$.

Take an example from our gallery.

Example

Let $f(x) = 2x^3 - 3x^2 + 5$. Find the stationary points of the graph $y = f(x)$.

Solution

We compute $f'(x) = 6x^2 - 6x$. To find stationary points we solve $6x^2 - 6x = 0$. Factorising to $6x(x-1) = 0$ gives $x = 0$ or $x = 1$. Substituting these values of x gives $f(0) = 5$ and $f(1) = 4$. So the stationary points are $(0,5)$ and $(1,4)$.

Note. Here f has degree 3, its derivative f' has degree 2, and so $f'(x) = 0$ is a quadratic equation.

The following exercise shows that a polynomial graph may have *no* stationary points. In this exercise, the polynomial f again has degree 3 and its derivative f' has degree 2, but the equation $f'(x) = 0$ has no solutions.

Exercise 8

Let $f(x) = x^3 + x - 2$. Show that $f(x)$ has no stationary points.

Next let's consider the number of stationary points of a polynomial graph $y = f(x)$, where $f(x)$ has degree n.

The stationary points are found by solving the equation $f'(x) = 0$, which has degree $n-1$, and hence has at most $n-1$ real solutions. Therefore the graph $y = f(x)$ has at most $n-1$ stationary points. This confirms our Conjecture 3.

Slightly more trickily, if the degree n is *even*, then the degree $n-1$ of the derivative $f'(x)$ is odd. So the graph of $f'(x)$ goes from $-\infty$ to $+\infty$, or vice versa. Therefore the graph of $f'(x)$ crosses the x-axis somewhere, changing sign from positive to negative or from negative to positive. This gives a turning point of $f(x)$. We have now confirmed Conjecture 4: when n is even, the graph of $f(x)$ has at least one turning point.

The next exercise gives a test for the number of stationary points of a cubic polynomial.

Exercise 9

Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are real numbers with $a \neq 0$. Show that:

a If $b^2 - 3ac < 0$, then $y = f(x)$ has no stationary points.

b If $b^2 - 3ac = 0$, then $y = f(x)$ has one stationary point.

c If $b^2 - 3ac > 0$, then $y = f(x)$ has two distinct stationary points.

Sketching polynomial functions

Although polynomial graphs come in many shapes and sizes, they can be sketched once we find a few of their features.

Example

Sketch the graph of $y = f(x)$ where $f(x) = 2x^3 - 6x^2 - 90x + 350$.

Solution

To find the x-intercepts, we solve $f(x) = 2(x^3 - 3x^2 - 45x + 175) = 0$. Trying factors of 175 we find that $x = 5$ is a solution, so $(x-5)$ is a factor. We can then completely factorise $f(x)$ as $2(x-5)^2(x+7)$. So the x-intercepts are at $x = 5, -7$. To find the y-intercept, we compute $f(0) = 350$.

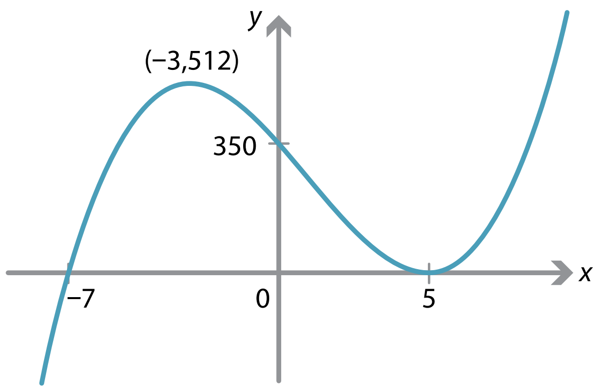
To find the behaviour as $x \to \pm\infty$, note that $f(x)$ behaves like the leading term $2x^3$. So $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

To find the stationary points, we solve $f'(x) = 0$. Differentiating gives

$$f'(x) = 6x^2 - 12x - 90 = 6(x^2 - 2x - 15).$$

Solving $x^2 - 2x - 15 = 0$ gives $x = -3, 5$. Substituting these values into f gives $f(-3) = 512$ and $f(5) = 0$. So the stationary points are $(-3, 512)$ and $(5, 0)$.

From this information, we can sketch the graph of $y = f(x)$.



Although it's unnecessary in the previous example, we could use a **sign diagram** to investigate stationary points. We choose values of x before and after each stationary point, and consider whether $f'(x) = 6(x+3)(x-5)$ is positive or negative, as shown below.

Value of x		-3		5	
Sign of $f'(x)$	$+$	0	$-$	0	$+$
Slope of graph $y = f(x)$	$/$	—	$\backslash$	—	$/$

From the sign diagram we see directly that $x = -3$ is a local maximum and $x = 5$ is a local minimum. However in the example this is deduced from other information.

Exercise 10

Sketch the graph of $y = x^3 - x$.

In summary, the following information may be useful when sketching the graph of a polynomial $y = f(x)$:

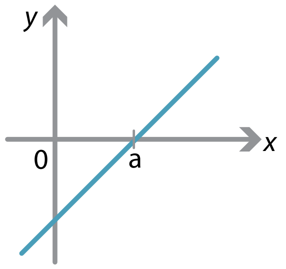
- x-intercepts, obtained by solving $f(x) = 0$
- y-intercept, obtained by evaluating $f(0)$
- behaviour as $x \to \pm\infty$, obtained by considering the leading term of $f(x)$
- stationary points, obtained by solving $f'(x) = 0$
- sign diagram for $f'(x)$, obtained by substituting values into $f'(x)$.

It's not always necessary to calculate all of these; often, as in the previous example, a graph can be sketched with less information.

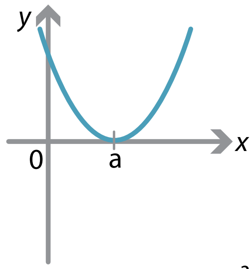
Repeated roots

When a polynomial has a factor $(x-a)$ to a power greater than 1, we say a is a **repeated root**. If $(x-a)^2$ is a factor of $f(x)$, we say a is a **double root** or a **root of multiplicity** 2. In general, if $(x-a)^m$ is a factor of $f(x)$, we say a is a **root of multiplicity** m. A root of multiplicity 1 is often called a **simple root**.

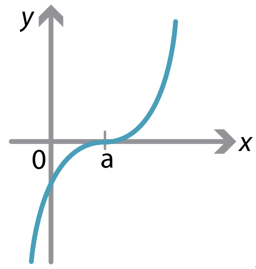
Consider the polynomial $f(x) = (x-a)^m$, where a is a real number and m is a positive integer. Obviously $f(x)$ has a root at $x = a$ of multiplicity m. The following three graphs show $y = f(x)$ when $m = 1, 2, 3$.



Graph of $f(x) = x - a$.



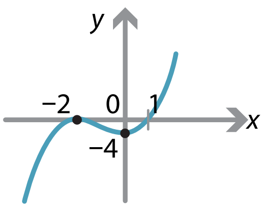
Graph of $f(x) = (x - a)^2$.



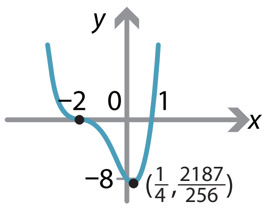
Graph of $f(x) = (x - a)^3$.

An important fact to note is that the *sign* of $f(x)$ changes at $x = a$ when m is odd, and does not change when m is even. (To see this, note that if $x > a$, then $x - a$ is positive and $(x - a)^m$ is also positive. If $x < a$, then $x - a$ is negative; if m is even, then $(x - a)^m$ is positive, while if m is odd, then $(x - a)^m$ is negative.)

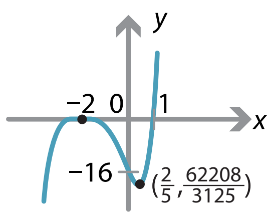
This fact becomes relevant when we consider more general polynomials with repeated roots. Consider the polynomials $f(x) = (x+2)^m(x-1)$, where m is a positive integer. Then $f(x)$ has a root at $x = -2$ of multiplicity m. The graphs for $m = 2, 3, 4$ are shown below.



Graph of
$f(x) = (x + 2)^2(x - 1)$.



Graph of
$f(x) = (x + 2)^3(x - 1)$.



Graph of
$f(x) = (x + 2)^4(x - 1)$.

First consider the polynomial $f(x) = (x + 2)^2(x - 1)$, which has a double root at $x = -2$. Then $f(-2) = 0$. Since the power 2 is even, the factor $(x + 2)^2$ does not change sign at $x = -2$; if x is slightly more or less than -2, then $(x + 2)^2$ is positive. For x close to -2, the factor $x - 1$ is negative, and so $f(x)$ is negative for x slightly more or less than -2. That is, $f(x)$ does not change sign at $x = -2$.

Next, consider $f(x) = (x + 2)^3(x - 1)$, which has a triple root at $x = -2$. As 3 is odd, the factor $(x + 2)^3$ does change sign at $x = -2$. For x near -2, the factor $x - 1$ is negative. So $f(x)$ changes sign at $x = -2$.

The above argument relies only on whether the multiplicity of the root $x = -2$ is even or odd. Using a similar argument, one can prove the following theorem.

Theorem

Let $f(x)$ be a polynomial with a root a of multiplicity m. Then $f(x)$ changes sign at $x = a$ if and only if m is odd.

If a polynomial $f(x)$ has a root at $x = a$ of *even* multiplicity m, then $f(x)$ does not change sign at $x = a$, and yet $f(a) = 0$. Hence there must be a turning point and we obtain the next theorem.

Theorem

Let $f(x)$ be a polynomial with a root a of even multiplicity. Then the graph of $y = f(x)$ has a turning point at $x = a$.

Additionally, we note from the above graphs that a root of multiplicity 2 *or more* always appears to be a stationary point. We can also prove this as a theorem.

Theorem

Let $f(x)$ be a polynomial with a root a of multiplicity 2 or more. Then the graph of $y = f(x)$ has a stationary point at $x = a$.

Proof

Since a has multiplicity at least 2, we know that $(x-a)^2$ is a factor of $f(x)$. (Maybe higher powers of $(x-a)$ divide into $f(x)$, but $(x-a)^2$ certainly does.)

We can then write $f(x) = (x-a)^2 g(x)$ for some polynomial $g(x)$. Differentiate $f(x)$ using the chain and product rules to obtain

$$f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x) = (x-a)\big[2g(x) + (x-a)g'(x)\big].$$

Thus $f'(a) = 0$ and so $x = a$ is a stationary point. $\square$

Exercise 11

Sketch the graph of $y = x^4 - x^2$.

Transformation methods

We've seen from the module *Functions II* that, if we take the graph of $y = f(x)$ and dilate or translate it in the x and y directions, or reflect in the axes, then the result is a graph of $y = af(b(x-c)) + d$, for some real numbers a, b, c, d.

We can sometimes use this idea in reverse, as in the following example. It's easier than using the standard method.

Example

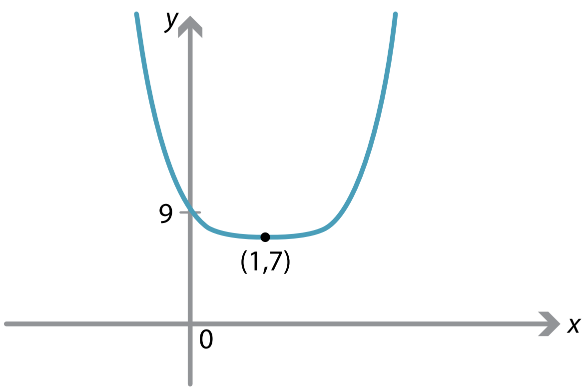
Sketch the graph of $y = 2(x-1)^4 + 7$.

Solution

We see that the graph is obtained from $y = x^4$ by successively:

- dilating by a factor of 2 from the x-axis in the y-direction
- translating 1 to the right
- translating 7 upwards.

Since we know the graph of $y = x^4$, the graph is easily obtained.



Exercise 12

Use the graph of $y = x^5$ to sketch the graph of $y = -4(x-3)^5 - 2$.

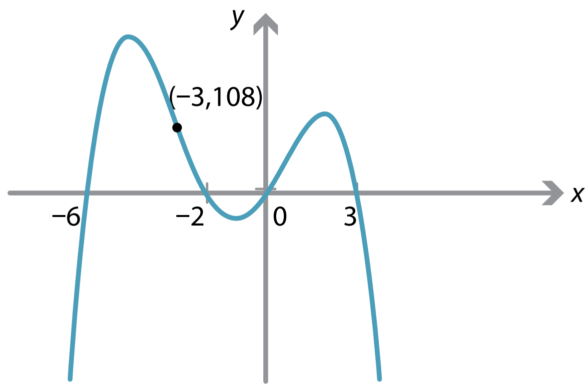
Polynomials from graphs

Given a polynomial, we have seen how to draw its graph. Now we do the reverse: given a graph, find the polynomial.

It is useful to remember that it takes two points to define a line, and three points to define a parabola. In general, it takes $n+1$ points to define a polynomial of degree n. This is not surprising, since a polynomial of degree n has $n+1$ coefficients.

Example

The graph $y = f(x)$ of the quartic polynomial $f(x)$ is drawn below. Find $f(x)$.



(Reality check: Five points are specified in the graph, namely four x-intercepts and one extra point. This is the right number of points to determine a quartic polynomial.)

Solution

The graph shows four x-intercepts at $x = -6, -2, 0, 3$. Hence $(x+6)$, $(x+2)$, x and $(x-3)$ are all factors of $f(x)$. Multiplying these factors together already gives a quartic; the only other possible factor is just a constant. So $f(x) = c(x+6)(x+2)x(x-3)$ for some constant c. Since $f(-3) = 108$, we have $108 = c \cdot 3 \cdot (-1) \cdot (-3) \cdot (-6)$, so $108 = -54c$ and $c = -2$. Thus $f(x) = -2(x+6)(x+2)x(x-3)$.

The above example demonstrates that knowing the roots is not enough information to determine a polynomial. Indeed, in the example, there is a whole family of polynomials $f(x) = c(x+6)(x+2)x(x-3)$ with the same roots, where c is any (non-zero) real number. We needed the extra data of a point on the curve to determine c.

In general:

- An x-intercept at $x = a$ implies that $(x-a)$ is a factor of $f(x)$.
- A y-intercept at $y = c$ implies that $f(0) = c$, i.e., the constant term in $f(x)$ is c.
- A given point (p,q) on the graph implies that $f(p) = q$.

If we are just given points on the graph of $y = f(x)$, one way to find the polynomial is by simultaneous equations, as the following example shows.

Example

Find the quadratic polynomial $f(x)$ whose graph passes through the three points $(1,-2)$, $(2,-1)$ and $(3,2)$.

Solution

Let the polynomial be $f(x) = ax^2 + bx + c$. We are given that $f(1) = -2$, $f(2) = -1$ and $f(3) = 2$, so we have the simultaneous equations

$$a + b + c = -2 \qquad (1)$$

$$4a + 2b + c = -1 \qquad (2)$$

$$9a + 3b + c = 2. \qquad (3)$$

Subtracting (2) − (1) and (3) − (2) gives $3a + b = 1$ and $5a + b = 3$. Subtracting these two equations gives $2a = 2$, so $a = 1$. Substituting $a = 1$ back into these equations gives $b = -2$ and then $c = -1$. So $f(x) = x^2 - 2x - 1$.

In principle we could use the above method to find, say, a quintic through six given points, or a degree-100 polynomial through 101 given points; it just becomes increasingly tedious to eliminate the variables and solve the simultaneous equations!

As it turns out (and beyond the secondary school curriculum), there is actually a formula for the polynomial through a given set of points, called the *Lagrange interpolation formula*, discussed in the *Appendix*.

Links forward

Approximating functions

If we are given a function $f(x)$, we can try to find a polynomial $p(x)$ which *approximates* that function. Indeed, if we know some of the values of f,

$$f(x_1) = y_1, \qquad f(x_2) = y_2, \qquad \dots, \qquad f(x_n) = y_n,$$

then we can find a polynomial $p(x)$ which takes those values, as we just saw.

That is one type of approximation. The following example shows another type of approximation: finding a polynomial which agrees with f, *and its derivatives*, at a point.

Example

Find a quadratic polynomial $p(x)$ which approximates the function $f(x) = \cos x$ in the sense that p agrees with f in value, and in its first two derivatives, at $x = 0$, i.e.,

$$p(0) = f(0), \qquad p'(0) = f'(0) \qquad \text{and} \qquad p''(0) = f''(0).$$

(Reality check: We're asked to find a quadratic polynomial satisfying three conditions. This makes sense since a quadratic has three coefficients.)

Solution

First of all we can compute $f'(x) = -\sin x$ and $f''(x) = -\cos x$, so $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -1$. Thus we want to find a quadratic polynomial $p(x)$ such that

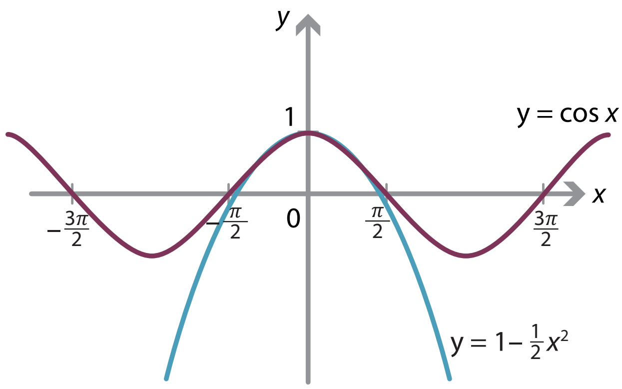
$$p(0) = 1, \qquad p'(0) = 0 \qquad \text{and} \qquad p''(0) = -1.$$

Let $p(x) = ax^2 + bx + c$, where a, b, c are real coefficients. Then we have $p'(x) = 2ax + b$ and $p''(x) = 2a$, so the three equations above become

$$c = 1, \qquad b = 0 \qquad \text{and} \qquad 2a = -1.$$

Hence $p(x) = -\frac{1}{2}x^2 + 1$.

We just showed that $p(x) = 1 - \frac{1}{2}x^2$ is a good approximation to $f(x) = \cos x$ near $x = 0$. A comparison of the graphs shows how good the approximation is.



Quadratic approximation to the function $\cos x$.

If we continue on, we can find a cubic approximation to $\cos x$ which agrees up to the third derivative at $x = 0$, a quartic which agrees up to the fourth derivative, and so on. In this way we get a sequence of polynomials which approximate $\cos x$ increasingly better.

$$\begin{aligned}
\cos x &\sim 1 && \text{‘constant’ approximation} \\
&\sim 1 - \frac{x^2}{2!} && \text{quadratic approximation} \\
&\sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} && \text{quartic approximation} \\
&\sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} && \text{degree-6 approximation}
\end{aligned}$$

Continuing these terms on to infinity results in an *infinite series* approximating $\cos x$. In fact this infinite series *equals* $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

A series like this is known as a **power series**. Many functions can be well approximated by power series. Some are approximated completely and ‘eventually exactly’ like this one. For others, the power series only gives a good approximation in an *interval of convergence*. This topic is studied extensively in university science and engineering courses.

Sums and products of roots

Suppose we have a monic cubic polynomial $f(x)$ with roots 2, 5, 7:

$$f(x) = (x-2)(x-5)(x-7).$$

If we expand out these brackets, we see something interesting:

$$f(x) = x^3 - (2+5+7)x^2 + (2\cdot5 + 2\cdot7 + 5\cdot7)x - 2\cdot5\cdot7.$$

Each coefficient is written in terms of the roots. In particular, the coefficient of x^2 is the (negative) sum of the roots, and the constant term is the (negative) product of the roots.

More generally, suppose we have a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with roots α, β, γ. Then

$$f(x) = c(x-\alpha)(x-\beta)(x-\gamma)$$

for some real number constant c. Expanding this out gives

$$f(x) = c\big[x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\big].$$

Comparing the two expressions for $f(x)$, we obtain

$$a_3 = c, \qquad a_2 = -c(\alpha+\beta+\gamma), \qquad a_1 = c(\alpha\beta+\beta\gamma+\gamma\alpha), \qquad a_0 = -c\alpha\beta\gamma.$$

From these equalities we obtain the following theorem, relating the sums and products of the roots to the coefficients of $f(x)$.

Theorem (Vieta's formulas for cubics)

Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a cubic polynomial with roots α, β, γ. Then

$$\alpha + \beta + \gamma = -\frac{a_2}{a_3}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_1}{a_3}, \qquad \alpha\beta\gamma = -\frac{a_0}{a_3}.$$

Technical notes.

1 The roots of $f(x)$ referred to in this theorem are all the *complex number* roots of $f(x)$. If we restrict attention to real roots, the result is not true.

2 In the theorem we must take all the roots of $f(x)$ with multiplicities.

Example

Let $f(x) = 2x^3 + 3x^2 - 4x + 7$.

1 What is the sum of the roots of $f(x)$?

2 What is the product of the roots of $f(x)$?

Solution

By Vieta's formulas, the sum of the roots is $-\frac{3}{2}$ and the product of the roots is $-\frac{7}{2}$.

(We can also deduce that, if the three roots of $f(x)$ are α, β, γ, then $\alpha\beta + \beta\gamma + \gamma\alpha = -2$.)

Vieta's formulas apply not just to cubics but to polynomials of any degree. For instance, consider the quartic polynomial

$$f(x) = 2(x-1)(x-3)(x-6)(x-7).$$

Expanding out this expression gives

$$f(x) = 2\Big(x^4 - (1+3+6+7)x^3 + (1\cdot3 + 1\cdot6 + 1\cdot7 + 3\cdot6 + 3\cdot7 + 6\cdot7)x^2 \\ - (1\cdot3\cdot6 + 1\cdot3\cdot7 + 1\cdot6\cdot7 + 3\cdot6\cdot7)x + 1\cdot3\cdot6\cdot7\Big).$$

The coefficient of x^3 is $-2(1+3+6+7)$, which is two (the leading coefficient) times the (negative) sum of the roots. And the constant term is two times the product of the roots. So again the coefficients of $f(x)$ can be described in terms of the sums and products of the roots.

For a general polynomial we have the following theorem. The proof is left to you.

Theorem (Vieta's formulas)

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$*. Then*

a *the sum of the roots of* $f(x)$*, counted with multiplicities, is* $-\dfrac{a_{n-1}}{a_n}$*, and*

b *the product of the roots, again counted with multiplicities, is* $(-1)^n \dfrac{a_0}{a_n}$.

Solving cubics

One idea for solving cubic equations is as follows. Suppose we take an expression

$$\sqrt[3]{A} + \sqrt[3]{B}$$

and cube it. We obtain

$$\begin{aligned}\left(\sqrt[3]{A} + \sqrt[3]{B}\right)^3 &= A + 3\left(\sqrt[3]{A}\right)^2 \sqrt[3]{B} + 3\sqrt[3]{A}\left(\sqrt[3]{B}\right)^2 + B \\ &= A + B + 3\sqrt[3]{AB}\left(\sqrt[3]{A} + \sqrt[3]{B}\right).\end{aligned}$$

This says that $\sqrt[3]{A} + \sqrt[3]{B}$ is a solution of the equation

$$x^3 = (A+B) + 3\sqrt[3]{AB}\, x.$$

So, given a cubic equation $x^3 + ax + b = 0$, we can find a solution as follows:

1 rewrite the equation in the form $x^3 = p + qx$

2 find A and B so that $A + B = p$ and $3\sqrt[3]{AB} = q$

3 a solution is then $\sqrt[3]{A} + \sqrt[3]{B}$.

Example

Find a solution to the equation $x^3 - 18x - 30 = 0$.

Solution

Rewriting the equation as $x^3 = 30 + 18x$, we must find A and B such that $A + B = 30$ and $3\sqrt[3]{AB} = 18$, i.e., $AB = 216$. Substituting $B = \frac{216}{A}$ into $A + B = 30$ gives

$$A + \frac{216}{A} = 30, \qquad \text{which is equivalent to} \quad A^2 - 30A + 216 = 0.$$

This quadratic has solutions $A = 12, 18$. So we obtain a solution $A = 12$, $B = 18$, and a solution to the original cubic is then

$$x = \sqrt[3]{12} + \sqrt[3]{18}.$$

We might note that this method only gives us *one* of the roots. However, at least in principle, knowing one root, we can factorise our cubic into linear and quadratic factors, and proceed. (In practice however, this method can lead to severe algebraic complications when there are three real roots, and complex numbers appear.)

Unfortunately, the above method only works when the cubic equation has no x^2 term. However, *you can always make a substitution to get rid of the x^2 term*!

For instance, suppose we want to solve the cubic equation

$$x^3 + 3x^2 - 15x - 47 = 0.$$

Let $z = x + 1$; then the above equation can be written in terms of z:

$$(z-1)^3 + 3(z-1)^2 - 15(z-1) - 47 = z^3 - 18z - 30 = 0.$$

So our equation reduces to the previous example. In general, given a cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

letting

$$z = x + \frac{a}{3}$$

and rewriting in terms of z, the quadratic term will vanish.

Exercise 13

Find a solution to the cubic equation $x^3 - 6x^2 + 27x - 58 = 0$.

Historical note. A general method of solving cubic equations dates back to the work of Tartaglia and Cardano in the early 16th century. It's quite interesting that this work was done a century before Descartes introduced coordinate geometry!

Insolvability of quintics

As mentioned earlier, while there is a general method to find the roots of any polynomial up to degree 4 in terms of radical expressions, there is no such general method for polynomials of degree 5 and above. This is the result of theorems discovered independently by Galois and Abel. The topic is well beyond our current scope but we can make a few comments about it.

The theorems of Galois and Abel are in particular about the *solvability of polynomial equations in radicals*. 'Radical' here refers to the radical sign $\sqrt[n]{\ }$. A **radical expression** is one that can be built out of integers by the usual operations of addition, subtraction, multiplication, division, and radical signs. When we use radical signs, we are allowed to take square roots, cube roots, fourth roots, and so on — any nth root where n is a positive integer. So

$$\frac{3}{\sqrt{7}} + \sqrt[3]{\frac{9-\sqrt{6}}{2+\sqrt[7]{11}+\sqrt[11]{42}} - \sqrt[3]{2}}$$

is an example of a radical expression, but

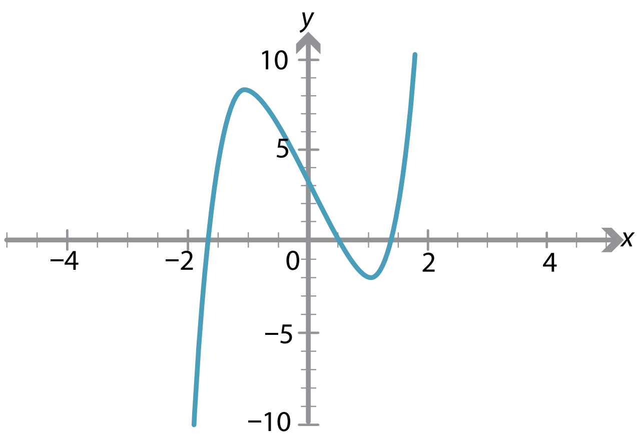
$$\pi + \sqrt{2}, \qquad \log 2, \qquad e^3$$

are not radical expressions. (So-called 'transcendental functions' sin, cos, tan, log, e^x cannot be resolved to radical expressions.) All the solutions we have found to polynomials so far have been radical expressions.

Theorem

The polynomial $x^5 - 6x + 3$ *has no roots which are radical expressions.*

Drawing the graph $y = x^5 - 6x + 3$, we see there are three x-intercepts. There are roots at approximately -1.7, 0.5 and 1.4. We can see the roots on the real number line — yet the theorem states that these numbers are not radical expressions.



This example is taken from Ian Stewart's text *Galois theory*. In fact the same applies to any quintic with rational coefficients and three real roots.

Aside. If we allow the coefficients of our polynomials to be non-radical, then it's easy to make non-radical roots. For example, $f(x) = x - \pi$ has a root at π. Here we are referring to polynomials with integer or rational coefficients.

Appendix

Lagrange interpolation formula

If you are given n points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ (with the x_i all distinct) and want to find the unique polynomial of degree $n-1$ whose graph goes through those points, there is a formula for it, called the *Lagrange interpolation formula*.

The idea behind the Lagrange interpolation formula is quite interesting. We first find a polynomial $p_1(x)$ such that $p_1(x_1) = y_1$, but for which $p_1(x_2) = p_1(x_3) = \cdots = p_1(x_n) = 0$. That is, the graph of $p_1(x)$ goes through (x_1, y_1) but has x-intercepts at $x_2, x_3, \dots, x_n$: we single out the first point and get $p_1(x)$ to go through it. Then we find a polynomial $p_2(x)$ which goes through (x_2, y_2) but has x-intercepts at all the other points $x_1, x_3, x_4, \dots, x_n$. And then a $p_3(x)$ which goes through (x_3, y_3) and so on. The trick is that, once we have worked out all of these polynomials going through one point each, *we can just add them up*. That is, we can take $f(x) = p_1(x) + p_2(x) + \cdots + p_n(x)$ and then $f(x_1) = y_1$, $f(x_2) = y_2$, $\dots$, $f(x_n) = y_n$ as desired.

Now, let's find $p_1(x)$. As in the previous examples, the x-intercepts at $x_2, x_3, \dots, x_n$ tell us factors. We have

$$p_1(x) = c(x - x_2)(x - x_3)\cdots(x - x_n),$$

where c is a constant. Since $p_1(x_1) = y_1$, we can work out c:

$$y_1 = c(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n), \qquad \text{so} \quad c = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n)}.$$

A formula for $p_1(x)$ is then, using product notation,

$$p_1(x) = \frac{y_1(x - x_2)(x - x_3)\cdots(x - x_n)}{(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n)} = y_1 \prod_{j \neq 1} \frac{x - x_j}{x_1 - x_j}.$$

To get the other polynomials $p_k(x)$ going through (x_k, y_k), we can use exactly the same method and we end up with a similar formula

$$p_k(x) = y_k \prod_{j \neq k} \frac{x - x_j}{x_k - x_j},$$

and adding these up to obtain $f(x)$ gives us the **Lagrange interpolation formula**

$$f(x) = \sum_{k=1}^{n} y_k \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}.$$

Rational root test

We saw in the section *Solving polynomial equations* that, to find the roots of a polynomial (with integer coefficients), a good place to look is the factors of the constant term. In fact, we proved that any integer root must be a factor of the constant term.

As long as we're prepared to try a few more possibilities, we can determine whether there are any roots which are *rational numbers* (i.e., fractions).

Theorem (Rational root test)

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. If $\dfrac{r}{s}$ is a rational root of $f(x)$, and $\dfrac{r}{s}$ is in simplest form, then r is a factor of a_0 and s is a factor of a_n.

Proof

Since $f\left(\frac{r}{s}\right) = 0$, we have

$$a_n\left(\frac{r}{s}\right)^n + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + a_{n-2}\left(\frac{r}{s}\right)^{n-2} + \cdots + a_1\left(\frac{r}{s}\right) + a_0 = 0.$$

Multiplying through by s^n gives

$$a_n r^n + a_{n-1} r^{n-1} s + a_{n-2} r^{n-2} s^2 + \cdots + a_2 r^2 s^{n-2} + a_1 r s^{n-1} + a_0 s^n = 0.$$

Examining this equation, we see that every term on the left-hand side except the last is divisible by r; so we have a multiple of r, plus $a_0 s^n$, equal to 0. Hence $a_0 s^n$ is a multiple of r; equivalently, r is a factor of $a_0 s^n$. But, as $\frac{r}{s}$ is in simplest form, r and s have no factor in common; hence r is a factor of a_0.

Similarly, every term on the left-hand side except the first is divisible by s, so $a_n r^n$ is also divisible by s; equivalently, s is a factor of $a_n r^n$. Again, since r and s have no common factor, s is a factor of a_n. $\square$

Fundamental theorem of algebra

We mentioned previously that, when we extend to *complex numbers*, all polynomial equations have a solution. This fact, the **fundamental theorem of algebra**, is proved in university-level mathematics courses, but we can say something about it here.

There are many different types of numbers. Perhaps the simplest is the set of **natural numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$. (For some mathematicians $\mathbb{N}$ also includes zero.) The natural numbers are good for counting, but not so good to solve an equation like $x + 2 = 0$. To solve that equation, we have to introduce a new exotic type of number called a *negative* number. When we add in these new numbers, we obtain a number system called the **integers** $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

However, if we ever come across an equation like $2x - 1 = 0$, the integers will not be sufficient to solve it. We will need to invent another exotic species of number called a *fraction* or *rational number* to solve it. The **rational numbers** are

$$\mathbb{Q} = \left\{ \frac{p}{q} \;:\; p, q \text{ are integers with } q \neq 0 \right\}.$$

This extension of the number system, however, is still not sufficient to solve an equation like $x^2 - 2 = 0$; as it turns out, $\sqrt{2}$ is not a rational number. The number system is thus extended to include *irrational numbers*, and we end up with an enormous number system called the **real numbers**.

Still, the real numbers are not enough to solve such an innocent quadratic equation as $x^2 + 1 = 0$. In a similar vein, we might add in a new number i such that $i^2 = -1$; so 'i is the square root of -1'. The numbers so obtained are called the **complex numbers**,

$$\mathbb{C} = \{ a + bi \;:\; a, b \text{ are real numbers} \}.$$

The amazing fact about the complex numbers is that we do not need to invent any new numbers to solve polynomial equations. Until now we found that very simple polynomial equations, such as $2x - 1 = 0$ and $x^2 - 2 = 0$, did not have solutions; now they do. Amazingly, now *any polynomial equation has a complex number solution*: for any polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the coefficients $a_0, \dots, a_n$ are complex numbers, we can find complex number roots $\alpha_1, \alpha_2, \dots, \alpha_n$ and factorise the polynomial completely as

$$f(x) = a_n (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n).$$

We don't need to go any further than complex numbers to solve polynomial equations. This property is called **algebraic closure**, and it's another way to state the fundamental theorem of algebra.

Theorem (Fundamental theorem of algebra)

The complex numbers $\mathbb{C}$ are algebraically closed.

That's not to say that we can't make up more numbers as we need! An important, even larger number system is called the **quaternions**; these involve *three* different square roots of -1 called i, j and k. The quaternions are very useful in 3-dimensional geometry and are important in higher mathematics and physics.

History

Évariste Galois

We have mentioned that Évariste Galois (and independently Abel) proved the general insolvability of quintic equations. Galois' life is possibly the most dramatic of any great mathematician and so is worth a mention here. Galois' life was a struggle against almost everyone and everything he encountered: polynomial equations, organised incompetence, political injustice, and deep questions of pure mathematics. By the time of his death at age 20, Galois had been tried and acquitted for threatening the king, armed himself for revolution in the Republican artillery, and — although nobody knew it for another decade — revolutionised mathematics. He died in a pistol duel.

Galois was born near Paris in 1811. Educated first by his mother, he entered a preparatory school at age 12 and, bored with school classes, started reading the works of great mathematicians. By age 15 he was reading original research memoirs. But it was not only mathematics that interested him: French society at the time was engaged in an ongoing and bitter struggle between the democratic ideals of the Republic and the conservative forces of the monarchy. Like his father, who was mayor of Bourg-La-Reine, Galois was profoundly opposed to tyranny.

Attempting the entrance exam to the top university, the *École Polytechnique*, a year early, Galois failed. It was to be the first of many episodes of mathematical incomprehension by his supposed superiors. Some teachers and lecturers saw his ability, but their word was not sufficient for admission. He enrolled in a private mathematics course instead.

By age 18, Galois was thinking deeply on the question of the solvability of polynomials. He submitted an article with some of his results on the topic. Some accounts suggest that the article was lost or thrown out; other accounts suggest he was asked to resubmit another version. In any case, the paper was never published and nothing became known of its results.

Around the same time, Galois' father committed suicide after a vicious political dispute turned personal and a political enemy published scurrilous material in his name. Shortly afterwards, Galois reattempted the Polytechnique entrance exam. The apocryphal story goes that he lost his temper and stormed out in disgust, hurling a blackboard eraser at the examiner; in any case he again failed. He enrolled at the *École Normale* instead.

In 1830 Galois again submitted his research, this time for the Grand Prize of the Academy of Sciences. The paper was received but the referee died before reading it; the paper again was lost. However, political developments overshadowed mathematics: the French parliament was dissolved by King Charles X; anti-monarchists gained a majority in the ensuing elections; the king attempted to suppress the press and parliament; and the so-called July Revolution broke out. After a mass uprising, the royalists and the Republicans reached a compromise, and Louis-Philippe was made king. Galois desperately wanted to join the uprising but the Director of the *École Normale* locked students in. Galois wrote a letter condemning the Director and was promptly expelled. He joined the Artillery of the National Guard; the artillerymen were almost entirely against the monarchy. But the king dissolved the Artillery as a security threat. At a banquet with his Republican colleagues in May 1831, Galois made a 'toast' to Louis-Philippe while holding a dagger. He was arrested and tried for threatening the king, but acquitted; sources report the jury was moved by his youth. Shortly afterwards he received notice that his manuscript on the solvability of equations was rejected as 'incomprehensible'.

On 14 July (Bastille day) 1831, Galois led a demonstration wearing the (now banned) uniform of the Artillery and heavily armed. Arrested, convicted and imprisoned, he was able to work on mathematics in jail until he was eventually paroled in early 1832.

Galois' freedom was not to last long. He became involved in a brief and tumultuous love affair, which ended with his rejection. Shortly afterwards, for his advances towards the woman, he was challenged to a duel. There has been a great deal of speculation over possible political motives for the duel; what is certain is that he lost. On the eve of the duel, he wrote the famous letter quoted at the start of this module, in which he attempted to explain his work. It was pistols at 25 paces; Galois was shot in the stomach and died the next day.

Rejected from the universities, a soldier of the Republic, contemptuous of the authorities, and finally slain by a comrade over (in his words) 'an infamous coquette', Galois died as tempestuously and as misunderstood as he lived. It was not until 1843 that the great mathematician Liouville announced that he had found, 'among the papers of Évariste Galois … a solution, as precise as it is profound, of this beautiful problem: whether or not there exists a solution by radicals …'

Answers to exercises

Exercise 1

Substituting $x = 3$ into $2x^3 - 8x^2 + 7x - 3$ gives

$$2 \cdot 3^3 - 8 \cdot 3^2 + 7 \cdot 3 - 3 = 54 - 72 + 21 - 3 = 0.$$

Exercise 2

From $f(x) = 2(x-5)^2(x+7)$, we expand to $f(x) = 2x^3 - 6x^2 - 90x + 350$ and so

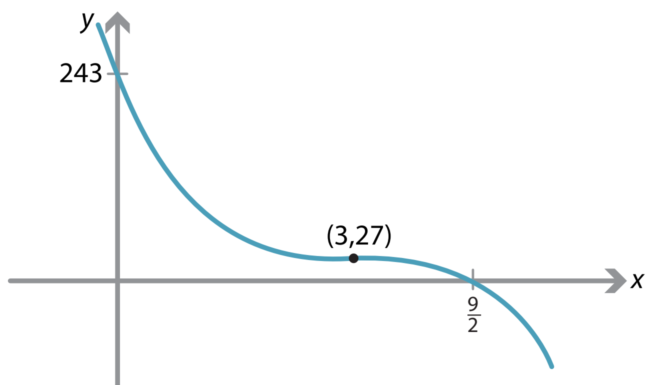
$$f'(x) = 6x^2 - 12x - 90 = 6(x^2 - 2x - 15).$$

Therefore $f'(x) = 0$ implies $x^2 - 2x - 15 = 0$, so $(x-5)(x+3) = 0$ and therefore $x = 5$ or $x = -3$. We check that $f(-3) = 512$ and $f(5) = 0$.

Exercise 3

a The graph of $y = a(x-h)^3 + k$ is obtained from $y = x^3$ by a dilation of factor a from the x-axis in the y-direction, then a translation of h units to the right and k units upwards.

For example, the graph of $f(x) = -8(x-3)^3 + 27$ is given below.



Graph of $f(x) = -8(x-3)^3 + 27$.

b If $f(x) = a(x-h)^3 + k$, expanding gives

$$\begin{aligned} f(x) &= a(x^3 - 3hx^2 + 3h^2x - h^3) + k \\ &= ax^3 - 3ahx^2 + 3ah^2x - ah^3 + k. \end{aligned}$$

Since $f(x)$ can also be written as $ax^3 + bx^2 + cx + d$, then $b = -3ah$ and $c = 3ah^2$. Hence $b^2 - 3ac = (-3ah)^2 - 3a(3ah^2) = 9a^2h^2 - 9a^2h^2 = 0$.

c Almost any example will work. For instance we can take $f(x) = x^3 + x$. Then $a = 1$, $b = 0$, $c = 1$, so $b^2 - 3ac = 0 - 3 = -3 \neq 0$.

Exercise 4

A polynomial of even degree has asymptotic behaviour like $y = a_n x^n$ where n is even. If a_n is positive, then as $x \to \pm\infty$, $f(x) \to \infty$; if a_n is negative, then as $x \to \pm\infty$, $f(x) \to -\infty$. The graph either begins by going down and ends by going up, or begins by going up and ends by going down. Therefore there must be an odd number of turning points.

On the other hand, a polynomial of odd degree has asymptotic behaviour like $a_n x^n$ where n is odd. So $f(x)$ goes from $-\infty$ to $+\infty$ or vice versa. Hence, it either begins and ends by going up, or begins and ends by going down. Therefore there must be an even number of turning points.

Exercise 5

We note that the quartic equation in x is a quadratic equation in x^2. So let $z = x^2$, and the equation becomes $z^2 - 5z + 6 = 0$. Factorising gives $(z-2)(z-3) = 0$, so $z = 2$ or 3. As $x = \pm\sqrt{z}$, we have the four solutions $x = \pm\sqrt{2}, \pm\sqrt{3}$.

Exercise 6

Since the coefficients are integers, if there is an integer solution, then it is a factor of 6, i.e., ±1, ±2, ±3 or ±6. Substituting these values we find that $x = -2$ is a solution:

$$(-2)^3 + 2(-2)^2 + 3(-2) + 6 = -8 + 8 - 6 + 6 = 0.$$

Therefore, by the factor theorem, $(x+2)$ is a factor of $x^3 + 2x^2 + 3x + 6$. Using polynomial division we find

$$x^3 + 2x^2 + 3x + 6 = (x+2)(x^2+3),$$

so it remains to find solutions to the quadratic equation $x^2 + 3 = 0$. This quadratic equation has no solutions. Thus the only solution to our original cubic equation is $x = -2$.

Exercise 7

Since r_1 is a zero of $f(x)$, we can use the factor theorem to write $f(x) = (x - r_1)p_1(x)$, for some polynomial $p_1(x)$. But r_2 is also a zero of $f(x)$, and so it must be a zero of $(x - r_1)$ or $p_1(x)$. As r_1 and r_2 are distinct, it follows that r_1 is a zero of $p_1(x)$. By the factor theorem we have $p_1(x) = (x - r_2)p_2(x)$, for some polynomial $p_2(x)$. Thus our original polynomial can now be written as $f(x) = (x-r_1)(x-r_2)p_2(x)$. Continuing in this way (by considering $r_3, r_4, \dots, r_k$ in turn), it follows that $(x-r_1)(x-r_2)\cdots(x-r_k)$ is a factor of $f(x)$.

Exercise 8

Differentiating gives $f'(x) = 3x^2 + 1$. The stationary points are the solutions of $3x^2 + 1 = 0$. But this quadratic equation has no real solutions, and so $f(x)$ has no stationary points.

Exercise 9

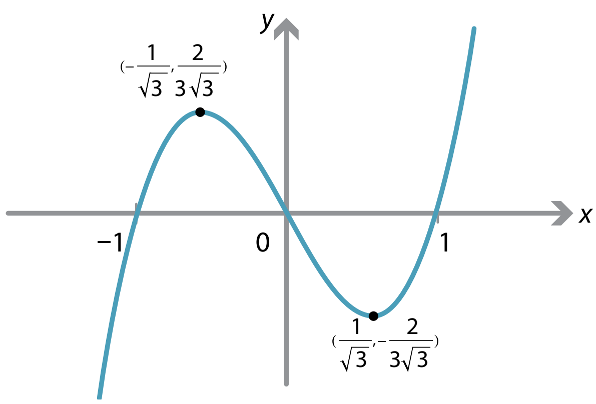
We find stationary points by solving $f'(x) = 0$, i.e., $3ax^2 + 2bx + c = 0$. This is a quadratic equation and its discriminant is $(2b)^2 - 4(3a)c = 4b^2 - 12ac$, which has the same sign as $b^2 - 3ac$. When $b^2 - 3ac < 0$, the discriminant is negative, $f'(x) = 0$ has no solutions, so there are no stationary points. When $b^2 - 3ac = 0$, the discriminant is 0, $f'(x) = 0$ has precisely one solution and there is one stationary point. When $b^2 - 3ac > 0$, the discriminant is positive, $f'(x) = 0$ has two distinct solutions and there are two distinct stationary points.

Exercise 10

The polynomial $f(x) = x^3 - x$ factorises as $x(x-1)(x+1)$, so x-intercepts are at $x = -1, 0, 1$. The y-intercept is $f(0) = 0$. Since the leading term is x^3, as $x \to \pm\infty$, $f(x) \to \pm\infty$. The derivative is $f'(x) = 3x^2 - 1$, so the stationary points can be found by solving the equation $3x^2 - 1 = 0$, i.e., $x = \pm\frac{1}{\sqrt{3}}$. We have $f(-\frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$ and $f(\frac{1}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$, so the stationary points are at $(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}})$. We can draw a sign diagram for $f'(x)$ as follows.

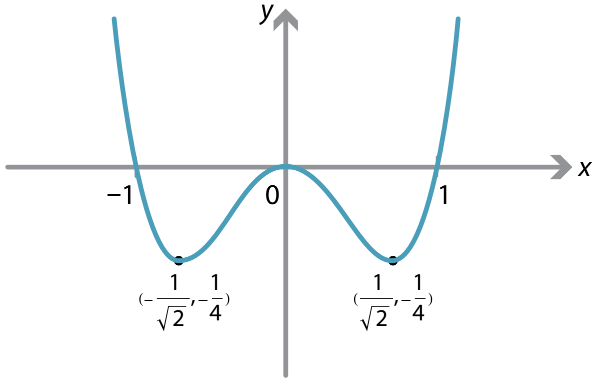
Value of x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
Sign of $f'(x)$	$+$	0	$-$	0	$+$
Slope of graph $y = f(x)$	/	—	\	—	/

The graph of $y = f(x)$ is as shown.



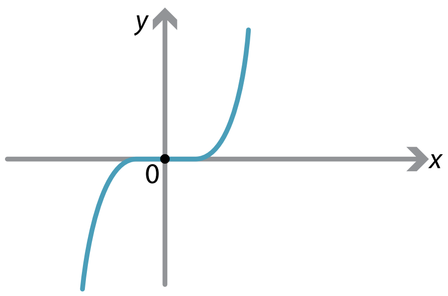
Exercise 11

Let $f(x) = x^4 - x^2$. Then we may factorise as $f(x) = x^2(x-1)(x+1)$ and the roots are $x = 0$ (with multiplicity 2) and $x = -1, 1$. As the root at $x = 0$ has multiplicity 2, we have a turning point. Since $f(0) = 0$, the y-intercept is 0. As the leading term is x^4, the graph of f is 'happy': $f(x) \to +\infty$ as $x \to \pm\infty$. Since $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$, the stationary points are at $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$. We can compute $f(-\frac{1}{\sqrt{2}}) = -\frac{1}{4}$ and $f(\frac{1}{\sqrt{2}}) = -\frac{1}{4}$, so the stationary points are $(-\frac{1}{\sqrt{2}}, -\frac{1}{4})$, $(0, 0)$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{4})$. Based on the above we can actually deduce the shape of the graph and see that all stationary points are turning points; alternatively we could draw a sign diagram. The graph is as follows.

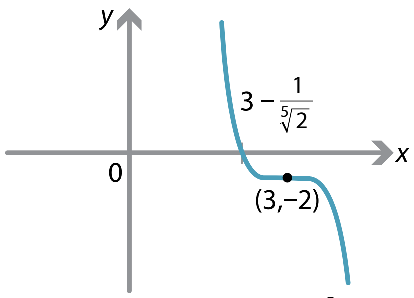


Exercise 12

The graph is obtained from $y = x^5$ by reflection in the x-axis, dilation by factor 4 from the x-axis in the y-direction, and translation of 3 to the right and 2 down.



Graph of $f(x) = x^5$.



Graph of $f(x) = -4(x-3)^5 - 2$.

Exercise 13

The suggested substitution is $z = x - 2$. Rewriting the equation in terms of z gives

$$(z+2)^3 - 6(z+2)^2 + 27(z+2) - 58 = 0,$$

which simplifies to $z^3 + 15z - 20 = 0$. This is now a cubic equation without a quadratic term, which we can rewrite as $z^3 = 20 - 15z$. To find a solution, following our method, we seek A and B such that $A + B = 20$ and $3\sqrt[3]{AB} = -15$, i.e., $AB = -125$. Substituting $B = -\frac{125}{A}$ into $A + B = 20$ gives

$$A - \frac{125}{A} = 20, \qquad \text{which is equivalent to} \quad A^2 - 20A - 125 = 0.$$

This quadratic equation factorises as $(A-25)(A+5) = 0$, so $A = 25$ or -5, and we obtain a solution $A = 25$, $B = -5$. Therefore a solution to the cubic equation is $z = \sqrt[3]{25} - \sqrt[3]{5}$. Substituting back for x gives a solution $x = 2 + z = 2 + \sqrt[3]{25} - \sqrt[3]{5}$.

References

- Tony Rothman, 'Genius and biographers: the fictionalization of Évariste Galois', *American Mathematical Monthly* 89 (1982), no. 2, 84–106.
- Ian Stewart, *Galois Theory*, 3rd edition, Chapman & Hall/CRC, 2004.

The second reference provides a proper mathematical account of the work of Galois on the solvability of polynomial equations and a good introduction to Galois theory, along with substantial historical background.

0 1 2 3 4 5 6 7 8 9 10 11 12