VCAA-AMSI maths modules

A guide for teachers – Years 11 and 12

Discrete Mathematics

Logic and Boolean algebra

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Years 11 & 12

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Logic and Boolean algebra – A guide for teachers (Years 11–12)

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History and Motivation

(http://en.wikipedia.org/wiki/Boolean_algebraHistory)

Given that much of this module is founded on truth and falsehood, the topic goes back to Ancient Greek philosophy and to Socrates and Aristotle in particular. However, Boolean algebra, on which the module heavily relies, began its development with Boole's work from the mid Nineteenth Century. Boole's aim in introducing his work was to provide a foundation for Aristotelian logic
(see http://en.wikipedia.org/wiki/George_Boole).

Before what we know today as abstract algebra had been developed, Boole produced a system that we recognise now as algebraic. But whereas the algebra with which we are more familiar is based on numerical relations, Boole's algebra is based on logical relations. Hence the variables of Boole take only the values 0 and 1, with 0 standing for false and 1 for truth. Boole's work was developed and refined until we get the axiomatised version that we discuss here.

It wasnâĂŹt until the 1930s that Claude Shannon realised the importance of Boolean algebra to switching circuits, the major topic of this module. Karnaugh maps

(http://en.wikipedia.org/wiki/Karnaugh_map)

arose in the 1950s as a means of simplifying Boolean expressions. These maps use the table form for representing switching circuits that we discuss later, followed by a clever use of the reductions that follow from the axioms of Boolean algebra. These map reductions can be achieved using the axioms alone, but Karnaugh's maps enable these simplifications to be determined in a neat way directly from the table form. It shouldbe noted though that the maps are only really efficient for small circuits. It is more practical to reduce anything of any size by computers algorithms.

Applications

The work that Boole originated was designed to consider matters of truth or validity and falsehood. Hence it can be used to determine the truth value of propositions and so decide if a chain of arguments is in fact valid. In the same way, Boolean algebra can be used to decide whether two statements are logically equivalent or not and so see if a tautology exists. From Shannon's work the same methods can be applied to switching circuits. The fact that Boolean algebra is an algebra often allows the expressions used in these applications to be simplified.

In addition the concepts of AND, OR and their negations are valuable in the work of current search engines

(see http://en.wikipedia.org/wiki/Boolean_searchBoolean_searches).

Contents

Lights in the Passage

The passage to our front door is lit by a light that is controlled by two switches, one at either end. When you switch a light on at one end you can switch it off from that same end or from the other end. This has obvious advantages for both human and electrical efficiency. But how would you wire such an arrangement?

Well, it's clearly not in series as we've shown in Figure 1, where the switches x and y are joined 'in line'.

1

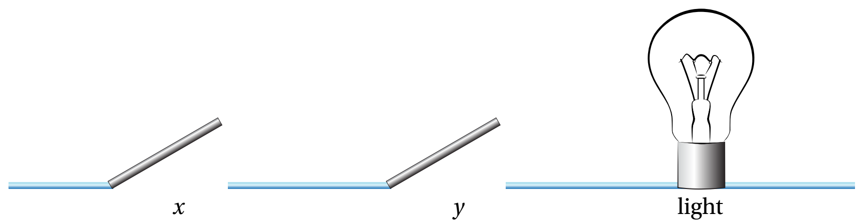

Figure 1: Two switches in series

Given the series set up you would need both x and y to be 'on' to get the light to work. Any other positioning of the switches would lead to no light which isn't so good if you've just come home to a dark house and you don't want to fall over the cat. Just to emphasise this, we've drawn up Table 1. By 'on' for a switch we visualise it as making a connection with the 'wire' shown in Figure 1.

Table 1: The effect of two switches in series

Switch x	Switch y	Light
on	on	on
on	off	off
off	on	off
off	off	off

That's obviously not how the electrician wired up the hall light. So the next guess is a wiring that is in parallel (see Figure 2).

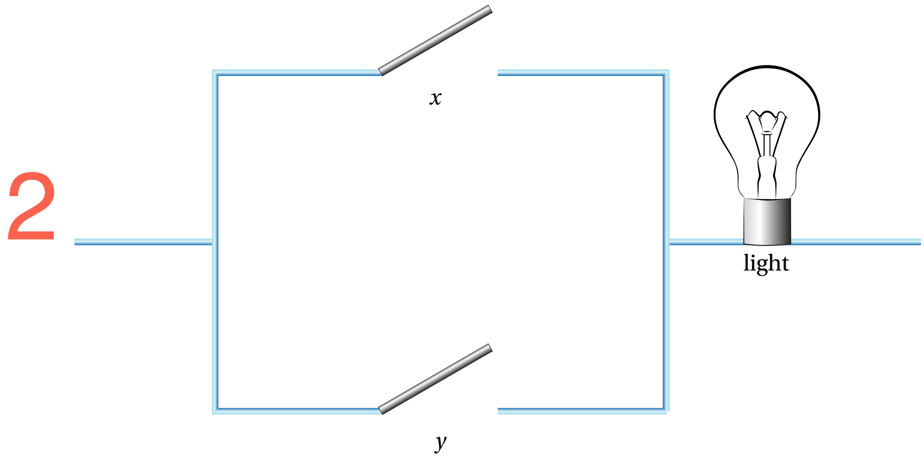

Figure 2: Two switches in parallel

All the possible situations are show in Table 2, but this is no good either because, for example, when x is on, the light is always on.

Table 2: The effect of two switches in parallel

Switch x	Switch y	Light
on	on	on
on	off	on
off	on	on
off	off	off

Now the only ways to wire things together are in series or parallel. So how did the electrician wire up the passage?

Questions

Questions

1 Draw up a table of off and on results for the switches in the systems in Figure 3. Simplify the systems where possible.

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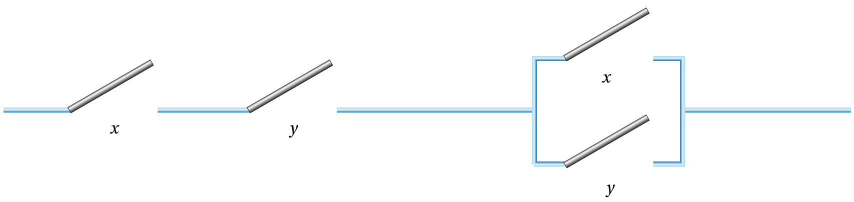

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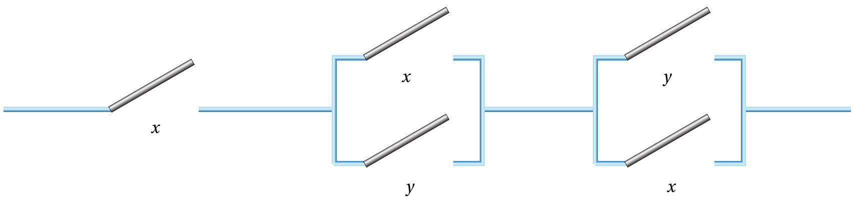

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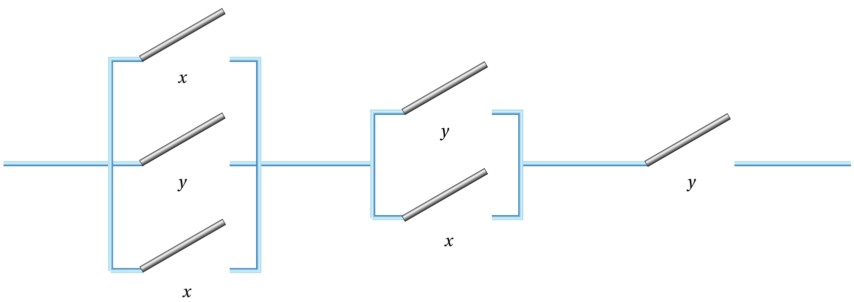

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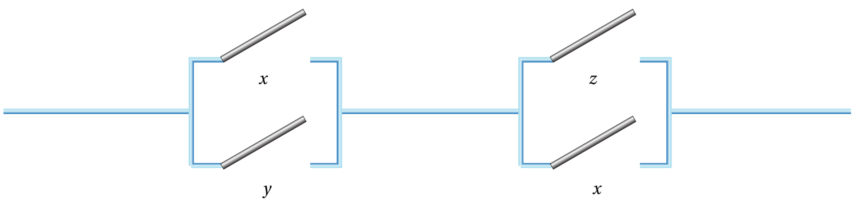

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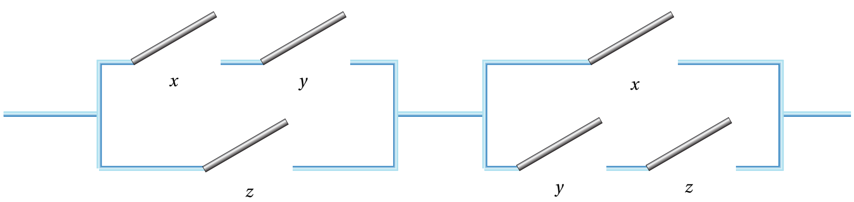

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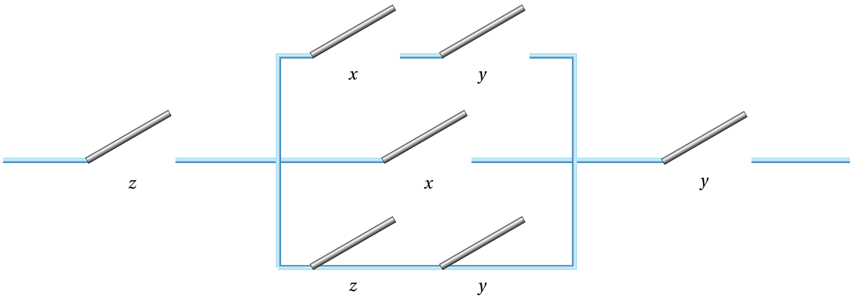


Figure 3: Some switching circuits

Questions

1 Voting for Two: Jamal and John have invented a system for getting a majority vote. If they both vote 'yes' by pressing a button a light comes on. Otherwise there is no light. Draw up a table that explains this system. What does this simplify to?

2 . Voting for Three: Veronique joins Jamal's and John's committee. Draw up a table for a system that produces a light only when there is a majority vote. What does this simplify to?

3 Passage Problem: Assume that switches x and y both start in the off position and the light is then off too. Draw up a table of the results we want to achieve so that we get the desired effect in the passage.

4 Experiment with systems that are combinations of two switches used in a variety of series and parallel arrangements. Produce on, off tables for each of them.

As we shall see shortly there are advantages to replacing the on and off of switches by 1 and 0. This means that Table 1 would become Table $1'$.

Table $1'$: The effect of two switches in series using ones and zeros

Switch x	Switch y	Light
1	1	1
1	0	0
0	1	0
0	0	0

Questions

6 In all the tables you have met so far, replace on and off with 1 and 0.

7 Table 1′ looks like multiplication. We could replace the whole of the table, and therefore the whole series system, by $x \times y$. What algebraic expression could you replace the parallel arrangement of switches by?

Problem here as I'm too lazy to use italics

8 But the arithmetic here is a little strange. In particular what is 1 + 1 in the parallel case? Where else in mathematics does this kind of thing happen?

9 . Think about the tables in Q6 from an algebraic standpoint. Can you express the results algebraically in terms of x, y and z?

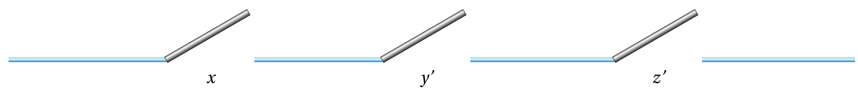
One final device that is useful here is the complement x' of a switch x. The effect of this is shown in Table 3. The idea here is that when x switch is on, its complement is off and vice versa.

Table 3: The complement of a switch

Switch x	Switch x'
1	0
0	1

4a

4b



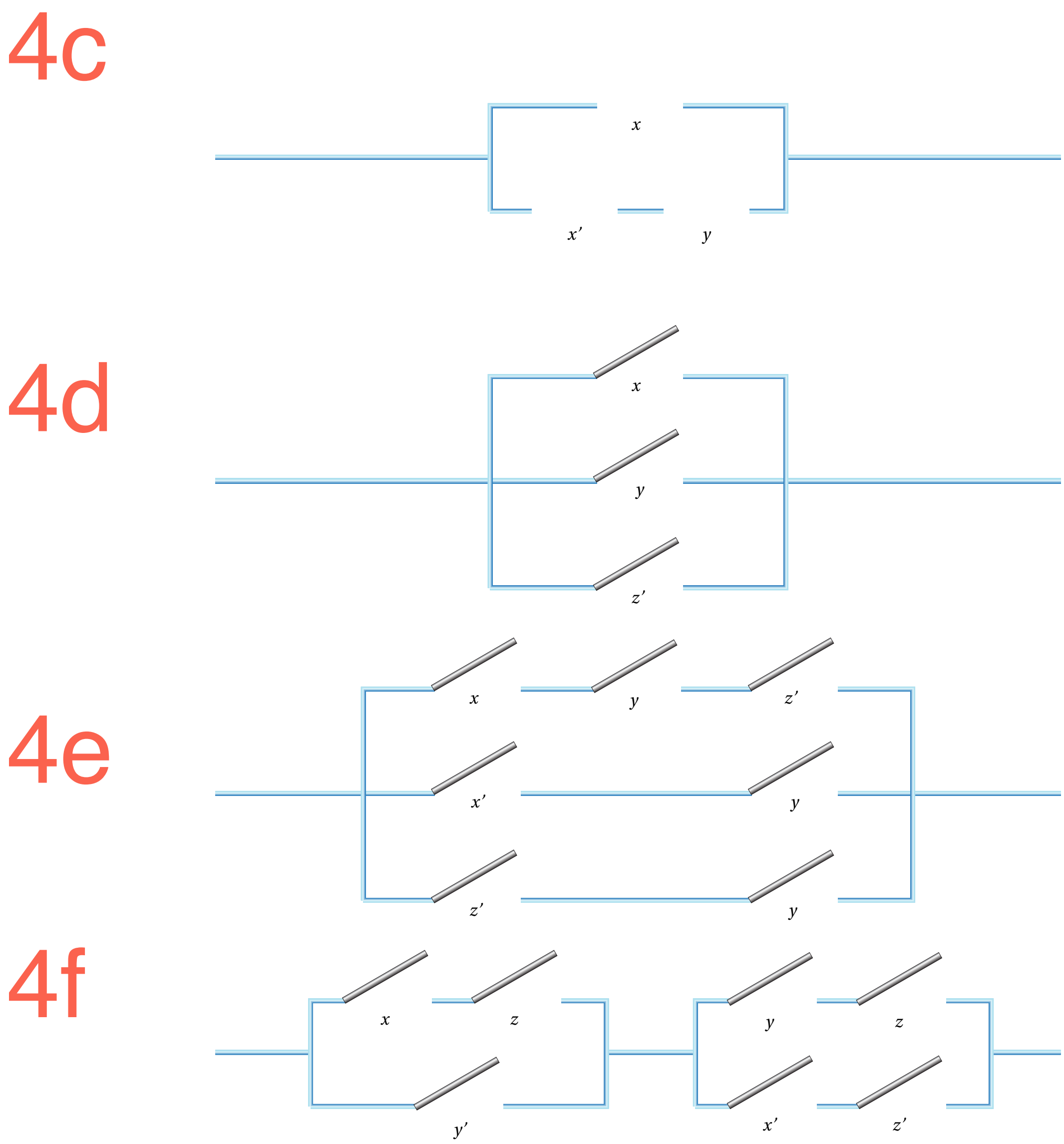


Figure 4: Some switching circuits

Questions

10 Draw up 0 and 1 tables for the systems in Figure 4. Produce algebraic expressions for each of them in terms of the + and × of parallel and series switches.

11 Now try to algebraicise the Passage Problem of Q4.

12 Continue to experiment as in Q5, but now allow for a switch and/or its complement to be present.

Boolean Algebra 2

It's algebra, Jim, but not as we know it

Although we appear to be going off in a completely new direction, we will get back to switching systems. When we do we will have powerful, algorithmic, tools for producing the simplest switching systems for any specified problem. We have already cheated by using these tools in some of the answers to the problems of the last section.

As we know, one rather odd thing happens with the zeros and ones that we used in the switching systems. Everything looks normal when you multiply. You would expect and get

$$0 \times 0 = 0; \quad 0 \times 1 = 0 = 1 \times 0; \quad \text{and } 1 \times 1 = 1$$

. But addition throws up something new. Sure

$$0 + 0 = 0; \quad 0 + 1 = 1 = 1 + 0;$$

But $1 + 1 = 1$! This is not what you might have guessed ahead of time. After all we normally get $1 + 1 = 2$ and in arithmetic base 2, $1 + 1 = 0$. However, $1 + 1 = 1$ is quite different.

On the other hand, there are other places where sort of $1 + 1$ is 1, but you might not have thought of it that way. Think about set theory. If you have the set of all sets contained in some set I, the power set of I, then the union of I with itself is I. So $I \cup I = I$. If we think of union as being addition does it give us the arithmetic above that we got for zeros and ones?

Ah, but $I \cap I = I$ too. Would we be better off using intersection for addition if we want to find another example of $1 + 1 = 1$?

Questions

13 Check out union and intersection to see which is the best parallel to addition of the zeros and ones in switching circuits.

14 Does either of union or intersection parallel multiplication?

15 If A is a subset of I, then what is $A \cup I$ and $A \cap I$. Do either of these look like addition or multiplication by multiplication by 1?

16 Is there anything else that our zeros and ones have that sets have? What properties does the 'anything else' have? Are these consistent with the zeros and ones?

It turns out that we have stumbled into a whole new ballgame. The zeros and ones we started with and the sets we looked at next, are all examples of a useful axiomatic system that contains lots of examples other than the two we have been playing with. This axiomatic system is called Boolean algebra (see http://en.wikipedia.org/wiki/Boolean_algebra). A Boolean algebra, B, is a set B on which operations $+, \times$ and complementation are defined along with a zero and a one, which satisfy the following axioms.

- **Axiom 1**

  B contains at least two elements.

- **Axiom 2**

  B is closed under $+$ and $\times$. That is for all x and y in B, $x+y$ and $x \times y$ are in B.

- **Axiom 3**

  $+$ and $\times$ are commutative. That is for all x and y in B, $x+y=y+x$ and $x \times y = y \times x$.

- **Axiom 4**

  $+$ and $\times$ are associative. That is, for all x, y and z in B, $(x+y)+z = x+(y+z)$ and $(x \times y) \times z = x \times (y \times z)$.

- **Axiom 5**

  $\times$ is distributive over $+$ and $+$ is distributive over $\times$. That is, for all x, y and z in B, $x \times (y+z) = (x \times y) + (x \times z)$, while $x + (y \times z) = (x+y) \times (x+z)$.

- **Axiom 6**

  $+$ and $\times$ have identities in B which are denoted, respectively, by 0 and 1. That is for all x in B, $x+0=x$ and $x \times 1 = x$.

- **Axiom 7**

  $+$ and $\times$ obey two absorption laws. That is, for all x and$y$ in B, $x + (x \times y) = x$ and $x \times (x+y) = x$.

- **Axiom 8**

  $+$ and $\times$ both have a common complement. That is, for all x in B, there is an x' in B such that $x + x' = 1$ and $x \times x' = 0$.

Questions

17 Test those axioms out on the 0 and 1 of switching circuits.

18 What are the complements of 0 and 1 in any Boolean algebra B?

19 Let $I = \{x, y\}$. Check that $B = \{\emptyset, \{x\}, \{y\}, I\}$ is a Boolean algebra. What are $+$ and $\times$?How does this change if I is any set?

20 Let $B = \{1, p, q, pq\}$ where p and q are distinct prime numbers. Let $+$ be the least common multiple of two members of B and $\times$ be their greatest common divisor. In order to make B a Boolean algebra, what are 0, 1 and the complements of all the elements of B?

How can this be generalised? Do you need p and q to be primes? Could you make a Boolean algebra from $\{1, p, \dots, pqr\}$ where p, q and r are all distinct prime numbers? Do you need the primes to be distinct?

21 . What other Boolean algebras can you discover?

Before you think that we are still right off track, we should say that this is actually going to be very useful for switching circuits. But there is more to deal with before we can get to that point. In fact, for a time, we are going to delve in the life of a pure mathematician. There may have been quite a lot of things that you have been assuming about Boolean algebras that are true but which need to be supported by the axioms. If they aren't, they are not true and you shouldn't have been using them. So what would a pure mathematician say? What would one of these rare creatures worry about?

First worry:$x + x = x$. It's clearly true in our switching circuits that two of the same switches in parallel are the same as just one on its own. It also works for the power set (see the answer to Q7) of a set. Does it work for all of the other Boolean algebras that you now know? Alright, BUT it's not an axiom! So how do we know it's true for all Boolean algebras? Well, is it supported by the axioms? Let's look.

$$\begin{aligned}
x + x &= (x + x) \times 1 && \text{(this is OK by axiom 6)} \\
&= (x + x) \times (x + x') && \text{(axiom 8)} \\
&= x + (x \times x') && \text{(axiom 5 - work backwards using the second form of absorption)} \\
&= x + 0 && \text{(axiom 8)} \\
&= x && \text{(axiom 6)}
\end{aligned}$$

The problem here is how do you choose the right path to go down? What axioms do you need and in what order? ThereâĂŹs no really good answer to this. You just have to play around for a start until you have a better intuition of the axioms. Then you have to play around till you get the right ones. It looks as if there isn't an algorithm that might help you.

Second worry: $x \times x = x$. Does it seem to be true? Try it.

$$\begin{aligned} x \times x &= x \times x + 0 \qquad \text{(axiom 6)} \\ &= x \times x + x \times x' \qquad \text{(axiom 8)} \\ &= x \times (x + x') \qquad \text{(axiom 5)} \\ &= x \times 1 \qquad \text{(axiom 8)} \\ &= x \qquad \text{(axiom 6)} \end{aligned}$$

There are more worries in the next set of questions.

Questions

22 Prove the following equalities are true for the Boolean algebras that you know. Prove that they must always be true in every Boolean algebra.

- i $x + 1 = 1$;
- ii $x \times 0 = 0$;
- iii $x \times (x + y) = x$;
- iv $x + (x \times y) = x$.

23 Which of the following are true? Prove those. Which of the following are false? Say why that is the case? Are they true in some Boolean algebra?

- i $x = x'$;
- ii $(x')' = x$;
- iii in some Boolean algebras x' is not unique

24 Prove that $(x + y)' = x'xy'$.

* * * *

A mathematical note

In your role as pure mathematician you may have noticed some symmetry in what has been going on. It looks as if we have a result in Boolean algebra and we interchange $+$ and $\times$, and 0 and 1, then we get a new result that it also true. This is called the principle of duality and it is sometimes useful as it reduces the number of things that you need to prove in Boolean algebra. Check that the axioms show this principle. So now worry on.

Questions

25 What is the dual of the equality in Q24? Check this for the Boolean algebras that you know.

26 Show that in a Boolean algebra

27 $x + (x \times b) = x$

a $x + (x' \times x) = x + x$

State and prove the dual of **a** and **b**. Show that each step in the proofs of the duals of **a** and **b** corresponds to a dual in the proofs of **a** and **b**.

27 Go through the axioms of Boolean algebra and apply the principle of duality. Hence by adding one new axiom, reduce the length of as many of the axioms as possible.

In Boolean algebra you may have met an abstract quantity defined by axioms for the first time. It turns out that there are a whole host of them in what is known as Abstract Algebra. Things such as groups, rings and fields are also defined by sets of axioms. All of these objects arose in attempts to generalise something that occurred naturally. For example, groups
(see http://en.wikipedia.org/wiki/Group_(mathematics))
have only one operation and could be thought of as generalisations of the integers with addition being the only operation; rings
(see http://en.wikipedia.org/wiki/Ring_(mathematics))
have two operations and are generalisations of the integers with two operations, addition and multiplication; and fields
(see http://en.wikipedia.org/wiki/Field_(mathematics))
again have two operations and have developed from the rational numbers with addition and multiplication. Like Boolean algebra, all of these generalisations have applications outside mathematics where they have been studied by pure mathematicians.

Using '×' for multiplication is beginning to get tedious. In normal arithmetic we avoid this tedium by replacing $x \times y$ by xy. Let's do this from here on.

Questions

28 Simplify

a $(x + y' + z')(x' + y)(x' + z)$

b $xy + xy' + x' + y$

c $xz' + xyz' + z$.

Switches again

Now you should be worrying about whether this Boolean algebra has anything to do with switches at all. In the section on switches we saw that we had a '+' and a 'x', which were defined in terms of the parallel and series organisation of switches. Given that the 0 and 1 of the smallest Boolean algebra developed from switches, it seems natural to worry about whether switch systems satisfy the axioms of a Boolean algebra. Let's work through them to see if any fail.

- **Axiom 1**

  The switches contain the elements 0 , no switches and 1, the line is always open.

  So that looks OK, if a bit fudged.

- **Axiom 2**

  B is closed under $+$ and $\times$. That is for all x and y in B, $x+y$ and xy are in$B$. Certainly if x and y are two switches we can put them in parallel and series to get another switching object.

- **Axiom 3**

  + and x are commutative. That is for all x and y in B, $x+y=y+x$ and $xy=yx$. The arrangement of the parallel pieces or the series pieces is irrelevant so this axiom is satisfied.

- **Axiom 4**

  + and $\times$ are associative. That is, for all x, y and z in B, $(x+y)+z=x+(y+z)$ and $(xy)z=x(yz)$.Again order is unimportant for switch arrangements.

- **Axiom 5**

  $\times$ is distributive over + and + is distributive over $\times$. That is, for all x, y and z in B, $x(y+z)=xy+xz$, while $x+yz=(x+y)(x+z)$.

Now the best way to check these is to use a table, see Table 4 below.

Table 4: A proof that $x(y+z)=xy+xz$

x	y	z	$x(y+z)$	$xy+xz$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

We leave the second part of this checking to the Questions below.

- **Axiom 6**

  $+$ and $\times$ have identities in B which are denoted, respectively, by 0 and 1. That is for all x in B, $x+0=x$ and $x\times 1=x$.

If we put a wire with no switches in parallel with no switch we get the same result as having just the switch. We leave the second of these to the Questions below.

- **Axiom 7**

$+$ and $\times$ obey two absorption laws. That is, for all x and y in B, $x+x\times y=x$ and $x(x+y)=x$. A proof of these is left to the Questions below.

- **Axiom 8**

  $+$ and $\times$ both have a common complement. That is, for all x in B, there is an x' in B such that $x+x'=1$ and $xx'=0$.

This is just the way that parallel and series switching works for x and its complement.

Questions

29 Show that for x switch x, $x\times 1=x$.

30 Show that for switches x, y and z, $x+yz=(x+y)(x+z)$.

31 Show that for switches x and y, $x+xy=x$ and $x(x+y)=x$.

32 Show that the principle of duality holds.

We now have all the axioms of a Boolean algebra, so a set of switches joined in parallel and series forms a Boolean algebra. BUT, what use is this?

Let's consider an example. Recall Q3, Voting for Three of the 'Light in the Passage' section (better to give a page reference). We have to set up switches so that a light goes on when two people out of three vote for an option by switching their switches on. Let x, y and z be the three options. Then $xy + xz + xz + xyz$ will give us a perfectly good switching system. However, we can simplify this using the properties of Boolean algebra.

$xy + xz + yz + xyz = x(y+z) + yz(1+x) = x(y+z) + yz.$

This produces the switching system of Figure 5.

5

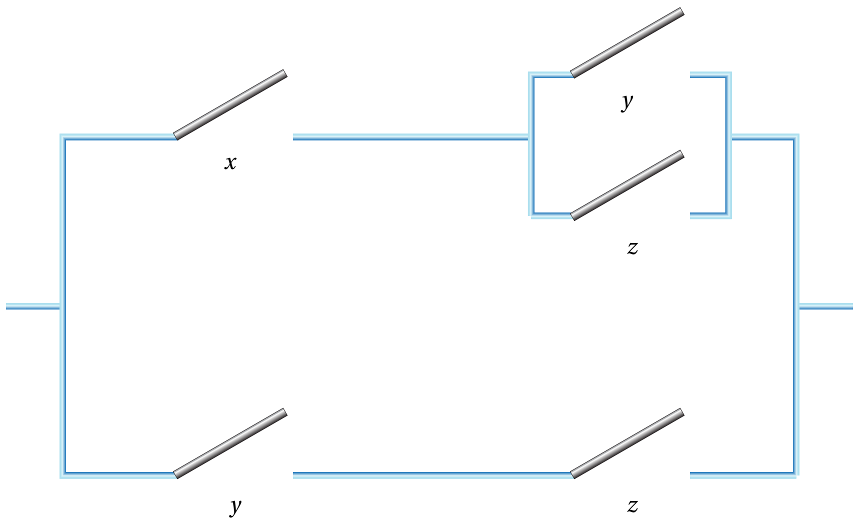


Figure 5:A switching circuit for voting for three

For what it's worth, this switching set up requires one less switch than the number needed using the obvious alternative arrangement of $ab + ac + bc$.

Questions

33 . Use Boolean algebra to simplify the system of switches from Q1.

34 Where possible, use Boolean algebra to simplify the switching systems from Q10.

35 Produce a switching system for the Passage problem (see Q11).

Functions for Boolean Algebras

Let's make it harder to make it simpler! It turns out that here is another and easier way to produce a switching system once you have the system's table. This comes about from knowing the function that will do the job for you. This function is a polynomial on the possible switches and their complements and polynomials are relatively simple to handle. As they say, now read on.

Your used to dealing with functions of real numbers, things like $f(x) = 3x + 5$ and $f(x) = x^2 - 5x - 1$, where there is only one variable x. But you can also have functions of two variables: $f(x, y) = x + xy - 2y$ and $f(x, y) = x^2 + y^2$. Functions with two variables work in more or less the same way as do functions of one variable. If you want to know the value of the function at particular values of x and y you just substitute them in the formula for $f(x, y)$.

Questions

36 Find the values of the following functions when $x = 1$ and $y = -1$.

- i $f(x, y) = x + xy - 2y$;
- ii $f(x, y) = x^2 + y^2$;
- iii $f(x, y) = e^x e^y$.

37 But sometimes we can reduce functions of two variables to functions of one variable. What happens to the two variable functions in Q36 if $y = 3x$?

But functions are not confined to being associated with real numbers. They work with whatever class of variables you choose. We now choose to work with variables from Boolean algebras. (Why are you not surprised?) But in some ways Boolean algebras make the algebra simpler! Like the functions of real numbers there is a point in introducing functions of Boolean algebras. We'll get to this in the next section.

Let's have a look at a couple of functions on the simplest Boolean algebra there is - the one consisting only of 0 and 1. We'll see what the functions looks like by drawing up a table of values.

Example

$f(x, y) = x + (x + y)$. The table we get from this is

x	y	$f(x,y)$
0	0	0
1	0	1
0	1	1
1	1	1

The result is not surprising because we can simplify the function to $f(x,y) = x + y$.

Example

$f(x,y) = x + (xy$. The table we get from this is

x	y	$f(x,y)$
0	0	0
1	0	1
0	1	0
1	1	1

Again this is not surprising as $x + xy = x$.

Questions

38 Produce a table that shows the values of the following functions for the smallest Boolean algebra. Check these values by simplifying the expression if possible.

- i $f(x,y) = x + y + 1$;
- ii $f(x,y) = (x+1)(y' + 0)$;
- iii $f(x,y) = xy$;
- iv $f(x,y) = (xy')(xy)'$;
- v $f(x,y) = x + x'y(x+y)$;

39 But there is no reason to be restricted to two variables. Produce a table that shows the values of the following functions of three variables for the smallest Boolean algebra.

$$f(x,y,z) = (x+y)(y+z)(z+x);$$

$$f(x,y,z) = x + x'y + y'z.$$

Questions

40 Show that any non-identity function of one variable on any Boolean algebra can be written in the form $f(x) = \alpha x + \beta x'$. What can be said about α and β?

41 Show that any non-identity function of two variables on any Boolean algebra can be written in the form $f(x, y) = +\beta x' y + \gamma x y' + \delta x' y'$. What can be said about α, β, γ and δ?

What is the corresponding situation for a polynomial function of three variables?

The form of a two variable non-identity function for switches x and y can be written as

$$f(x, y) = f(1,1)xy + f(0,1)x'y + f(1,0)xy' + f(0,0)x'y'$$

and we know the values of the constants by the answer to Q41. This form is called the disjunctive normal form. It can be used more readily to produce switching circuits than the system we used in the Switches Again section. We know from the table of the situation what the values of $f(0,0), f(1,0), f(0,1)$and $f(1,1)$ are and so we can substitute these in the disjunctive normal form to get the function we need so that we can immediately draw up the switching system for a particular problem. But you will get a simpler circuit if you use the various methods from Boolean algebra to simplify the polynomial you have.

Questions

42 **a** Devise a switching circuit for two people, where a light goes on if and only if they both vote for a proposal.

b devise a switching circuit for three people, where a light goes on if and only if there is a majority in favour of the proposal.

c Able, Baker and their children Charlie and Delta need a voting system too. However, Able and Baker are prepared to accept a majority vote but decide that if they both vote for a proposal or they both vote against it, their opinion should prevail. Devise a switching circuit for this situation.

43 . Design the switching circuit for the Light in the Passage using a disjunctive normal form approach.

44 Three passages meet and there is a strong light at this junction that lights up all the passages. Each passage has a switch at the end away from the light. A switching circuit is needed that will only go on when an odd number of switches is on. Design this circuit.

A Reasoning Retreat

There used to be an advert that talked about 'the pause that refreshes'. This section is meant to be a refreshing reasoning pause. Its aim is to give a background to reasoning before we start on the more serious business looking at truth.

Reasoning is a fundamental human process. Very little is achieved in life without thinking that involves reasoning even at the level of deciding whether the pair of shoes we have just put on our feet is what we would like to purchase or not. (If we buy this could I wear it with this on such and such occasions?) And it's not unknown in animals even. (If I fetch this stick I'll probably get a nice biscuit.) Certainly the first Australian Raven (Crow) who discovered how to eat a Cane Toad without dying of the toad's poison almost certainly gave the matter some thought.

Now there are several types of reasoning that we commonly use. These are discussed in more depth in Holton, D., Stacey, K., FitzSimons, G. (2012). Reasoning: A dog's tale. The Australian Mathematics Teacher, 68(3), 22-26.

The main type of reasoning for the purposes of this module is deduction
(http://en.wikipedia.org/wiki/Deductive_reasoning).
This is the 'if ... then' procedure that is required in mathematics all the time. It can be seen in alternate forms that either involve 'if' and 'then', 'this implies that' or use 'because', 'so' and the like, or any of these implicitly. This form of reasoning involves some statement, proposition, and some rule. If the statement holds then the rule allows us to make a true conclusion. 'Because these lines are parallel and this angle is 30°, then this other angle is 30°.' This is an example where corresponding angles are involved and there is a rule about corresponding angles that enables us to make a deduction.

Given a true, valid, proposition and a proven rule, deduction will give a correct result.

But other ways of reasoning are not so tight and predictable. We'll consider two of these. First let's look at the detective's method. Someone has been murdered with a knife. Jake has a bloodstained knife in his pocket (proposition). So Jake is the murderer. But poor old Jake has just been taking a stone out of his horseâĂŹs hoof, the knife slipped and he cut himself. In the second scene of the TV programme someone checks the DNA of the blood and Jake is released though maybe he actually used another knife for killing. There is likely to be a knife somewhere with at least a minute amount of blood on it and it will probably be found before the end of the next 120 minutes.

This kind of reasoning is called abduction
(http://en.wikipedia.org/wiki/Abductive_reasoning) and is used a lot especially when we think we are using deduction. 'If Jake has a knife, then he killed the victimâĂŹ is a

deduction but it relies on the rule that âĂŸsomeone who has a knife may have killed the victim'.

It turns out that much of the false reasoning that we do is based on abduction. With an apology to Socrates, let's look at 'all men are liars', 'Helen is a liar', so 'Helen is a man'. Again we are putting things the wrong way round and getting a false conclusion (although there is a Johnny Cash song about a boy named Sue!)

However, abduction has its uses in mathematics. It is a very good source for ideas. We observe something. That observation leads to a conclusion under a certain rule. That conclusion may be true or false but it might be true and it might lead us to some new ideas and new ways to approach old ideas.

Then there is the data collection method that finds a pattern for the data collected. This leads to the pattern being assumed for all situations. But sometimes this conclusion is false. The regions problem works this way. Take a circle and choose some points on its boundary. In joining up all the points with straight lines you produce a number of regions (see Figure 5). The number of regions appears to double as the number of points increases. This is certainly what the data up to 6 points suggests. However, it all goes horribly wrong with 6 points.

6

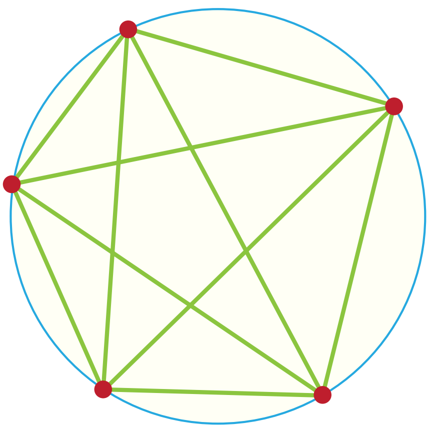


Figure 6:Regions formed in a circle

Again this is a method we all use all of the time. Sometimes this works for every case; sometimes, as in the regions problem, it doesn't. Whichever way it is food for thought and possibly new ideas. If it is true we need to prove it; if it is false we need to find a counterexample. Either way might be useful.

The kind of reasoning used here is called induction

(http://en.wikipedia.org/wiki/Inductive_reasoning.

(This is not to be confused with Mathematical Induction which is a well-established and valuable even valid proof method.) Like abduction, induction is a great way to produce new ideas and, possibly, new results in mathematics.

But sometimes we reason incorrectly by looking at the converse of an implication, a deduction. But let's take a maths example. If you are stacking cans as in Figure 6, with one less can in each row as you move up the stack, you can easily show that if a stack has seven rows, the number of cans in the stack is divisible by 3. This can be done by letting the top row have m cans, the next have $m+1$, and so on. Then the number of cans is $m+(m+1)+(m+2)+\cdots+(m+6)=7m+21$. In the heat of the moment you might think that if a number is divisible by 7, then that number of cans can be stacked in seven rows. In this case though this converse of the original correct implication is false. The only way you can stack 14 cans is shown in Figure 6. However, looking at the converse we might get some useful ideas about stacking cans.

7

Figure 7:Stacking 14 cans

In the next section we'll concentrate on deduction and related issues. However it is worth knowing that this is not the only way that we reason and to know the value of each type of reasoning. Deduction gives watertight results. Abduction and induction give new ideas that have to be tested before they can be trusted.

Having got that off our chests we can now get on with the serious things of life such as propositional logic.

Propositions

There are two aims for the rest of this module. These are to find tautologies and to simplify logical statements. When are linking statements true for all truth values of the propositions involved and how can we make the statements less obfuscating? Can we simplify the statements? We move simultaneously toward these goals.

In http://www.iep.utm.edu/prop-log/, we have the following definition: Propositional logic, also known as sentential logic and statement logic, is the branch of logic that studies ways of joining and/or modifying entire propositions, statements or sentences to form other propositions, statements or sentences. Hopefully these new propositions are of some value to us.

Reading between the lines you will see that propositions are sometimes called sentences or statements. Elsewhere they may be called cases. Anyway by a proposition we mean a statement that can be assigned a truth value of T, if it is true and F, if it is false. Because we want to combine these from time to time in algebraic type ways, we'll mostly use the value 1 for T and 0 for F.

Going back to the last section we know that the following are propositions:

1. These lines are parallel;
2. This angle is 30°;
3. Jake has a knife;
4. Jake killed the victim;
5. All men are liars;
6. Helen is a liar;
7. Helen is a man.

Not all sentences are propositions though. Look at these. Do any of them lack an important property??

1. Go home!
2. Can I have a drink?
3. Advance Australia fair.
4. This sentence is false.
5. These peas are horrible.

Question

45 Consider the sentences (viii) to (xii) above and say why each is a proposition or not.

It may already have entered your head that there is a similarity here with Boolean algebra. The emphasis that we have so far on zeros and ones may have got your antennae going. Let us move further in this direction by defining the negation of a proposition. If p is a proposition, then the negation of p, written p', is true when p is false and false when$p$ is true. This can be shown in a truth table, see Table 5. A truth table is one where we give all possible truth vales to all propositions and determine the resulting truth values of all of the operations on these truth values.

Table 5: The truth table for negation

p	p'
0	1
1	0

Question

46 Write down the negations of propositions (i) to (vii) and (xii) in the text above.

More on truth tables

In this section we concentrate on truth tables for two ways of combining propositions. These ways are important in certain searches, for example on the web.

The first method of combination is when we are looking for both a proposition p and a proposition q. So we want both p and q. We'll represent this as the proposition pq and show its truth table in Table 6. You can see from the table, perhaps, why we write the combine proposition as pq.

Table 6: The truth table for pq

p	q	pq
0	0	0
0	1	0
1	0	0
1	1	1

As you would expect, the result for pq is only 1 (true) when both p and q are true.

You might want to search for female politicians so p might be 'the person is a woman' and q 'the person is a member of parliament'. Clearly you will get quite a different, and perhaps less useful result, if you searched for 'either the person is a woman or the person is a member of parliament'. But you might want to search for flights that 'go from Melbourne to London either through Singapore or Hong Kong'. So the combination either p or q is worth considering. This proposition is represented by $p+q$. We show the truth table of this combination in Table 7.

Table 7: The truth table for $p+q$

p	q	$p+q$
0	0	0
0	1	1
1	0	1
1	1	1

This table has only one false entry and that is for flights that go neither of the ways stip-

Questions

47 Which of the following propositions are of the form pq and which of the form $p+q$? What are p and q in each case?

i May does the washing and Ria does the drying;

ii Fred watches sport or detective programmes;

iii Today it will be 25° and there will be showers;

iv Marilyn will wear a skirt or a pair of jeans;

v Leanne will wear a skirt with jeans underneath;

vi Spot is a black dog.

Questions

48 Suppose that p is the proposition 'Alex swam at the Olympics' and q is 'Alex won a gold medal'. What do the following propositions mean in words?

i pq;

ii $p+q$;

iii pq';

iv $p+q'$;

v $p+p'q$;

vi $p'q+pq'$.

49 Write truth tables for the compound propositions (iii) to (vi) of Q48. What is interesting about (v)?

Deduction

We talked about the reasoning process of deduction earlier (page reference) and now is the time to look at its truth table (Table 8). This comes with perhaps a little surprise or two.

Table 8: The truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

It seems reasonable to think that if p and q were both true that you would get a correct implication proposition; hence the entry 1 in that part of the table. But some of the other entries may be a little mystifying, so let's go through them one by one.

So what if p and q are both false. How can we use a true argument to go from p to q? Suppose that p is 'Alyssa was prime minister of Australia in 2015' and q is 'Alyssa was elected to parliament before 2016'. Now neither p nor q is true but if $p \rightarrow q$ may be true, to be a prime minister of Australia you have to be elected first. Because there is the chance of the implication being true we take that as the preferred option.

So what if p is false and q is true. What can be said about a deduction from p to q? An example using words is not easy to find but look at p is '$5 = 7$'and q is '$0 = 0$'. Then p is false and q is true, but if we multiply both sides of p by 0 we correctly get q.

Finally, if p is true and q is false we would expect $p \to q$ to be false. If not we would have to rethink all of our known theorems!

Now in many ways this is unsatisfactory. It is possible that we might come up with counterexamples to some of the cases in the truth table. In that case it might be better to think of the truth here as founded on the fact that it is possible for a true implication to take us between p and q except in the one case where p is true and q is false.

Questions

50 Make up your own examples to show that the truth table for deduction is correct.

51 Let p, q and r be three propositions. Produce truth tables for the following:

- i $p' \to q$;
- ii $p \to qr$;
- iii $p + q \to p'$;
- iv $(p + q') \to pq$;
- v $(q \to p) + (p \to q)$.

52 What is the truth table for p if and only if q?

53 Can you find a proposition that has the same truth table as that of $p \to q$?

Tautologies

There are certain truth tables where all the entries are true (1). We found such an example in Q51(v): the proposition $(q \to p) + (p \to q)$is always true. Propositions that are always true or valid are called tautologies.

It is through tautologies that we can decide whether or not an argument is valid. If the truth table for the argument shows that the overall proposition is a tautology, then the argument is valid. Suppose that p is the statement 'it is hot'. Then $p + p'$ tells us that either it is hot or it is not hot'. As Q54(i) is a tautology, then, as we would expect, either it is hot or it's not to be one too. In discussions or even written statements, things often get complicated so quickly that it is hard to see if the overall argument is valid. The truth tables of such arguments allow us to decide if they are true or not by seeing whether or not they are tautologies.

Question

55 Decide which of the following arguments is valid:

i When it rains the roof gets wet. The roof is wet. Therefore it has rained;

ii In a right angled triangle with sides a, b and c with c the hypotenuse, $a^2 + b^2 = c^2$. Therefore the triangle with sides 3, 4 and 5 is a right angled triangle;

iii In a right angled triangle with sides a, b and c with c the hypotenuse, $a^2 + b^2 = c^2$. Therefore the triangle with sides 3, 4 and 6 is not a right angled triangle.

Boolean Algebra Again?

There has always been a hint in the air that what we have been doing with the propositions mirrors what we did with Boolean algebras. Let's see what evidence we have so far on this connection.

- Propositional logic is based on 0 and 1
- There are operations $p+q$, pq and p'
- $p + p' = 1; (p')' = p; (pq)' = p' + q'$;and $(p+q)' = p'q'$;

All of these appear in Boolean algebra. To be sure that we have a Boolean algebra all we have to do is to make sure that the axioms of Boolean algebra hold for propositions (see insert page numbers). Let's look at a few for a start.

- **Axiom 1**

  B contains at least two elements. Propositional logic contains at least true and false. These hover in the background just as on and off do for switching circuits.

- **Axiom 2**

  B is closed under $+$ and $\times$. That is for all x and y in $B, x + y$ and $x \times y$ are in B. For propositions p and q, $p + q$ andpq are both propositions.

- **Axiom 3**

  $+$ and $\times$ are commutative. That is for all x and y in $B, x + y = y + x$ and $x \times y = y \times x$. So we have to show that $p + q = q + p$ and $pq = qp$ are tautologies. We do the first in Table 9.

Table 9: Towards a proof of commutativity

p	q	$p+q$	$q+p$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

From here we can see that $(p+q) \to (q+p)$and $(q+p) \to (p+q)$. So + (either or) in propositional logic are the same.

Question

56 Prove that $pq = qp$.

57 Prove that the remaining axioms are also true.

Gates

Given propositional logic and switching circuits are related via Boolean algebra it makes sense to see if we can automate operations in logic through circuits. When you think about it $p+q$ must be linked in parallel as we need both propositions to be true in order for $p+q$ to be true. In the same way pq should be a parallel connection.

Question

58 Find a logic circuit for $p \to q$.

59 Draw circuits for $q(p+q)$ and $p+pq$ and show that they are equivalent using two methods.

But logic circuits have a simplification that doesn't occur in switching circuits. This is the concept of gates that will for specific jobs. So we have AND gates, OR gates, NAND gates NOR gates and NOT gates. Let's take these one at a time.

AND gates

These represent the 'both and' operation in logic. In the simplest case the inputs are p and q and the outputspq. However, to deal with a number of proposition, we may have inputs $p, q, ., r$and outputs $pq \dots r$.

8a

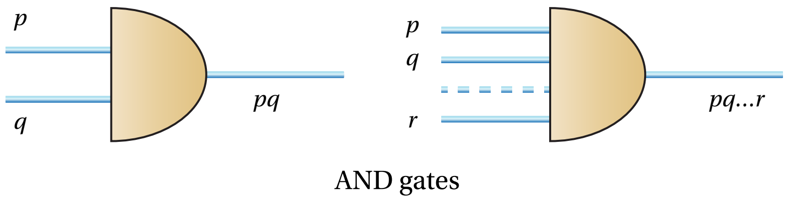
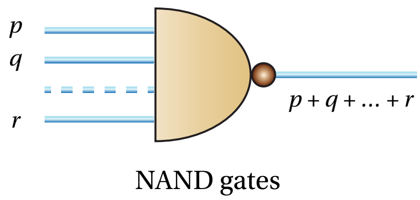


Figure 8a: AND gates

NAND gates

These provide the negation of AND gates. So inputs $p, q, \ldots, r$ produce outputs $p'q' \ldots r'$

8b



p'q'...r'

Figure 8b: NAND gates

Or gates

As you might expect, the inputs for OR gates are $p, q, \ldots, r$ and the outputs $p + q + \cdots + r$.

8c

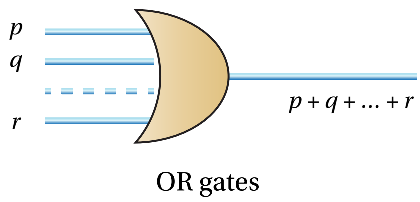


Figure 8c: OR gates

NOR gates

Here we have the negation of OR to give outputs of $(p + q + \cdots + r)'$

8d

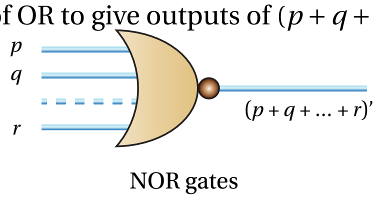
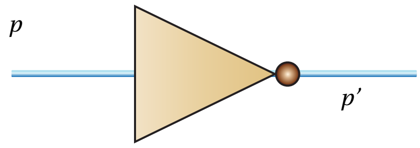


Figure 8d: NOR gates

NOT gates

These are for negation. So p goes in and p' comes out.

8e



NOT gates

Figure 8e: NOT gates

These gates allow a considerable reduction in the diagrams necessary to produce the corresponding logic circuits. As well as that they are on tap for actually constructing the logic circuits themselves.

Question

60 Produce logic circuits for the logical expressions below. Then represent these using the gates of Figure 8.

- i $(p'q)' + r$;
- ii $((pq)'r + s)(pq + r's)$.

61 . What proposition does the following gate diagrams represent?

9a

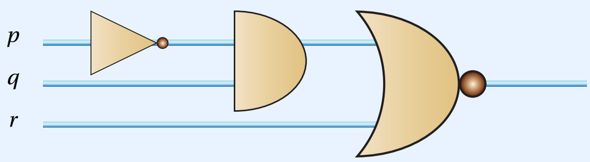


Figure 9a: Question 61

9b

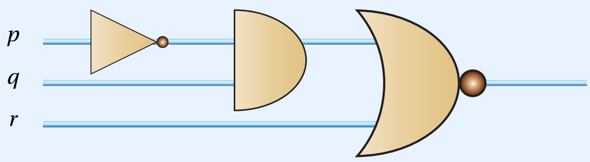


Figure 9b: Question 61

Search engine queries also employ the logic gates we have been considering. To do this searching, each web page on the Internet can be thought of as an 'element' of a 'set'.

Elements in this set that re combined with a space between them are accepted as being connected by AND. So typing 'blue dogs' should get you sites where the words 'blue' and 'dogs' are next to each other.

On the other hand 'blue OR dogs' should give you sites that contain either 'blue' or 'dogs'.

If you are looking for blue objects that are not dogs you should get this by using 'blue NOT dogs'. Alternatively you can insert a minus sign before the second word (or phrase) as in 'blue-dogs' (with no space between the minus sign and the second object.

Karnaugh Maps

We have already seen two methods for simplifying switching circuits. These are manipulation using the properties of Boolean algebra and the function approach that reduces the problem down to looking at values of the function at specific points. (Of course these two methods can be used together.) Karnaugh maps

(see http://en.wikipedia.org/wiki/Karnaugh_map and

http://www.facstaff.bucknell.edu/mastascu/eLessonsHTML/Logic/Logic3.htmlKMapsIntro)

are a third method that is more efficient than the function approach and which are often used for relatively small numbers of switches (or propositions). Once the number of switches gets past a certain point, however, computer algorithms need to come into play.

Example of a Karnaugh Map: The switching circuit for Voting for Three (see Q3) can be rewritten as

x	y	z	Light
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Table 9: Voting for three

The Karnaugh map (K-map) of this is presented below. It contains all the information of the original table but that information is represented differently. The result is a decrease in third number of cells that need to be considered.

BC

A

	00	01	11	10
0	0	0	1	0
1	0	1	1	1

Table 10: an example of a K-map

Note:

- The entries 00, 01 etc. along the top of table 10 represent the values of y and z together. So the entry in column 4 row 2 is the value for $x = 1, y = 1, z = 0$.
- The headings of the columns differ by 1 in each place as you move from left to right. (The same is true for rows regardless of how many rows there are.)
- The headings increase in an unnatural way in that they do not increase in the way that binary numbers would. However, this enables a term and its complement to be next to each other and so reductions of the form $w + w'$ are easier to spot and combine.
- K-maps can be produced for any number of switches or propositions. It is probably more efficient to divide the numbers so that about half are along the top and the rest down the side.

Now the entry in Table 10 in column 2 row 2, represent the value of $xy'z$ and the entry in column 3 row 2 represent ABC. But as these have both got a value of 1 (and in the disjunctive normal form) they give $xy'z + xyz$ which simplifies to xz. This always happens when two ones are next to each other in a K-map. From Table 10 above we can see that we could have combined xyz with xyz' to get xy. This happens whenever there are two adjacent ones in the same row or column. So working with columns the third column of Table 10 shows that $x'yz + xyz = yz$. As a result the switching circuit can be simplified to $xz + xy + yz$.

For groups of 1s that have area that is a power of 2, more substantial savings can be made in the simplification process (see http://en.wikipedia.org/wiki/Karnaugh_mapSolution).

Question

62 Use the K-map method to simplify the switching circuits for the Q42(c) (page reference).

63 Use the K-map method to simplify the logic circuits in Q60.

Note that other questions can be found on the site

http://www.facstaff.bucknell.edu/mastascu/eLessonsHTML/Logic/Logic3.htmlKMapsIntro.

64 Simplify the following partial table using the Karnaugh approach. Check that the disjunctive normal form gives the same result.

yz

wx

	01	11
01	1	1
11	1	1

Logic and Boolean Algebra - Answers

Switching Systems: Answers

1 **a**

Switch x	Switch y	Result
on	on	on
on	off	off
off	on	off
off	off	off

Simplifies to x and y in series.

b

Switch x	Switch y	Result
on	on	on
on	off	on
off	on	off
off	off	off

Simplifies to just the switch x.

c

Switch x	Switch y	Result
on	on	on
on	off	off
off	on	on
off	off	off

Simplifies to just the switch y.

d

Switch x	Switch y	Switch z	Result
on	on	on	on
on	on	off	on
on	off	on	on
off	on	on	on
off	on	on	on
off	on	off	off
off	off	on	off
off	off	off	off

Simplifies to the pair y and z in series and the pair in parallel with x.

e

Switch x	Switch y	Switch z	Result
on	on	on	on
on	on	off	on
on	off	on	on
on	off	off	off
off	on	on	on
off	on	off	off
off	off	on	off
off	off	off	off

Simplifies to x and y in series, y and z in series and z and x in series. Then these three series pieces in parallel.

f

Switch x	Switch y	Switch z	Result
on	on	on	on
on	on	off	off
on	off	on	off
on	off	off	off
off	on	on	on
off	on	off	off
off	off	on	off
off	off	off	off

Simplifies to y and z in series.

2

Jamal	John	Light
on	on	on
on	off	off
off	on	off
off	off	off

This is the same as $x$and y in series.

3

Veronique	Jamal	John	Light
on	on	on	on
on	on	off	on
on	off	on	on
on	off	off	off
off	on	on	on
off	on	off	off
off	off	on	off
off	off	off	off

Does this remind you of Q1(e)?

4

Swich x	Switch y	Light
on	on	off
on	off	on
off	on	on
off	off	off

This is the same as x and y in series.

5 We leave this to you.

6 This should be straightforward. However, have a look at the arithmetic as you go along.

7 Addition, +, almost.

8 $1+1=1$. Not in arithmetic nor even modulo arithmetic.

9 What we've done below assumes that you will allow $1+1=1$.

Q1(**a**) $x+y$; **Q1**(**b**) x: **Q1**(**c**) y; **Q1**(**d**) $x+yz$

Q1(**e**) $y+yz+zx$; **Q1**(**f**) yz **Q2** xy; **Q3** $xy+yz+za$;

Q4: This is problematic. We'll come back to this after we've done complements.

10 Note that if x is on (1), then x' is off (0) and vice-versa.

Check the algebraic answers we have given hear to make sure that they are correct. We will give an efficient way to arrive at these results later. (REF) From the results here can you guess an efficient method?

a

Switch x	Switch y	Result
1	1	0
1	0	0
0	1	1
0	0	0

This is $x'y$.

b

Switch x	Switch y	Switch z	Result
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

This is $xy'z'$.

c

Switch x	Switch y	Result
1	1	1
1	0	1
0	1	1
0	0	0

This is $x+y$.

d

Switch a	Switch b	Switch c	Result
1	1	1	0
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	11
0	1	0	1
0	0	1	0
0	0	0	1

$x+y+z'$.

e

Switch x	Switch y	Switch z	Result
1	1	1	0
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	0

$x'y + yz' = y(x' + z')$.

f

Switch x	Switch y	Switch z	Result
1	1	1	0
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

$xyz + x'y'z'$.

11 $xy' + x'y$

12 We leave this to you.

13 First we want 0 + 0 = 0. So what might 0 be? What's the nothing set? How about the empty set $\emptyset$.

- **Union**

  $\emptyset \cup \emptyset = \emptyset$- fine so far as a parallel for 0 + 0 = 0. $\emptyset \cup I = I \ldots$ good; and$I \cup I = I$ So union seems to tie in nicely with +.

- **Intersection**

  $\emptyset \cap \emptyset = \emptyset; \emptyset \cap I = \emptyset$ and $I \cap I = I$. Clearly intersection isn't the same as + in switching circuits. But what does intersection parallel?

14 Intersection looks pretty good for multiplication.

15 $A \cup I = I$ and this mirrors addition from switching circuits, at least for $A = \emptyset$.

$A \cap I = A$ and this mirrors multiplication or what you might think multiplication might be.

16 Complementation. This is defined naturally on sets. It has the property that $A \cup A' = I$ and $A \cap A' = \emptyset$. These are consistent with $0 + 1 = 1$ and $1 + 0 = 1$; $0 \times 1 = 0$ and $1 \times 0 = 0$. We have consistency!

17 This should be straightforward by looking through the text.

18 $0' = 1$ and $1' = 0$.

19 Yes. You will need to use: $0 = \emptyset$, $1 = I, \{x\}' = \{y\}$and $\{y\}' = x$. The + is the union and the $\times$ is intersection.

For any set I, the corresponding Boolean algebra is the set of all subsets of I along with union and intersection as the two operations. (This is known as the power set of I.) If A is any subset of I, then A' is the complement of A in$I$.

20 $0 = 1; 1 = pq, p' = q$and $q' = p$.

In general you need the Boolean algebra of the biggest number, n, along with of all its factors. In this algebra, the complement of the number a is n/a, $0 = 1$ and $1 = n$.

It is not possible for n to have repeated prime factors. (Why not? You will certainly have problems with$p + p'$ and $p \times p$ if p^2 is a factor of n.)

21 Whatever you come up with it must satisfy all of the ten axioms we have given for a Boolean algebra.

You might consider the Boolean algebra consisting of all finite and cofinite sets of integers. By a cofinite set we mean any subset of integers whose complement is finite. In this case, + is union of sets and $\times$ is the intersection of sets, $0 = \emptyset$, $1 =$ the set of integers, and the complement of a finite set is a cofinite one and vice versa.

Does this idea work if you replace 'integers' with 'fractions'? How about real numbers? If A and B are Boolean algebras, then so is $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$. What is are zero, the one and the complement of (x, y)? What about $A \times B \times C$?

You may find other examples of sets that qualify for Boolean algebras on the web. Can you change the definition of + and $\times$ to make such a set work?

22 **i** $x + 1 = (x + 1) \times 1 = (x + 1) \times (x + x') = x + (1 \times x') = x + x' = 1$.

ii $x \times 0 = x \times (x \times x') = (x \times x) \times x' = x \times x' = 0$.

iii $x \times (x+y) = (x+0) \times (x+y) = x + (0 \times y) = x + 0 = x$.

iv $x + (x \times y) = (x \times 1) + (x \times y) = x \times (1+y) = x \times 1 = x$.

Are there quicker ways to do any of these?

23 i false: a can never equal $a'.x + x' = 1$, but $x + x = x$. So $x = 1$. However $1 \times 1 = 1$ and $1 \times 1' = 0$.

ii True: $1 = 1 \times 1 = (x' + x) \times (x' + x'') = x' + (x \times x'')$ so $x = x \times x''$ and $x'' = x$.

iii False by assuming that y is also an inverse of $x : x' = x' \times 1 = x' \times (x+y) = (x' \times y) + (x' \times y) = x' \times y$. but we can replace x' by y and y by x' in this series of equations to get $x' = x' \times y = y \times x' = y$.

24 To do this we need to prove that $(x+y) + (x' \times y') = 1$ and $(x+y) \times (x' \times y') = 0$. In the former case, $x + y + (x' \times y') = x + [(y + x') \times (y + y')] = x + (y + x') \times 1 = x + 1 = 1$. In the latter case, $(x+y) \times (x' \times y') = x \times x' \times y' + y \times x' \times y' = 0$.

25 $(x \times y)' = x' + y'$. For example, $(0 \times 1)' = 0' = 1$ and $0' + 1' = 1 + 0 = 1$. The general result can be proved by going through the steps of Q24 and replacing them by their duals.

26 i $x + (x \times y) = (x \times 1) + (x \times y) = x \times (1+y) = x \times 1 = x$;

ii $x + (x' \times y) = (x + x') \times (x+y) = 1 \times (x+y) = x + y$.

The duals are (i) $x \times (x+y) = x$; and (ii) $x \times (x' + y) = x \times y$.

27 Strictly speaking the principle of duality becomes a new axiom and we can reduce Axioms 2, 3, 4, 5, 6, 7 and 8 'by half'.

28 i $x'z + y'$;

ii 1;

iii $x + z$.

29 A switch in series with a wire is the same as the switch alone.

30 Note that we use the table method of proof. Why?

31

x	y	z	$x+yz$	$(x+y)(x+z)$
1	0	0	0	0
0	0	1	0	0
0	1	0	0	0
1	0	1	0	
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

32

Switch x	y	x+xy	x(x+y)
0	0	0	0
1	0	1	1
1	1	0	1
0	1	0	0
1	1	1	1

33 This is because we can interchange + and x etc. in the above axioms.

34
- $xy(x+y) = xxy + xyy = xy + xy = xy$;
- $x(x+y)(y+x) = x(x+y) = xx + xy = x + xy = x(1+y) = x$;
- $(x+y+x)[(x+y)y] = (x+y)y = xy + y = y$;
- $(x+y)(x+z) = x + xz + yx + yz = x + yz$;
- $(xy+z)(x+yz) = xy + xyz + xz + yz = x(y+z) + yz = xy + yz + zx$;
- $z(xy+x+yz)y = xyz + xz + xyz = xyz + yz = yz$.

35
- $x'y$ - no simplification;
- $xy'z'$ - no simplification;
- $x + x'y = x + y$;
- xyz'- no simplification;
- $xyz' + x'y + yz' = xyz' + yz' + x'y = x'y + yz' = y(x'+z')$;
- $(xz+y')(yz+x'z') = xyz + xzx'z' + y'yz + x'y'z' = xyz + x'y'z'$.

36 Does $xy' + x'y$ work?

Functions

37 i 2;

ii 2;

iii 1.

38 i $3 \times 2 - 5x$;

ii 10×2;

iii e^{4x}.

39 i $f(x, y)$ simplifies to 1.

x	y	$f(x,y)$
1	1	1
1	0	1
0	1	1
0	0	0

ii $f(x, y)$ simplifies to y'.

x	y	$f(x,y)$
0	0	1
1	0	1
0	1	0
1	1	0

iii

x	y	$f(x,y)$
0	0	0
1	0	0
0	1	0
1	1	1

iv $f(x, y)$ simplifies to xy'.

x	y	$f(x,y)$
0	0	0
1	0	1
0	1	0
1	1	0

v $f(x,y)$ simplifies to $x+y$.

x	y	$f(x,y)$
0	0	0
1	0	1
0	1	1
1	1	1

40 i

x	y	z	$f(x,y,z)$
0	0	0	0
0	1	0	0
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	1
1	0	1	1
1	1	1	1

Why do we get that answer? Is it because the expression simplifies to something simple? If so, what

ii

x	y	z	$f(x,y,z)$
0	0	0	0
0	1	0	1
0	0	1	1
0	1	1	1
1	0	0	1
1	1	0	1
1	0	1	1
1	1	1	1

Why do we get that answer? What does the expression simplify to?

41 In a Boolean algebra, $x^2 = x \times x = x$. So every power of an element reduces to the original element. Hence there are no powers in a polynomial function. On the other hand $x \times x' = 0$, so there are no possible products of the form xx'. There are no other possibilities for the potential powers and products of x and x' so $f(x) = \alpha x + \beta x'$. This assumes that there are no constants. Any non-zero constant has to be 1 which would make the function equivalent to 1.

The coefficients α and β must come from the Boolean algebra in question. But we can be more precise than this. Now $f(0) = \alpha \times 0 + \beta \times 1 = \beta$ and similarly $f(1) = \alpha$. So $f(x) = f(1)x + f(0)x'$.

42 The answer here is similar to that in Q40, but we have more combinations of variables. The constants can be defined in terms of f(0, 0), f(0, 1), f(1, 0) and f(1, 1).

Three variables is a little more messy because it will have more terms. But the coefficients can be found in the same way via terms like $f(1,0,1)$.

43 a $f(x,y) = xy$, so we need two switches in series.

b $f(x,y,z) = xy + z(xy' + x'y)$.

c $f(w,x,y,z) = wxy + wxz + wyz + xyz$.

d $f(w,x,y,z) = wx$, w and x represent Able and Baker.

44 $f(x,y) = xy' + x'y$.

45 $f(x,y,z) = xyz + xy'z' + x'yz' + x'y'z$.

46 Only (xii) is a proposition because it is the only one that is a statement that can be given a value of true (1) or false (0). In particular exclamations (viii), questions (ix), and declamations (x) can never be propositions. On the other hand, (xii) is a statement about peas. For some peas the statement may be true and for others it may be false. So (xii) is a proposition.

But (xi) is a worry. If the sentence is false then it has to be true, and if the sentence is true it is false. So (xi) can't be assigned either 0 or 1. The self-reflective nature of this sentence is a hurdle against it being a proposition. But 'this sentence is true' is a proposition because it can be given the truth value T (1). And 'this other sentence is false' is also a proposition, because it can be true or false depending on the other sentence. So you have to tread carefully with propositions. A small word change can mean a great difference.

47 i These lines are not parallel;

ii This angle is not 30°;

iii Jake doesn't have a knife;

iv Jake didn't kill the victim;

v Not all men are liars (but not all men are not liars);

vi Helen is not a liar;

vii Helen is not a man; (xii) these peas are not horrible.

48 pq: (i), (iii), (v), (vi) (black and dog). $p+q$: (ii) (iv). The 'and' and the 'or' usually give the game away.

49 (i) Alex swam at the Olympics and won a gold medal;

(ii) Alex swam at the Olympics or won a gold medal;

(iii) Alex swam at the Olympics and didn't win a gold medal;

(iv) Alex swam at the Olympics or didn't win a gold medal;

(v) Alex swam at the Olympics or didn't swim at the Olympics and won a gold medal;

(vi) Alex didn't swim at the Olympics and won a gold medal or did swim at the Olympics and didn't win a gold medal.

50 **iiii**

p	q	q'	pq
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

iv

p	q	q'	$p+q$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

v

p	p'	q	$p'q$	$p+p'q$
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1

Note that $p+p'q$ is the same as $p+q$. Why might this be the case?

vi

p	p'	q	q'	$p'q$	pq'	$p'q+pq$
0	1	0	1	0	0	0
0	1	1	0	1	0	1
1	0	0	1	0	1	1
1	0	1	0	0	0	0

51 This will depend on what you have written.

52 (i) only false if $p=0=q$;

(ii) only false if $p=q=1$ and $r=0$;

(iii) only false if $p=0$ and $q=1$;

(iv) only false if $p=q=0$ and$p=1$ and $q=0$;

(v) always true.

is false for $p=1$ and $q=0$ and $p=1$ and $q=0$.

53 Try $p'+q$.

54 (i), (ii), (iii), (iv) are tautologies but the last two are not.

55 Let p =it rains and q =roof wet. Now $q \to p$ is not a tautology so the conclusion is false.

Although a 3, 4, 5 triangle is right angled it cannot be proved using this argument because it is not a tautology;

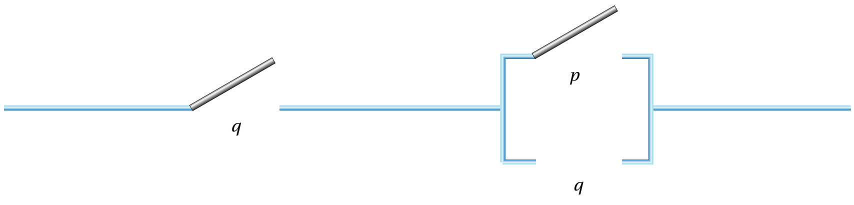
Let p = right angled triangle and $q = a^2 + b^2 + c^2$. Does $(p \to q) \to (q' \to p')$? This is a tautology, so the 3, 4, 6 triangle is not a right angled one.

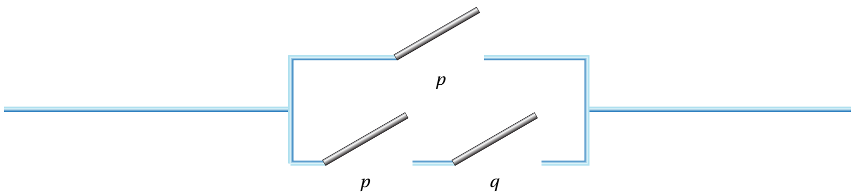
56 Again the truth tables of pq and qp are the same. truth tables for all of the axioms do what you hope they would do.

57 See Q53.

58 The circuits are shown below.

10

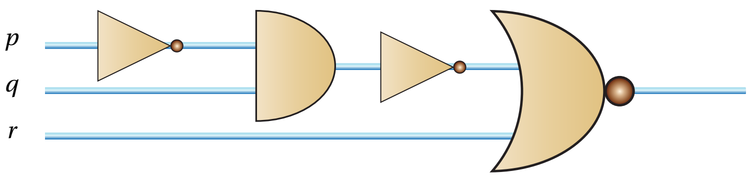




The equivalence can be shown using truth tables or Boolean algebra.

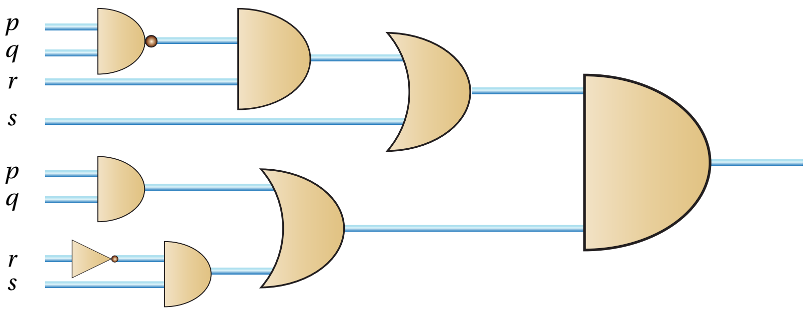
59 **i**

11a



ii

11b



60 **a** $(p'q + r)'$; **b** $(p'q' + r)s$.

61 The table for Q42(c) is

w	x	y	z	vote
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

One corresponding K-map is shown below.

yz

	00	01	11	10
00	0	0	0	0
01	0	0	1	0
11	0	1	1	1
10	0	0	1	0

Going along the third row gives

$$wxy'z + wxyz = wxz \text{ and } wxyz + wxyz' = wxy;$$

going down the third column gives

$$w'xyz + wxyz = xyz \text{ and } wxyz + wx'yz = wyz$$

. The combined result is $wxy + wxz + wyz + xyz$. Check this with the answer in Q42(c).

62 The K-map can be written straight from the proposition.

qr

	00	01	11	10
0	1	1	1	0
1	1	1	1	1

The first row gives:

$$p'q'r' + p'q'r = p'q' \text{ and } p'q'r + p'qr = p'r$$

. The second row gives:

$$pq'r' + pq'r = pq', pq'r + pqr = pr, and pqr + pqr' = pq$$

. But as the row contains four ones we can get a simplification more quickly. Since p remains 1 throughout and $q$and r change from 0 to 1, then the whole row reduces to p. (Check that $pq' + pr + pq = p$ using Boolean algebra.)

The columns give: $q'r', q'r$ and qr.

Altogether this simplifies to $p + q' + r$.

But this can be done more efficiently by noting that we have several 2×2 rectangles of ones. The first one consists of columns 1 and 2 and rows 1 and 2. As a result, this block simplifies to q' as this is the only term that is unchanged throughout the block. Similarly the only constant term in the block made up of columns 2 and 3 and rows 1 and 2. The only constant term here is r. Then row 2 is also constant on p. This method produces $p + q' + r$ much more efficiently than the adjacent terms method above.

rs

	00	01	11	10
00	0	1	0	0
01	0	1	0	0
11	0	1	1	0
10	0	1	0	0

The second column is a block of four and so gives a simplification of $r's$.

The second and third columns of the third row gives pqs.

Overall this gives $s(r' + pq)$.

63 By the K-map approach, since x and z are the only terms that are all 1 in the 2 x 2 square, and the other terms are both 0 and 1, we get xz.

By the disjunctive normal form we get $w'xy'z + w'xyz + wxy'z + wxyz$. Now the first two terms together give $w'xz$ and the last two give wxz. Then $w'xz + wxz = xz$ as Karnaugh predicts.

Years

11 & 12