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A guide for teachers – Years 11 and 12

Algebra and coordinate geometry: Module 2

Coordinate geometry

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Coordinate geometry – A guide for teachers (Years 11–12)

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Coordinate geometry

Assumed knowledge

- Familiarity with the coordinate plane.
- Facility with algebra — see the module *Algebra review*.
- Knowledge of elementary trigonometry.

Motivation

Coordinate geometry is where algebra meets geometry. In secondary school mathematics, most coordinate geometry is carried out in the coordinate plane $\mathbb{R}^2$, but three-dimensional geometry can also be studied using coordinate methods.

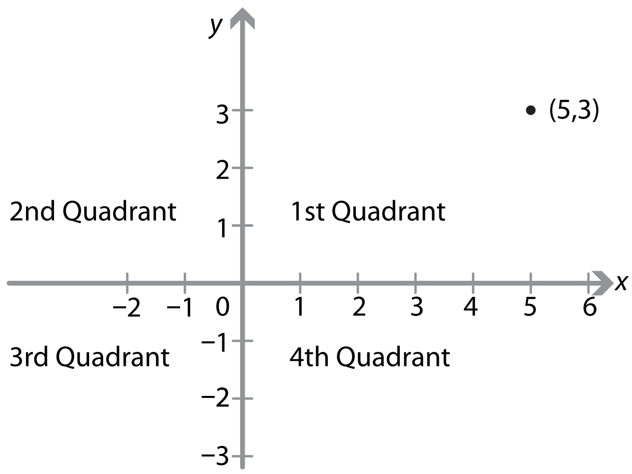
Classical Euclidean geometry is primarily about points, lines and circles. In coordinate geometry, points are ordered pairs (x, y), lines are given by equations $ax + by + c = 0$ and circles by equations $(x-a)^2 + (y-b)^2 = r^2$. Thus the simplest, most useful and most often met application of coordinate geometry is to solve geometrical problems. Parabolas, ellipses and hyperbolas also regularly arise when studying geometry, and we shall discuss their equations in this module.

Readers not familiar with coordinate geometry may find the TIMES module *Introduction to coordinate geometry* (Years 9–10) useful as a gentle introduction to some of the ideas in this module.

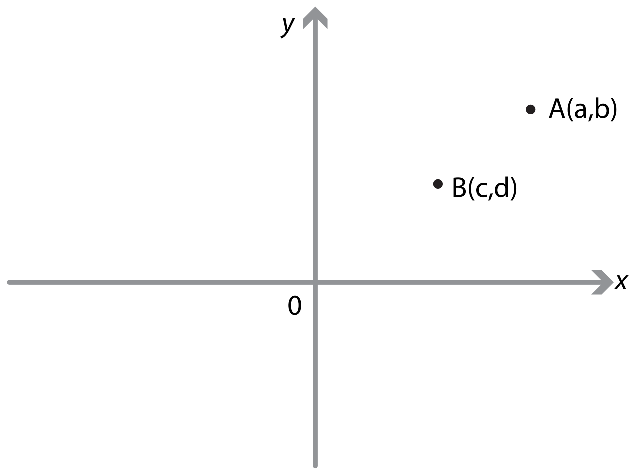
Content

The Cartesian plane

The number plane, or Cartesian plane, is divided into four **quadrants** by two perpendicular lines called the x-**axis**, a horizontal line, and the y-**axis**, a vertical line. These axes intersect at a point called the **origin**. Once a unit distance has been chosen, the position of any point in the plane can be uniquely represented by an **ordered pair** of numbers (x, y). For the point $(5,3)$, for example, 5 is the x-coordinate and 3 is the y-coordinate, sometimes called the first and second coordinates. When developing trigonometry, the four quadrants are usually called the first, second, third and fourth quadrants as shown in the following diagram.



There are a number of elementary questions that can be asked about a pair of points $A(a, b)$ and $B(c, d)$.



- What are the coordinates of the midpoint of the interval AB?
- What is the distance between the two points?
- What is the gradient of the interval joining the two points?
- What is the equation of the line joining the two points?

The answers to all of these questions will be discussed in this module, as will answers to other questions such as:

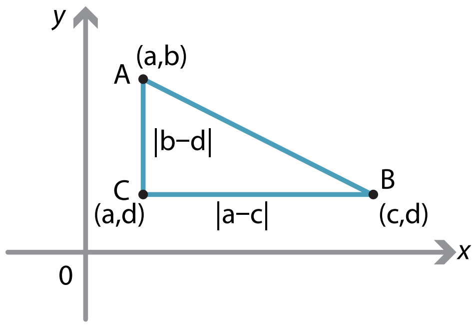
- What is the equation of the line through the origin parallel to AB?
- What is the equation of the perpendicular bisector of AB?
- At what angle does the line AB meet the x-axis?

These questions give some idea of the scope of coordinate geometry.

The distance between two points

Distances in geometry are always positive, except when the points coincide. The distance from A to B is the same as the distance from B to A.

In order to derive the formula for the distance between two points in the plane, we consider two points $A(a,b)$ and $B(c,d)$. We can construct a right-angled triangle ABC, as shown in the following diagram, where the point C has coordinates (a,d).



Now, using Pythagoras' theorem, we have

$$\begin{aligned} AB^2 &= |b-d|^2 + |a-c|^2 \\ &= (a-c)^2 + (b-d)^2. \end{aligned}$$

So

$$AB = \sqrt{(a-c)^2 + (b-d)^2}.$$

A similar formula applies to three-dimensional space, as we shall discuss later in this module.

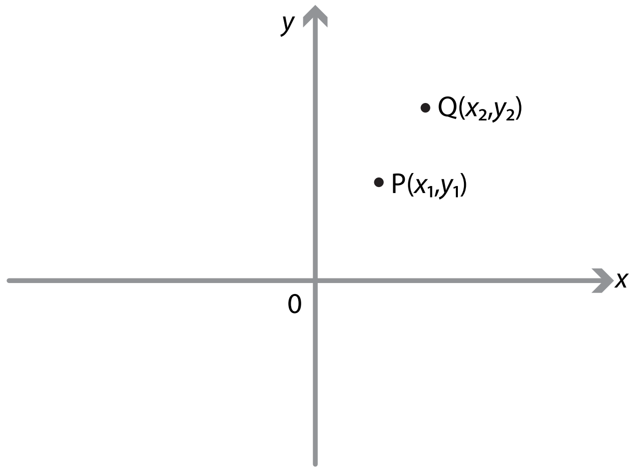
The distance formula

Suppose $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *are two points in the number plane. Then*

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and so

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

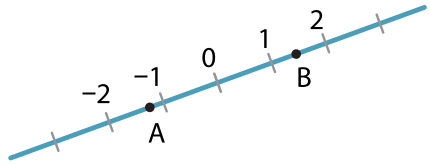


It is clear from the distance formula that:

- $PQ = QP$
- $PQ = 0$ if and only if $P = Q$.

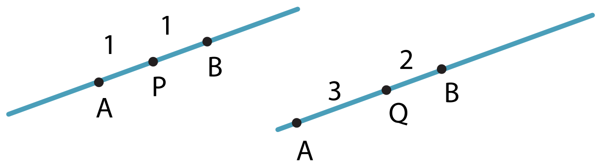
Midpoints and division of an interval

Given two points A, B in the plane, it is clearly possible to create a number line on AB so as to label each point on AB with a (real) number.



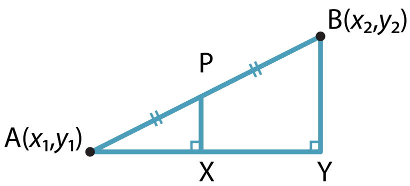
There are (infinitely) many ways in which this can be done, but it turns out not to be particularly useful for geometrical purposes.

On the other hand, in the following diagrams, we can say that the point P divides the interval AB in the ratio $1:1$ and the point Q divides the interval AB in the ratio $3:2$.



Midpoint of an interval

The midpoint of an interval AB is the point that divides AB in the ratio $1:1$.



Assume that the point A has coordinates (x_1, y_1) and the point B has coordinates (x_2, y_2). It is easy to see, using either congruence or similarity, that the midpoint P of AB is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

This is shown in the module *Introduction to coordinate geometry* (Years 9–10).

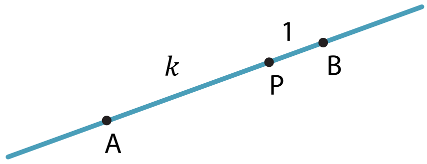
Internal division of an interval

We now generalise the idea of a midpoint to that of a point that divides the interval AB in the ratio $k:1$.

Definition

Suppose $k > 0$ is a real number and let P be a point on a line interval AB. Then P divides AB in the ratio $k:1$ if

$$\frac{AP}{PB} = \frac{k}{1} = k.$$



Note. As $k \to 0$, $P \to A$ and as $k \to \infty$, $P \to B$.

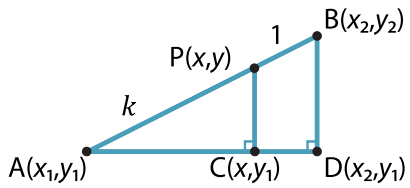
Theorem

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be two points in the plane and let $P(x,y)$ be the point that divides the interval AB in the ratio $k:1$, where $k>0$. Then

$$x=\frac{x_1+kx_2}{1+k} \qquad \text{and} \qquad y=\frac{y_1+ky_2}{1+k}.$$

Proof

If $x_1=x_2$, then it is clear that x is given by the formula above. So we can assume that $x_1\neq x_2$. Consider the points $C(x,y_1)$ and $D(x_2,y_1)$, as shown in the following diagram.



The triangles ACP and ADB are similar (AAA). So

$$\begin{aligned}\frac{AP}{AB}&=\frac{AC}{AD}\\ \frac{k}{k+1}&=\frac{x-x_1}{x_2-x_1}\\ kx_2-kx_1&=kx-kx_1+x-x_1\\ (k+1)x&=kx_2+x_1\\ x&=\frac{x_1+kx_2}{1+k}.\end{aligned}$$

Similarly,

$$y=\frac{y_1+ky_2}{1+k}. \qquad \square$$

Exercise 1

If $P(x,y)$ lies on the interval $A(x_1,y_1)$, $B(x_2,y_2)$ such that $AP:PB=a:b$, with a and b positive, show that

$$x=\frac{bx_1+ax_2}{b+a} \qquad \text{and} \qquad y=\frac{by_1+ay_2}{b+a}.$$

The formulas of exercise 1 are worth learning.

Example

Let A be $(-3,5)$ and B be $(5,-10)$. Find

1. the distance AB
2. the midpoint P of AB
3. the point Q which divides AB in the ratio $2:5$.

Solution

1. $AB^2 = \big(5-(-3)\big)^2 + \big(-10-5\big)^2 = 8^2 + 15^2 = 17^2$, so $AB = 17$.
2. P has coordinates $\big(1, -\frac{5}{2}\big)$.
3. Q has coordinates

$$\left(\frac{5\times -3+2\times 5}{5+2}, \frac{5\times 5+2\times -10}{5+2}\right) = \left(-\frac{5}{7}, \frac{5}{7}\right).$$

Exercise 2

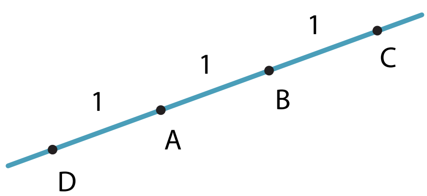
Give an alternative proof of the previous theorem on internal division of an interval. Use the fact that $AC = AP\cos\theta$ in the diagram from the proof, where $\theta = \angle PAC$.

Exercise 3

Suppose points A, B and C are not collinear and have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). Let D be the midpoint of BC and suppose G divides the median AD in the ratio $2:1$. Find the coordinates of G and deduce that the medians of $\triangle ABC$ are concurrent.

External division of an interval

Dividing an interval AB internally in a given ratio produces a point between A and B. External division produces a point outside the interval AB.

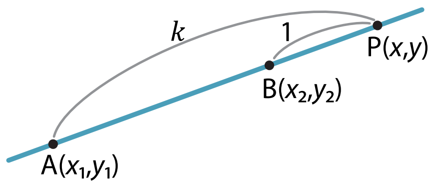


Suppose D, A, B and C are collinear and $DA = AB = BC$, as in the above diagram. Then

$$\frac{AC}{CB} = \frac{2}{1} = 2 \qquad \text{and} \qquad \frac{AD}{DB} = \frac{1}{2}.$$

We say that C divides AB externally in the ratio $2:1$ and that D divides AB externally in the ratio $1:2$. Clearly this is different from the internal division of an interval discussed in the previous subsection.

In general, suppose that $P(x, y)$ is on the line AB but is external to the interval AB and that $\frac{AP}{PB} = \frac{k}{1}$, for some $k > 0$.



Then

$$\begin{aligned} \frac{x - x_1}{k} &= \frac{x - x_2}{1} \\ x - x_1 &= kx - kx_2 \\ x(1-k) &= x_1 - kx_2 \\ x &= \frac{x_1 - kx_2}{1-k}. \end{aligned}$$

This suggests that we make the following definition.

Definition

Suppose $k < 0$ is a real number with $k \neq -1$ and let P be a point on a line AB. Then P divides the interval AB in the ratio $k:1$ if P is external to the interval and

$$\frac{AP}{PB} = -k.$$

With this convention we have

$$x = \frac{x_1 + kx_2}{1+k} \qquad \text{and} \qquad y = \frac{y_1 + ky_2}{1+k}$$

for external division, exactly as for the internal division of an interval! These formulas are algebraically the same as for internal division, but here k is negative.

Example

Find the coordinates of the point P which divides the interval $A(-3,-7)$, $B(-1,-4)$ externally in the ratio $4:3$.

Solution

Here $k = -\dfrac{4}{3}$ and

$$x = \frac{-3 + \left(-\frac{4}{3}\right)(-1)}{1 - \frac{4}{3}} = \frac{-9+4}{3-4} = 5,$$

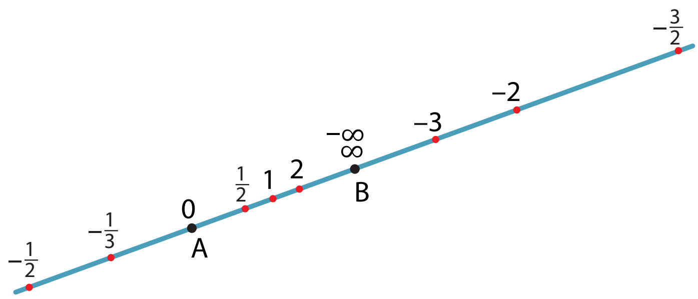
$$y = \frac{-7 + \left(-\frac{4}{3}\right)(-4)}{1 - \frac{4}{3}} = \frac{-21+16}{3-4} = 5.$$

The point P has coordinates $(5,5)$.

Now, for each point P on the line AB, we have an associated number k:

- for P in the interval AB, we take $k = \dfrac{AP}{PB}$
- for P on the line AB external to the interval AB, we take $k = -\dfrac{AP}{PB}$.

The following diagram shows the values of k for the marked points.



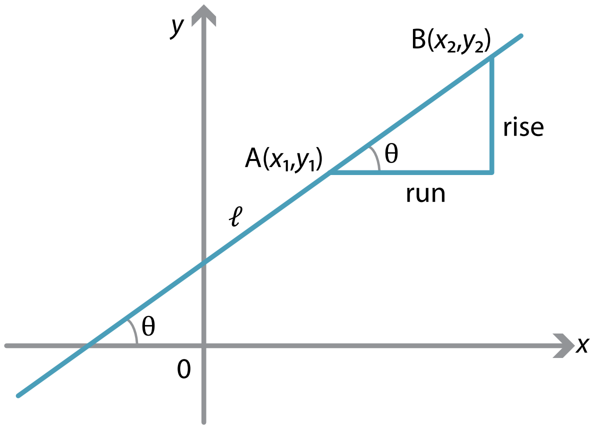
Clearly, as $P \to \infty$ in either direction, $\dfrac{AP}{PB} \to -1$. (We are now close to a concept called the real projective line.)

Exercise 4

Suppose M is the midpoint of AB, where A is $(8,10)$ and B is $(18,20)$. Further suppose that P divides AB internally and Q divides AB externally in the ratio $2:3$. Show that $MP \cdot MQ = MB^2$.

Gradients and the angle of inclination

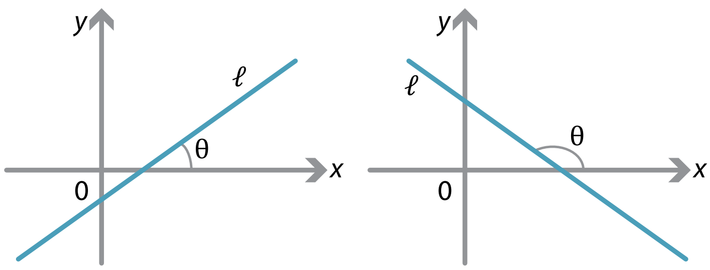
Suppose l is a line in the number plane not parallel to the y-axis or the x-axis.



Let θ be the angle between l and the positive x-axis, where $0^\circ < \theta < 90^\circ$ or $90^\circ < \theta < 180^\circ$. Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on l. Then, by definition, the gradient of the interval AB is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

From the diagram above, this is equal to $\tan\theta$. So $\tan\theta$ is the gradient of the interval AB. Thus the gradient of any interval on the line is constant. Thus we may sensibly define the **gradient** of l to be $\tan\theta$.



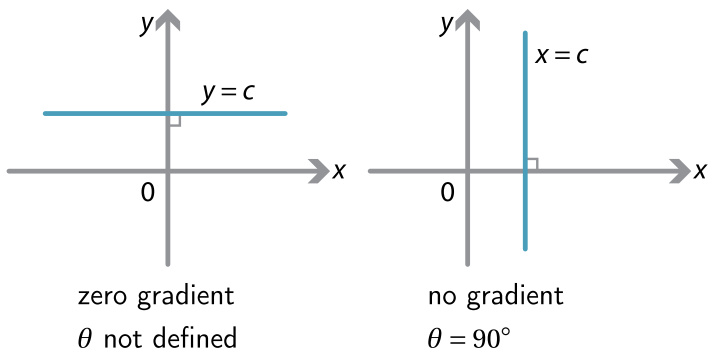
positive gradient

$0^\circ < \theta < 90^\circ$

negative gradient

$90^\circ < \theta < 180^\circ$

In the case where the line is parallel to the x-axis, we say that the gradient is 0. In the case where the line is parallel to the y-axis, we say that $\theta = 90^\circ$.



Example

Find the gradients of the lines joining the following pairs of points. Also find the angle θ between each line and the positive x-axis.

1 $(2,3)$, $(5,3)$ **2** $(1,2)$, $(7,8)$ **3** $(1,2)$, $(7,-4)$ **4** $(2,3)$, $(2,7)$

Solution

1 The line through $(2,3)$ and $(5,3)$ is parallel to the x-axis and so the gradient is 0.

2 The gradient is $\dfrac{8-2}{7-1} = 1$ and $\theta = 45^\circ$.

3 The gradient is $\dfrac{-4-2}{7-1} = -1$ and $\theta = 135^\circ$.

4 The line is parallel to the y-axis, so the line has no gradient and $\theta = 90^\circ$.

Exercise 5

Find the gradients of the lines joining the following pairs of points:

a $\left(cp, \dfrac{c}{p}\right)$, $\left(cq, \dfrac{c}{q}\right)$

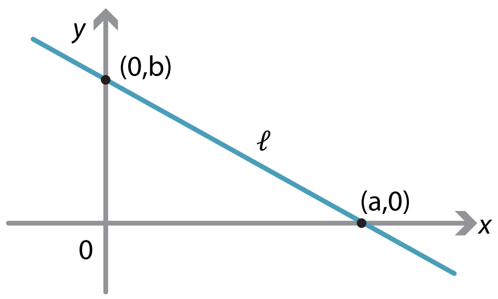
b $(ap^2, 2ap)$, $(aq^2, 2aq)$

c $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$.

Intercepts and equations of lines

Intercepts

All lines, except those parallel to the x-axis or the y-axis, meet both coordinate axes. Suppose that a line l passes through $(a,0)$ and $(0,b)$. Then a is the **x-intercept** and b is the y-**intercept** of l. The intercepts a and b can be positive, negative or zero. All lines through the origin have $a = 0$ and $b = 0$.

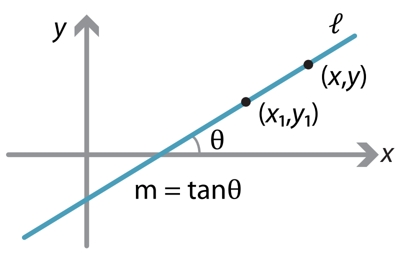


Equations of lines

One of the axioms of Euclidean geometry is that two points determine a line. In other words, there is a unique line through any two fixed points. This idea translates to coordinate geometry and, as we shall see, all points on the line through two points satisfy an equation of the form $ax + by + c = 0$, with a and b not both 0. Conversely, any 'linear equation' $ax + by + c = 0$ is the equation of a (straight) line. This is called the **general form** of the equation of a line.

Point–gradient form

Consider the line l which passes through the point (x_1, y_1) and has gradient m.



Let $P(x, y)$ be any point on l, except for (x_1, y_1). Then

$$m = \frac{y - y_1}{x - x_1},$$

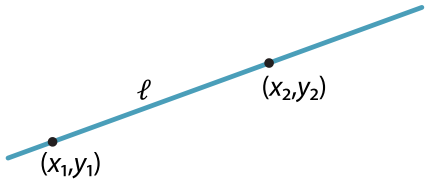
and so

$$y - y_1 = m(x - x_1).$$

This equation is called the **point–gradient form** of the equation of the line l.

Suppose that $(x_1, y_1) = (0, c)$. Then the equation is $y - c = mx$ or, equivalently, $y = mx + c$. This is often called the **gradient–intercept form** of the equation of the line.

Line through two points

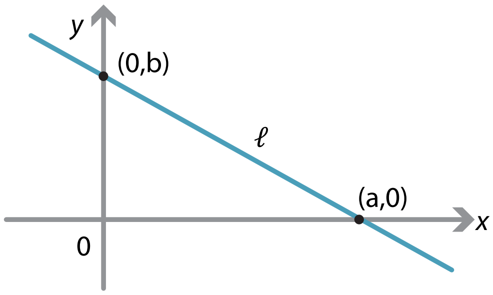


To find the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and then use the point–gradient form $y - y_1 = m(x - x_1)$.

A special case is the line through $(a,0)$ and $(0,b)$, where $a, b \neq 0$.



In this case, the gradient is

$$m = \frac{b-0}{0-a} = -\frac{b}{a}.$$

Thus the equation of the line is

$$y - 0 = -\frac{b}{a}(x-a)$$

$$ay + bx = ab$$

or, equivalently,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

which is easy to remember. This is called the **intercept form** of the equation of a line.

Summary

There are four different forms of the equation of a straight line which we have considered.

Equation of a straight line

General form	$ax + by + c = 0$
Point–gradient form	$y - y_1 = m(x - x_1)$
Gradient–intercept form	$y = mx + c$
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$

Example

1 Find the equation of the line through $(3, 4)$ with gradient 5.

2 Find the equation of the line through $(3, 4)$ and $(-2, -3)$.

3 Find the equation of the line with x-intercept 3 and y-intercept -5.

Solution

1 The equation is $y - 4 = 5(x - 3)$ or, alternatively, $y = 5x - 11$.

2 The gradient is

$$m = \frac{4 - (-3)}{3 - (-2)} = \frac{7}{5}.$$

So the equation of the line is

$$y - 4 = \frac{7}{5}(x - 3)$$

or, alternatively, $7x - 5y - 1 = 0$.

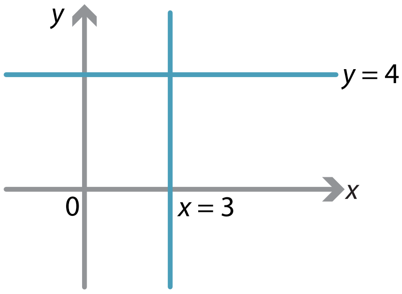
3 The equation is

$$\frac{x}{3} - \frac{y}{5} = 1$$

or, alternatively, $5x - 3y - 15 = 0$.

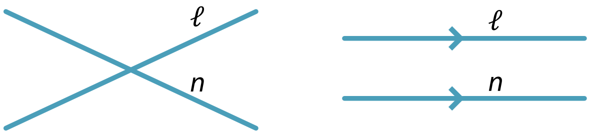
Vertical and horizontal lines

The equation of the vertical line through $(3,4)$ is $x = 3$. The equation of the horizontal line through $(3,4)$ is $y = 4$.

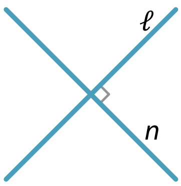


Parallel, intersecting and perpendicular lines

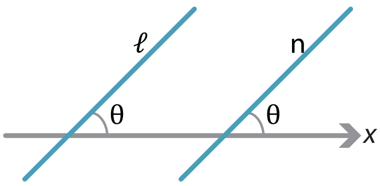
The axioms for Euclidean geometry include: *Two lines meet at a point or are parallel.*



Amongst those pairs of lines which meet, some are perpendicular.



If lines l and n are not vertical, then they are parallel if and only if they have the same gradient $m = \tan\theta$.



Clearly, two horizontal lines are parallel. Also, any two vertical lines are parallel.

If lines l and n are not parallel, then their point of intersection can be found by solving the equations of the two lines simultaneously.

Example

Determine whether the following pairs of lines are parallel or identical and, if they are not, find the (unique) point of intersection:

1 $y = x + 3$	**2** $y = x + 3$	**3** $y = x + 3$	**4** $y = x + 3$
$y = x + 6$	$3y = 3x + 7$	$5y = 5x + 15$	$2y = x - 2.$

Solution

1. Both lines have gradient $m = 1$, but they have different y-intercepts. So the two lines are parallel.
2. The lines $y = x + 3$ and $y = x + \frac{7}{3}$ have the same gradient and are therefore parallel.
3. The line $5y = 5x + 15$ can also be written as $y = x + 3$. So the two lines are the same.
4. Number the equations:

$$y = x + 3 \qquad (1)$$

$$2y = x - 2. \qquad (2)$$

Substituting (1) into (2) gives $2(x + 3) = x - 2$, so $2x + 6 = x - 2$. Therefore $x = -8$ and $y = -5$, and the point of intersection is $(-8, -5)$.

Exercise 6

a Find the equation of the line parallel to the x-axis which passes through the point where the lines $4x + 3y - 6 = 0$ and $x - 2y - 7 = 0$ meet.

b Find the gradient of the line which passes through the point $(2, 3)$ and the point of intersection of the lines $3x + 2y = 2$ and $4x + 3y = 7$.

Perpendicular lines

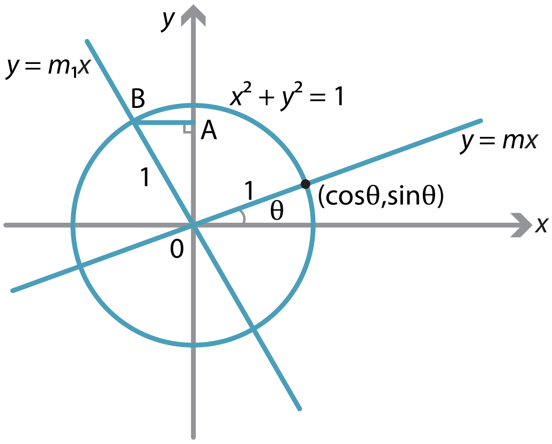
All vertical lines $x = a$ are perpendicular to all horizontal lines $y = b$. So, in the following, we discuss lines with equations $y = mx + b$, where $m \neq 0$.

Theorem

The lines $y = mx + b$ and $y = m_1x + c$ are perpendicular if and only if $mm_1 = -1$.

Proof

It is sufficient to prove that $y = mx$ is perpendicular to $y = m_1x$ if and only if $mm_1 = -1$.



Suppose $y = mx$ meets the x-axis at angle θ, that is, $m = \tan\theta$. For convenience, assume that $m > 0$. Then $y = mx$ meets the unit circle at $(\cos\theta, \sin\theta)$ in the first quadrant. If $y = m_1x$ is perpendicular to $y = mx$, then $\angle AOB = \theta$, $AB = \sin\theta$, $OA = \cos\theta$ and B is $(-\sin\theta, \cos\theta)$. So

$$m_1 = \frac{\cos\theta}{-\sin\theta} = -\frac{1}{\tan\theta} = -\frac{1}{m},$$

as required.

Conversely, suppose $mm_1 = -1$ and $m = \tan\theta$. Then

$$m_1 = -\frac{1}{\tan\theta} = -\frac{\cos\theta}{\sin\theta} = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{\cos\left(\frac{\pi}{2} + \theta\right)} = \tan\left(\frac{\pi}{2} + \theta\right).$$

Hence $y = mx$ is perpendicular to $y = m_1x$. $\square$

This result is often stated as:

The line $y = m_1x + c_1$ is perpendicular to the line $y = m_2x + c_2$ if and only if $m_1m_2 = -1$.

Exercise 7

Use the angle sum formulas

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

to show that

$$\tan(90^\circ + \theta) = -\frac{1}{\tan\theta}.$$

Then use this identity to give an alternative proof that lines $y = mx$ and $y = m_1 x$ are perpendicular if and only if $mm_1 = -1$.

Example

Find the equation of the line l through $(1,3)$ perpendicular to the line $2x+3y=12$. Find the equation of the line through $(4,5)$ parallel to l.

Solution

The gradient of the line $2x+3y=12$ is $-\frac{2}{3}$, so l has gradient $\frac{3}{2}$. The equation of l is

$$\begin{aligned} y-3 &= \frac{3}{2}(x-1) \\ 2y-6 &= 3x-3 \\ 3x-2y+3 &= 0. \end{aligned}$$

The other line has equation $y-5 = \frac{3}{2}(x-4)$ or, equivalently, $3x-2y-2=0$.

Example

For each pair of lines, determine whether they are parallel, identical or meet. If they meet, find the point of intersection and whether they are perpendicular.

1. $l\colon\ 3x = 2y+5, \quad n\colon\ 2x+3y=7$
2. $l\colon\ 6x = 5y+7, \quad n\colon\ 12x = 10y+13$
3. $l\colon\ 3y = 2x-1, \quad n\colon\ 4x = 5y-7$
4. $l\colon\ 5y = 2x+2, \quad n\colon\ 15y-6x-6=0$

Solution

1 The gradient of l is $\frac{3}{2}$ and the gradient of n is $-\frac{2}{3}$. Since

$$\frac{3}{2} \times -\frac{2}{3} = -1,$$

the lines are perpendicular.

We have

$$\begin{aligned} 6x &= 4y + 10 \qquad \text{from } l, \\ 6x + 9y &= 21 \qquad \text{from } n. \end{aligned}$$

Substituting, we obtain $13y + 10 = 21$ and so

$$y = \frac{11}{13}, \qquad x = \frac{29}{13}.$$

The point of intersection is $\left(\frac{29}{13}, \frac{11}{13}\right)$.

2 The lines l and n have the same gradient $\frac{6}{5}$. The line l has x-intercept $\frac{7}{6}$, and the line n has x-intercept $\frac{13}{12}$. So l is parallel to n, but $l \neq n$.

3 The gradient of l is $\frac{2}{3}$ and the gradient of n is $\frac{4}{5}$. So l meets n, but l is not perpendicular to n.

We have

$$\begin{aligned} 15y &= 10x - 5 \qquad \text{from } l, \\ 12x &= 15y - 21 \qquad \text{from } n. \end{aligned}$$

Substituting, we obtain $12x = 10x - 5 - 21$. So $x = -13$ and $y = -9$. The point of intersection is $(-13, -9)$.

4 Rearranging $15y - 6x - 6 = 0$ gives $5y = 2x + 2$. So l and n are the same line.

Exercise 8

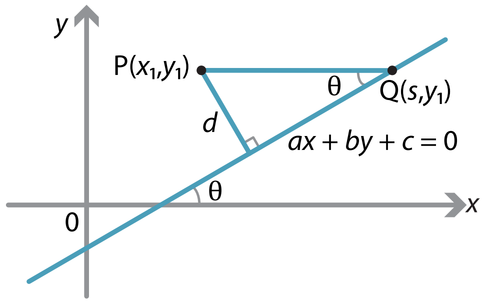
Consider the line l with equation $ax + by + c = 0$ and the point $P(x_1, y_1)$.

a Show that the line through P parallel to l is given by $ax + by = ax_1 + by_1$.

b Show that the line through P perpendicular to l is given by $bx - ay = bx_1 - ay_1$.

Perpendicular distances

Given a line l and a point P not on the line, it is natural to ask the question: What is the distance d of P from l?



Assume that l has equation $ax+by+c=0$, with $a \neq 0$ and $b \neq 0$. We will also assume that the gradient m of l is positive. So the angle θ between l and the positive x-axis is acute.

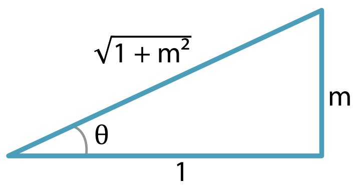
Let $P(x_1, y_1)$ be any point, and let $Q(s, y_1)$ be the point where the line l meets the line through P parallel to the x-axis. Since Q lies on the line l, we have $as+by_1+c=0$ and so

$$s = \frac{-by_1 - c}{a}.$$

Thus

$$PQ = |x_1 - s| = \left|\frac{ax_1 + by_1 + c}{a}\right|.$$

We can draw the following triangle, as $m = \tan\theta$ with θ acute.



From the triangle, we have

$$\sin\theta = \frac{m}{\sqrt{1+m^2}}.$$

Since $m = -\dfrac{a}{b}$, this implies that

$$\sin^2\theta = \frac{m^2}{1+m^2} = \frac{a^2}{a^2+b^2}.$$

We are assuming that θ is acute, so $\sin\theta$ is positive and therefore

$$\sin\theta = \frac{|a|}{\sqrt{a^2+b^2}}.$$

From the first diagram, we now have

$$d = PQ\sin\theta = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Theorem

Let $P(x_1, y_1)$ be a point and let l be the line with equation $ax+by+c=0$. Then the distance d from P to l is given by

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Proof

We have checked that the formula holds when $a \neq 0$, $b \neq 0$ and $m > 0$. Check that the formula also holds when $a = 0$, when $b = 0$ and when $m < 0$. □

Example

What is the distance d of the point $P(-6,-7)$ from the line l with equation $3x+4y=11$?

Solution

Here $a=3$, $b=4$, $c=-11$, $x_1=-6$ and $y_1=-7$. So

$$d = \frac{|-18-28-11|}{\sqrt{3^2+4^2}} = \frac{57}{5}.$$

Exercise 9

A circle has centre $(1,2)$ and radius $\sqrt{5}$.

a Find the perpendicular distance from the centre of the circle to the line with equation $x+2y-10=0$ and hence show this line is a tangent to the circle.

b Find the perpendicular distance from the centre of the circle to the line with equation $x+2y-12=0$ and hence show the line does not meet the circle.

Exercise 10

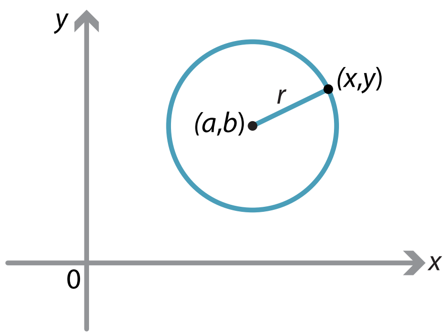
Consider a line l with equation $ax+by+c=0$. Show that the expression ax_1+by_1+c is positive for (x_1, y_1) on one side of the line l and negative for all points on the other side.

Geometrical proofs using coordinate methods

If a point $P(x, y)$ lies on the circle with centre (a, b) and radius r, then Pythagoras' theorem gives

$$(x-a)^2 + (y-b)^2 = r^2.$$

Conversely, if $P(x, y)$ satisfies the equation above, then P lies on the circle with centre (a, b) and radius r.



This is an example of a locus problem.

A **locus** is the set of all points satisfying a geometrical condition. For example, a circle is the locus of all points a fixed distance r from a fixed point (a, b).

Example

Let $P(a, b)$ and $Q(c, d)$ be two points in the plane. Find the equation of the line l that is the perpendicular bisector of the line segment PQ.

Solution

First assume that $a \neq c$ and $b \neq d$. The gradient of PQ is $\dfrac{b-d}{a-c}$, and so the gradient of the perpendicular line l is

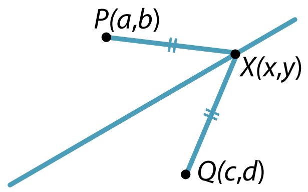
$$m = \frac{c-a}{b-d}.$$

The line l goes through the midpoint $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$ of PQ. Thus the equation of l is

$$\begin{aligned}
y - \frac{b+d}{2} &= m\left(x - \frac{a+c}{2}\right) \\
(b-d)(2y-b-d) &= (c-a)(2x-a-c) \\
2(b-d)y + (d^2-b^2) &= 2(c-a)x + (a^2-c^2) \\
2(b-d)y + 2(a-c)x &= a^2+b^2-c^2-d^2.
\end{aligned}$$

The final equation above is a sensible symmetric form for the equation of the perpendicular bisector of PQ. This equation is also correct in the case that $a = c$ or $b = d$.

Alternatively, the question can be answered by finding the locus of all points $X(x, y)$ equidistant from $P(a, b)$ and $Q(c, d)$.



$$\begin{aligned} & PX = QX \\ \iff\ & PX^2 = QX^2 \\ \iff\ & (x-a)^2 + (y-b)^2 = (x-c)^2 + (y-d)^2 \\ \iff\ & -2ax - 2by + a^2 + b^2 = -2cx - 2dy + c^2 + d^2 \\ \iff\ & 2(b-d)y + 2(a-c)x = a^2 + b^2 - c^2 - d^2. \end{aligned}$$

Note that the locus method involves less algebra.

Exercise 11

Use the locus method illustrated in the previous example to find the equation of the perpendicular bisector of the line segment PQ, where the coordinates of P and Q are $(-1,6)$ and $(3,4)$.

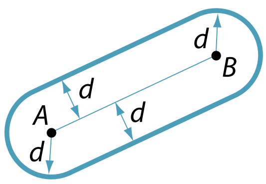
Example

What is the locus of all points distance d from the fixed interval AB?

Solution

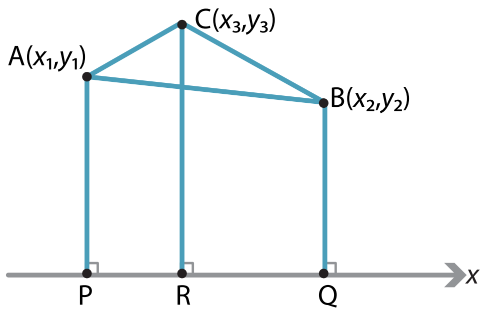
This is not meant to be a problem in coordinate geometry!

The locus is two intervals parallel to AB together with two semicircles of radius d with centres A and B.



Example

What is the area of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$? Assume that C is above the interval AB as shown.



Solution

Let $P(x_1, 0)$, $Q(x_2, 0)$ and $R(x_3, 0)$ be the projections of A, B and C onto the x-axis. We express the area of the triangle ABC in terms of three trapezia:

$$\begin{aligned}\text{Area } \triangle ABC &= \text{Area } APRC + \text{Area } CRQB - \text{Area } APQB \\ &= \frac{1}{2}\big((y_1+y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)\big) \\ &= \frac{1}{2}\big(y_1(x_3-x_1-x_2+x_1)+y_2(x_2-x_3-x_2+x_1)+y_3(x_3-x_1+x_2-x_3)\big) \\ &= \frac{1}{2}\big(y_1(x_3-x_2)+y_2(x_1-x_3)+y_3(x_2-x_1)\big).\end{aligned}$$

The formula found in the previous example gives a negative value if C is below AB. In general, the area of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is

$$\text{Area } \triangle ABC = \left|\frac{1}{2}\big(y_1(x_3-x_2)+y_2(x_1-x_3)+y_3(x_2-x_1)\big)\right|.$$

Note. For readers familiar with determinants, the area of the triangle is $\left|\frac{1}{2}\Delta\right|$, where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Exercise 12

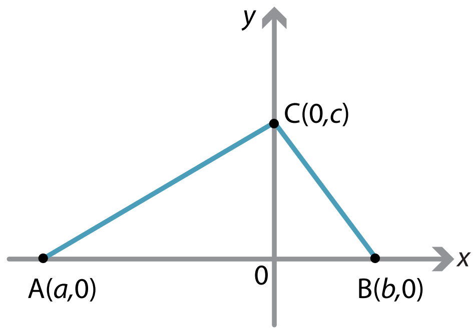
Three points $A(3,6)$, $B(7,8)$ and $C(5,2)$ are the vertices of $\triangle ABC$. Find the area of this triangle using the formula given above.

Example

Use coordinate geometry to show that the altitudes of a triangle ABC are concurrent (at the **orthocentre** of the triangle).

Solution

By choosing the x-axis to be the line AB and the y-axis to be the line OC, we can assume $A(a,0)$, $B(b,0)$ and $C(0,c)$ are the vertices of the triangle ABC, as shown in the following diagram.



The gradient of BC is $-\frac{c}{b}$, so the gradient of the altitude through A is $\frac{b}{c}$. Thus the equation of the altitude through A is

$$y = \frac{b}{c}(x-a).$$

To find the y-intercept of the altitude, take $x=0$, which implies $y = -\frac{ba}{c}$. By swapping a and b, we get $-\frac{ab}{c} = -\frac{ba}{c}$. So the altitude through B has the same y-intercept. The altitude through C is the y-axis. Thus $H = \left(0, -\frac{ab}{c}\right)$ lies on all three altitudes.

Exercise 13

Find a proof that the three altitudes of a triangle are concurrent using classical Euclidean geometry.

Exercise 14

Let M be the point $(2,3)$ and let l be a line through M which meets $2x+y-3=0$ at A and meets $3x-2y+1=0$ at B. If M is the midpoint of AB, find the equation of l.

Links forward

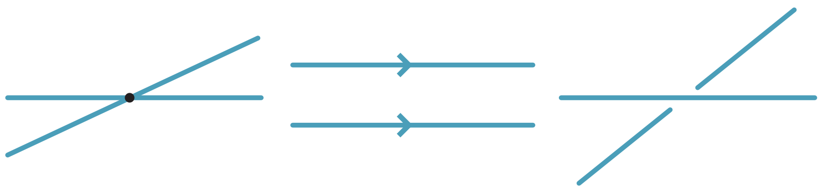
Three-dimensional coordinate geometry

The principal elements in solid geometry (3-dimensional geometry) are points, lines and planes. In the plane (2-dimensional geometry), the 'incidence' properties are very straightforward:

- two points determine a line
- two lines are parallel or meet at a unique point.

In 3-dimensional space, things are more complicated:

- two points still determine a line
- two lines meet or are parallel or are **skew**.

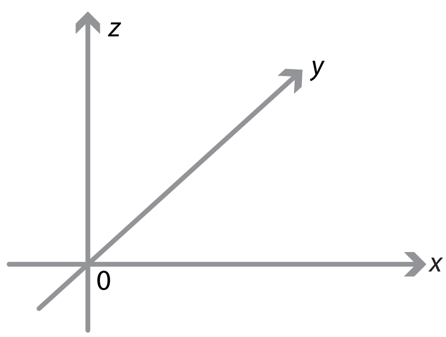


To visualise a pair of skew lines, think of one as being 'behind' the other.

Three points which are not collinear determine a unique plane. But there are an infinite number of planes through a fixed line or three collinear points.

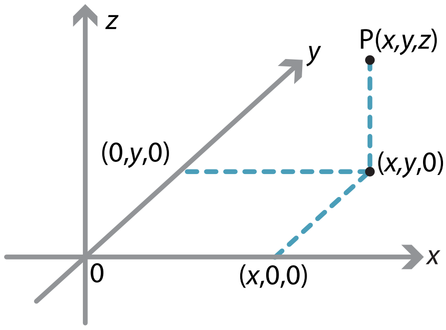
Two planes are parallel or meet in a line. Indeed, if two planes have a point in common, they must have a whole line of points in common (this is not obvious).

To set up a coordinate system for solid geometry, choose a plane $0xy$ and then fix axes $0x$ and $0y$ as in plane geometry.



The axis $0z$ is the normal to the plane $0xy$ through the origin. Usually we choose $0z$ to come out of the standard x–y plane. (This is called a right-handed coordinate system and is very important when vectors are used — especially in applications in physics.)

Thus $0x \perp 0y$, $0x \perp 0z$ and $0y \perp 0z$.



Every point in 3-space is a certain distance z above the x–y plane (if $z < 0$, the point is below). The points in the $0xy$ plane are given the coordinates $(x, y, 0)$ and, in general, P is (x, y, z).

Midpoints and distance

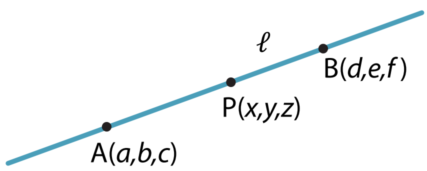
Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then:

- The midpoint of the line segment PQ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.
- The distance between P and Q is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$.

So, for example, in the above diagram $OP^2 = x^2 + y^2 + z^2$.

Lines in $\mathbb{R}^3$

To describe the points on a line, it is easiest to use a parameter which varies as a point moves along the line.



Consider the line l through the fixed points $A(a,b,c)$ and $B(d,e,f)$. Let $P(x,y,z)$ be any point on l. Then the vector $(x-a, y-b, z-c)$ is a scalar multiple t of the vector $(d-a, e-b, f-c)$. So

$$x-a = t(d-a)$$

$$y-b = t(e-b)$$

$$z-c = t(f-c)$$

and thus

$$x = td + (1-t)a$$

$$y = te + (1-t)b$$

$$z = tf + (1-t)c,$$

for $t \in \mathbb{R}$. This is the parametric form for a line in $\mathbb{R}^3$. If $t = 0$, then $P = A$ and, if $t = 1$, then $P = B$.

We can obtain the Cartesian form for the line by eliminating t. It is

$$\frac{x-a}{d-a} = \frac{y-b}{e-b} = \frac{z-c}{f-c}.$$

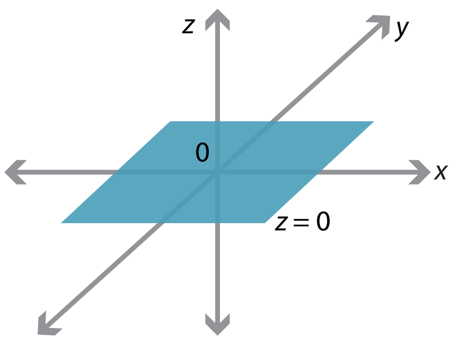
Parametric forms for curves in $\mathbb{R}^2$ are discussed in the module *Quadratics*.

Planes in $\mathbb{R}^3$

There are three principal planes in $\mathbb{R}^3$. The x–y plane, which contains the axes $0x$ and $0y$, is given by

$$\Pi_z = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}.$$

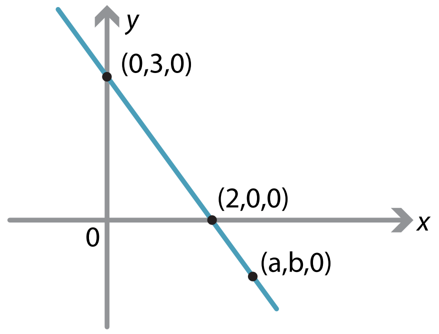
The other two principal planes are $x = 0$ and $y = 0$.



Consider the set

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 2y = 6\}.$$

If $z = 0$, then this restricts to a line in the x–y plane.

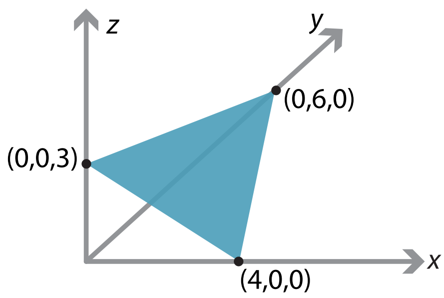


However, if $(a, b, 0)$ is any point on this line, then $(a, b, z) \in \Pi$ for any value of z. Thus Π is a plane in $\mathbb{R}^3$. Indeed Π is a plane parallel to the z-axis. In a similar way, $3y + 2z = 6$ is a plane parallel to the x-axis, and $3x + 2z = 6$ is a plane parallel to the y-axis.

Now consider the set

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 2y + 4z = 12\}.$$

Given the previous examples, it should be no surprise that this is a plane. Clearly it passes through the points $(4, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 3)$, so the plane has three intercepts!



The general equation for a plane is

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\},$$

where at least one of a, b, c is non-zero.

So, in some ways, the general plane in $\mathbb{R}^3$ is analogous to the line in $\mathbb{R}^2$. To prove the above facts about planes, it is best to use vectors to represent the points in the plane.

Quadratic equations and the conics

The general linear equation

$$ax + by + c = 0,$$

where a or b is non-zero, is the equation of a line in $\mathbb{R}^2$. So it is natural to ask the question:

> What curves in $\mathbb{R}^2$ occur as the graph of the general quadratic equation in two variables?

Consider the set

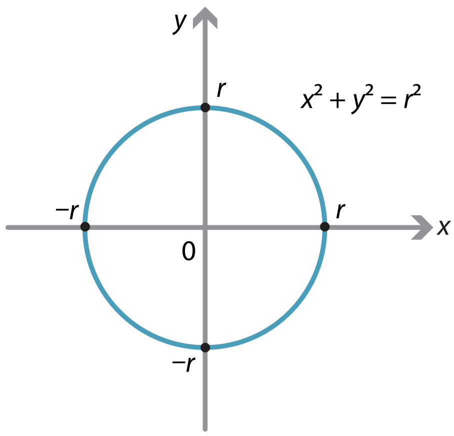
$$S = \{(x, y) \in \mathbb{R}^2 \mid ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\},$$

where a, b, c, f, g, h are constants and at least one of a, h and b is non-zero. The 2's in the equation are somewhat mysterious, but are related to completing the square.

There are several approaches to studying such sets S, including conic sections and the focus–directrix definition, but we will leave them to the module *Quadratics*. In this section, we will simply give a picture gallery of typical examples.

Example 1 (Circle)

The graph of $x^2 + y^2 = r^2$ is the circle with centre the origin and radius r.

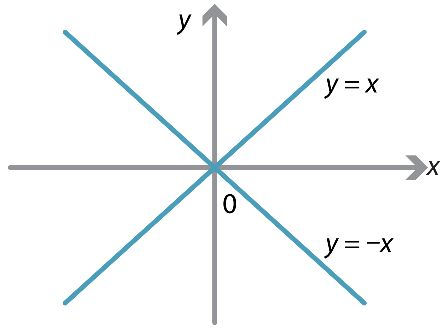


Example 2 (Empty set)

The graph of $x^2 + y^2 = -1$ is the empty set.

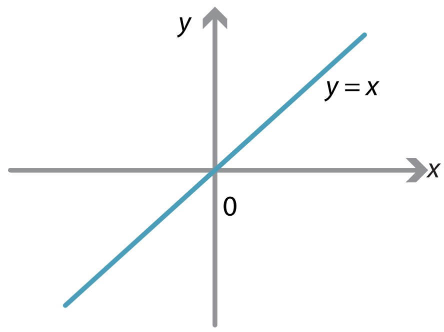
Example 3 (Two lines)

The equation $x^2 - y^2 = 0$ factorises as $(x + y)(x - y) = 0$, so $y = -x$ or $y = x$. So this example is the union of two lines.



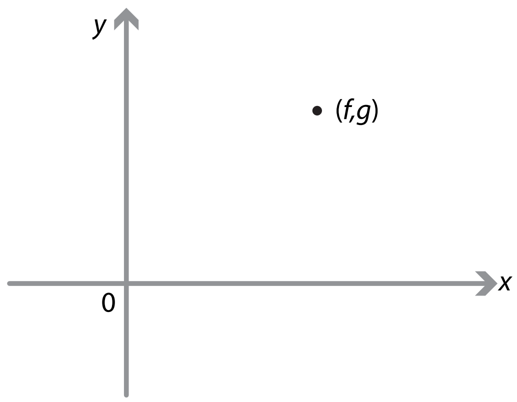
Example 4 (One line)

The equation $x^2 - 2xy + y^2 = 0$ factorises as $(y - x)^2 = 0$, which is just one line!



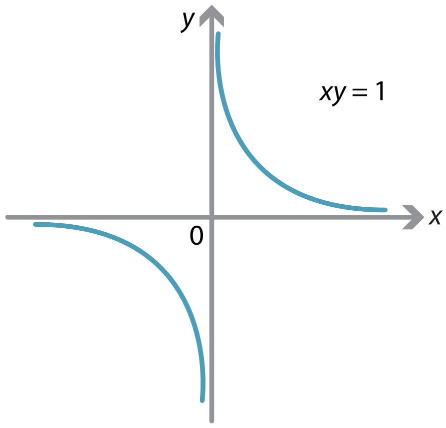
Example 5 (One point)

The graph of $(x - f)^2 + (y - g)^2 = 0$ is just the single point (f, g).



Example 6 (Rectangular hyperbola)

The graph of $xy = 1$ is called a rectangular hyperbola.

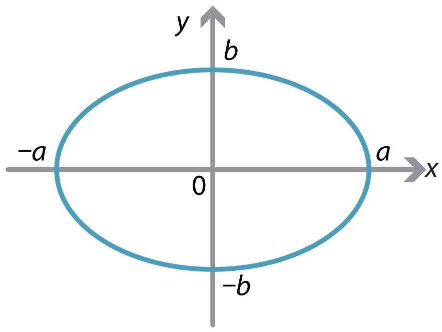


Example 7 (Ellipse)

The graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{for } a > 0,\ b > 0,$$

is an ellipse. It is a circle if $a = b$.



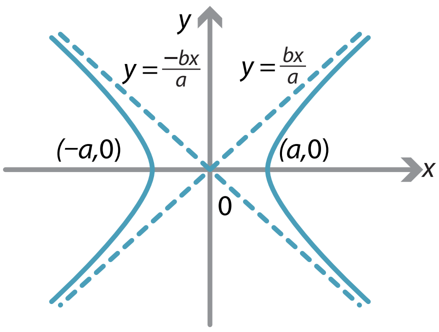
Example 8 (Hyperbola)

Consider the quadratic equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{for } a > 0,\ b > 0.$$

By setting $x = 0$, we see there are no y-intercepts. If $y = 0$, then $x = \pm a$.

As $x \to \infty$, $y \to \pm\infty$ and the curve approaches the lines $y = \dfrac{bx}{a}$ or $y = -\dfrac{bx}{a}$, which are called the asymptotes of the curve.

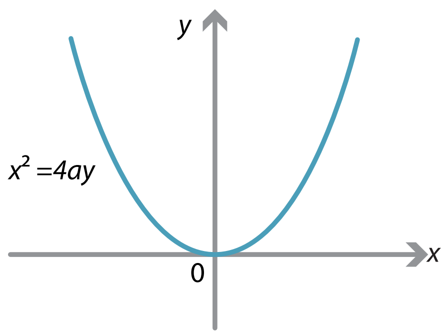


This is a (general) hyperbola. It is a rectangular hyperbola if $a = b$.

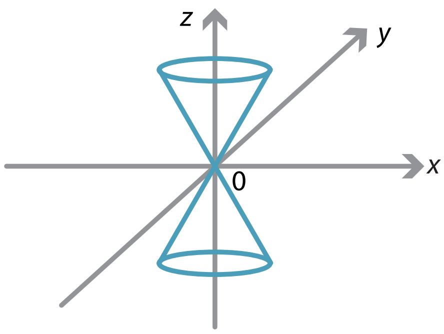
If the rectangular hyperbola $xy = 1$ from Example 6 is rotated through 45°, it becomes $\frac{x^2}{2} - \frac{y^2}{2} = 1$, which is recognisable as a hyperbola.

Example 9 (Parabola)

The graph of $x^2 = 4ay$, for $a > 0$, is a parabola. It has no asymptotes.



The set of points in $\mathbb{R}^3$ satisfying $x^2 + y^2 = z^2$ forms a cone (perhaps better described as a double cone), as shown in the following diagram.



A horizontal plane meets the cone in either one point or a circle. All the other examples in our picture gallery (except the empty set) occur when different planes intersect the cone!

It is educative to try and visualise them all. For this reason the ellipse, the parabola, the hyperbola and the circle are often called the conic sections. Several of the Greek mathematicians followed this approach to the conics.

The general quadratic in two variables x and y is

$$S = \{(x, y) \in \mathbb{R}^2 \mid ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\}.$$

If we apply a translation, a rotation or a reflection to S, then we do not change the intrinsic geometrical nature of S. Unless $a = b = 0$ (in which case S is a rectangular hyperbola), a rotation can be found which reduces S to either

$$a_1x^2 + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0 \qquad \text{or} \qquad a_1x^2 + 2g_1x + 2f_1y + c_1 = 0.$$

In the second case, completing the square in x leads to

$$a_2x^2 + 2f_2y + c_2 = 0,$$

which is a parabola. In the first case, completing the square in both x and y leads to

$$a_2x^2 + b_2y^2 = c_2,$$

which is either a hyperbola, an ellipse, a circle, two lines, one line, one point or the empty set, depending on the values of a_2, b_2 and c_2.

History and applications

There were three facets to the development of coordinate geometry:

- the invention of a system of coordinates
- the recognition of the correspondence between geometry and algebra
- the graphic representation of relations and functions.

The Greek mathematician Menaechmus (380–320 BCE) proved theorems using a method that was very close to using coordinates, and it has sometimes been maintained that he had introduced coordinate geometry.

Apollonius of Perga (262–190 BCE) dealt with problems in a manner that may be called a coordinate geometry of one dimension, with the question of finding points on a line that were in a ratio to the others. The results and ideas of the ancient Greeks provided a motivation for the development of coordinate geometry.

Coordinate geometry has traditionally been attributed to René Descartes (1599–1650) and Pierre de Fermat (1601–1665), who independently provided the beginning of the subject as we know it today.

Answers to exercises

Exercise 1

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points. Assume that $P(x, y)$ is on the interval AB with

$$AP : PB = a : b = \tfrac{a}{b} : 1.$$

So P divides AB in the ratio $\frac{a}{b} : 1$. Thus

$$x = \frac{x_1 + \frac{a}{b}x_2}{1 + \frac{a}{b}} = \frac{bx_1 + ax_2}{b + a}$$

and, similarly,

$$y = \frac{by_1 + ay_2}{b + a}.$$

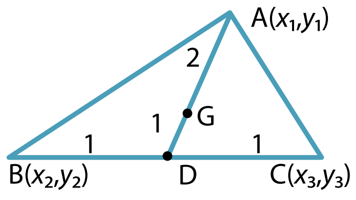
Exercise 2

Without loss of generality, assume $PB = 1$. Then

$$\begin{aligned} AC &= AP \cos\theta \\ x - x_1 &= k\,\frac{x_2 - x_1}{k + 1} \\ (x - x_1)(k + 1) &= kx_2 - kx_1 \\ x &= \frac{x_1 + kx_2}{k + 1}. \end{aligned}$$

From the same diagram, we have $CP = AP \sin\theta$. Proceed in a similar way to find y.

Exercise 3



The midpoint D of BC is

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right).$$

The point G that divides the median AD in the ratio $2 : 1$ is

$$\left(\frac{1}{3}\left(x_1 + \frac{2(x_2 + x_3)}{2}\right), \frac{1}{3}\left(y_1 + \frac{2(y_2 + y_3)}{2}\right)\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

This is symmetric in x_1, x_2, x_3 and in y_1, y_2, y_3. So G lies on the other medians BE and CF, and the medians are concurrent.

Exercise 4

M is $(13, 15)$.

P is $\left(\dfrac{3\times 8 + 2\times 18}{3+2}, \dfrac{3\times 10 + 2\times 20}{3+2}\right) = (12, 14)$.

Q is $\left(\dfrac{3\times 8 - 2\times 18}{3-2}, \dfrac{3\times 10 - 2\times 20}{3-2}\right) = (-12, -10)$.

Thus

$$MB^2 = 5^2 + 5^2 = 50,$$

$$MP^2 = 1^2 + 1^2 = 2,$$

$$MQ^2 = 25^2 + 25^2 = 2\times 25^2.$$

Hence

$$MP \cdot MQ = \left(2\times 2\times 25^2\right)^{\frac{1}{2}} = 50 = MB^2.$$

Exercise 5

a Gradient $= \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \dfrac{p-q}{pq(q-p)} = -\dfrac{1}{pq}$

b Gradient $= \dfrac{2aq - 2ap}{aq^2 - ap^2} = \dfrac{2(q-p)}{(q-p)(q+p)} = \dfrac{2}{p+q}$

c Gradient $= \dfrac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta} = \dfrac{2b\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)}{2a\sin\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)} = -\dfrac{b}{a}\cot\left(\dfrac{\phi+\theta}{2}\right)$

Exercise 6

a The lines $4x + 3y - 6 = 0$ and $x - 2y - 7 = 0$ meet where $11y + 22 = 0$, so $y = -2$ and $x = 3$. The line through $(3, -2)$ parallel to the x-axis is $y = -2$.

b The lines $3x + 2y = 2$ and $4x + 3y = 7$ meet where $y = 13$ and $x = -8$. So the gradient is $\dfrac{13-3}{-8-2} = -1$.

Exercise 7

The angle sum formulas give

$$\sin(\theta+90^\circ) = \sin\theta\cos 90^\circ + \cos\theta\sin 90^\circ = \cos\theta$$

$$\cos(\theta+90^\circ) = \cos\theta\cos 90^\circ - \sin\theta\sin 90^\circ = -\sin\theta.$$

Thus

$$\tan(90^\circ+\theta) = \frac{\cos\theta}{-\sin\theta} = -\frac{1}{\tan\theta}.$$

Suppose $y = mx$ is perpendicular to $y = m_1 x$. Then $m = \tan\theta$ and $m_1 = \tan(90^\circ+\theta)$. So

$$m_1 = -\frac{1}{m}$$

and therefore $mm_1 = -1$.

Conversely, suppose $mm_1 = -1$, where $m = \tan\theta$. Then

$$m_1 = -\frac{1}{\tan\theta} = \tan(90^\circ+\theta).$$

So $y = mx$ is perpendicular to $y = m_1 x$.

Exercise 8

a All lines parallel to $ax+by+c=0$ have equation $ax+by+d=0$, for some d. If $P(x_1, y_1)$ lies on the line, then $ax_1+by_1+d=0$. So the equation is $ax+by = ax_1+by_1$.

b All lines perpendicular to l have equation $bx-ay+d=0$, for some d. If $P(x_1, y_1)$ lies on the line, then $bx_1-ay_1+d=0$. So the equation is $bx-ay = bx_1-ay_1$.

Exercise 9

a The perpendicular distance of $(1,2)$ from the line $x+2y-10=0$ is

$$\frac{|1+4-10|}{\sqrt{1^2+2^2}} = \sqrt{5},$$

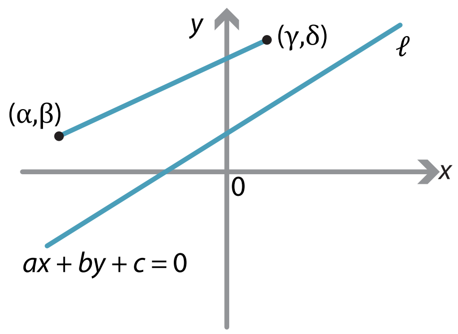
which equals the radius. Therefore the line is tangent to the circle.

b The perpendicular distance of $(1,2)$ from the line $x+2y-12=0$ is

$$\frac{|1+4-12|}{\sqrt{1^2+2^2}} = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5},$$

which is greater than the radius $\sqrt{5}$. Hence the line does not touch the circle.

Exercise 10



Note that a point (x_1, y_1) lies on the line l if and only if $ax_1 + by_1 + c = 0$.

Suppose that points (α, β) and (γ, δ) are on the same side of l. Then the line segment from (α, β) and (γ, δ) does not cross l. So $ax_1 + by_1 + c \neq 0$ for each point (x_1, y_1) on the line segment. Thus the expression $ax_1 + by_1 + c$ cannot change sign as the point (x_1, y_1) moves along the line segment from (α, β) to (γ, δ). Hence the signs of $a\alpha + b\beta + c$ and $a\gamma + b\delta + c$ are the same.

Points on different sides of l must have different signs, as it is possible to choose (x_1, y_1) so that $ax_1 + by_1 + c > 0$, and choose (x_2, y_2) so that $ax_2 + by_2 + c < 0$.

Exercise 11

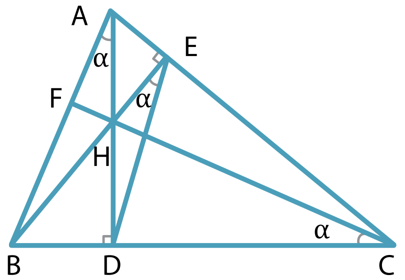
$$\begin{aligned}
(x+1)^2 + (y-6)^2 &= (x-3)^2 + (y-4)^2 \\
x^2 + 2x + 1 + y^2 - 12y + 36 &= x^2 - 6x + 9 + y^2 - 8y + 16 \\
2x - 12y + 37 &= -6x - 8y + 25 \\
8x - 4y + 12 &= 0 \\
2x - y + 3 &= 0
\end{aligned}$$

Exercise 12

The area of the triangle is

$$\left|\frac{1}{2}\big(6(5-7) + 8(3-5) + 2(7-3)\big)\right| = \left|\frac{-12-16+8}{2}\right| = 10.$$

Exercise 13



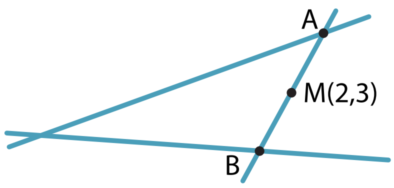
Let AD be the altitude perpendicular to BC, let BE be the altitude perpendicular to AC and let AD meet BE at H. Finally let CH meet AB at F. We must prove FC is perpendicular to AB.

Join DE. Let $\alpha = \angle BCF$. Then $ECDH$ is a cyclic quadrilateral (supplementary opposite angles), so $\angle DEH = \alpha$ (angles in the same segment).

Similarly, $AEBD$ is a cyclic quadrilateral and $\angle DAB = \alpha$.

From $\triangle ABD$, $\angle ABD = 180° - 90° - \alpha = 90° - \alpha$. Finally, in $\triangle CBF$, $\angle BFC + \alpha + 90° - \alpha = 180°$, so $\angle BFC = 90°$, as required.

Exercise 14



Let $A = (a, b)$ and $B = (c, d)$. The midpoint M of AB is $(2, 3) = \left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$, so

$$a + c = 4 \qquad (1)$$

$$b + d = 6. \qquad (2)$$

Since the point $A(a, b)$ lies on the line $2x + y - 3 = 0$ and the point $B(c, d)$ lies on the line $3x - 2y + 1 = 0$, we also have

$$2a + b = 3 \qquad (3)$$

$$3c - 2d = -1. \qquad (4)$$

From (1) and (2), we have $c = 4 - a$ and $d = 6 - b$. Substituting these into (4) gives

$$3a - 2b = 1. \qquad (5)$$

We can now solve (3) and (5) to obtain $a = 1$ and $b = 1$.

We have found that A is $(a, b) = (1, 1)$. So the gradient of the line l is $m = \dfrac{1-3}{1-2} = 2$. Thus l has equation $y - 1 = 2(x - 1)$ or, equivalently, $y = 2x - 1$.

0 1 2 3 4 5 6 7 8 9 10 11 12