MENTAL COMPUTATION: A STRATEGIES APPROACH

MODULE 1
introduction

Alistair McIntosh

Mental Computation: A strategies approach

Module 1 Introduction

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This is one of a set of 6 modules providing a structured strategies approach to mental computation.

Module 1 Introduction
Module 2 Basic facts addition and subtraction
Module 3 Basic facts multiplication and division
Module 4 Two-digit whole numbers
Module 5 Fractions and decimals
Module 6 Ratio and percent

Second Edition, Hobart Tasmania 2005

University of Tasmania
Department *of* Education Tasmania
Catholic Education Office Hobart
Department of Education and Training, Australian Capital Territory

This set of modules was prepared as part of Strategic Partnership with Industry – Research and Training (SPIRT) scheme project (C00107187): Assessing and Improving the Mental Computation of School-age Children.

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The contribution of all partners to the project is acknowledged, including
The Industry Partners:
Department *of* Education, Tasmania
Catholic Education Office, Hobart
Australian Capital Territory Department of Education and Training
and The Research Partner:
The University of Tasmania

The significant contribution of the Department *of* Education, Tasmania in terms of funding and time is acknowledged by all of the partners.

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Printed by: Printing Authority of Tasmania

Publisher: Department *of* Education, Tasmania

ISBN: 1 920865 20 9

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ABOUT THIS SERIES

1.1 CONTENTS OF THE SERIES

This series of Modules provides a suggested sequence of activities for developing the mental computation of students. The activities are aimed mainly at developing specific strategies for processing mental calculations. The complete series is intended to serve as a practical classroom resource for teaching thinking strategies for mental computation in a coherent and sequential manner.

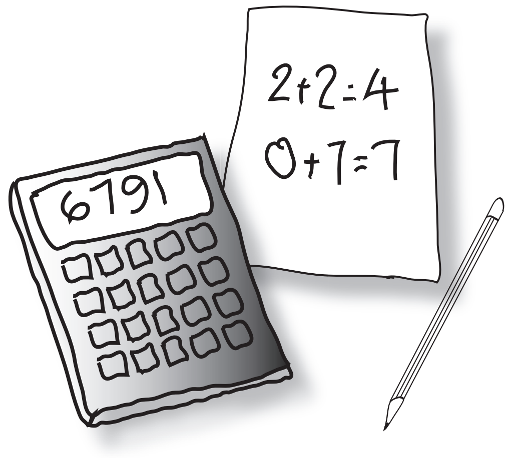
The aims of the project were to:
- Describe levels of achievement in mental computation.
- Provide a simple means whereby teachers can assess mental computation levels of individual students.
- Provide sequential modules of activities for improving the mental computation of students in grades 3 to 10.

The series consists of the following:

Module 1	Introduction
Module 2	Basic facts addition and subtraction
Module 3	Basic facts multiplication and division
Module 4	Two-digit whole numbers
Module 5	Fractions and decimals
Module 6	Ratio and percent

1.2 BACKGROUND TO THE SERIES

This resource is one of the products of a three-year (2001-2003) Strategic Partnerships with Industry Research and Training (SPIRT) project of the University of Tasmania in partnership with the Department *of* Education, Tasmania, the Catholic Education Office, Tasmania and the Department of Education and Training in the Australian Capital Territory. Associated funding was provided from the Australian Research Council.



1.3 WHAT IS MENTAL COMPUTATION?

Mental computation has been defined as 'the ability to calculate exact numerical answers without the aid of calculating or recording devices' (Reys,1984). Activities in this book have been designed with this as the immediate goal – the development of students' ability to calculate mentally within a range of calculations for which they do not require additional aids. However, it is worth bearing in mind that mental computation facility can be assisted through the use of pencil-and-paper recording for memory support.

The ultimate goal is that students have instant recall of basic facts (single digit addition and multiplication), have developed flexible strategies for a wider range of mental calculations (most two-digit computations and simple fractions, decimals and percents) and turn to paper and pencil or a calculator for more complex computations.

In the case of written computation most students are taught one particular method for each operation. This is not the case with mental computation. Good mental calculators use a range of mental strategies when performing calculations, drawing on conceptual understanding of numbers and operations and known relationships with the particular numbers and operations involved. Number sense and conceptual understanding are therefore essential pre-requisites for competency in mental computation.

In the sections involving calculations with whole numbers we have assumed that the necessary conceptual understanding has been developed and therefore concentrated on activities for calculating mentally. Our analysis of common errors students made in the testing strongly suggests that the large majority of the errors were due to conceptual misunderstandings. In the later sections (fractions, decimals, ratio and percentage) we have built in activities designed to develop the conceptual understanding on which the computation strategies are built.

Although estimation is a valuable real-life maths skill (we constantly estimate: costs in a supermarket, time left before we need to leave for work, measurements of all kinds), we have not specifically included the development of estimation strategies within these Modules. Rather, we have confined ourselves to the mental computation competence that underpins estimation.

1.4 THE PLACE OF MENTAL COMPUTATION

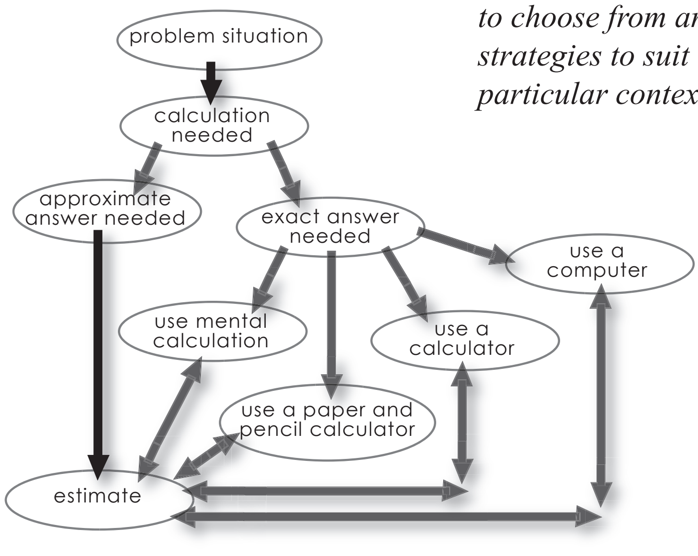
The Australian Education Council in 'The National Statement on Mathematics for Australian Schools' (1991) stated that:

'Students should develop the ability to… use mental, calculator and paper-and-pencil strategies effectively and appropriately in different situations… This requires that they:
- *Decide what operations to perform (formulate the calculation);*
- *Select a means of carrying out the operation (choose a method of calculation);*
- *Perform the operation (carry out the calculation);*
- *Make sense of the answer (interpret the results of the calculation). (p.108)*

The process of choosing an appropriate method of computation has been depicted in diagrammatic form as presented in the National Council for Teachers of Mathematics (NCTM) in the *Curriculum and Evaluation Standards* (1989, p.9)

The *National Statement* and the *Curriculum and Evaluation Standards* documents both clearly state that students should become as proficient as possible in all three computation methods (mental, paper-and-pencil and calculator). Students should be aware of their capabilities in all three methods and faced with the need to calculate should make a sensible choice of computational method.

Further, with regard to mental computation, the *National Statement* states:
'People need to carry out straightforward calculations mentally, and students should regard mental computation as a first resort in many situations where a calculation is needed. Strategies associated with mental computation should be developed explicitly throughout the schooling years, and should not be restricted to the recall of basic facts. People who are competent in mental computation tend to use a range of personal methods, which are adapted to suit the particular numbers and situation. Therefore, students should be encouraged to develop personal mental computation strategies, to experiment with and compare strategies used by others, and to choose from amongst their available strategies to suit their strengths and the particular context (p.109).'



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1.5 WHY IS MENTAL COMPUTATION IMPORTANT?

Reys (1984) describes the history of mental computation in (United States) schools as having progressed through the following stages:

- an integral part of arithmetic instruction for many years;
- a de-emphasis on mental computation early in the twentieth century as a strong, negative reaction to the theory of mental discipline that prevailed during the latter part of the nineteenth century;
- a revival of interest in the 1930s and 1940s, when the social utility of mathematics was emphasised;
- a de-emphasis in the late 1950s and early 1960s, when mathematics instruction began to focus on structural properties of the mathematical systems taught in schools;
- an increased attention to mental computation in the 1980s, partly resulting from the back-to-basics movement, but primarily caused by the growing availability of technology and a recognition of the important role mental computation plays in the efficient use of technology.

Some 20 years on, at the beginning of the 21st century, we can see that this latest 'swing of the pendulum', if such it is, has maintained or even increased momentum. Mental computation is now widely advocated as an important form of computation for students to learn in school. Reasons for this sustained emphasis on the importance of mental computation can be summarised as follows:

- It is the most common form of computation used by adults in their daily lives.

In 1957, Wandt and Brown found that 75% of calculations done by adults in their daily lives involved mental computation. Northcote and McIntosh (1999) found that, in an age in which calculators are commonplace, this percentage had increased to 85%. In comparison, paper-and-pencil calculations and calculators were involved in only 12% and 8% of calculations respectively. This alone would suggest that the balance between school instruction in mental and written computation should be re-examined.

- It is used for estimation.

Approximate computations are based on single or double-digit mental arithmetic, together with an understanding of place value (Australian Education Council, 1991, p.108).

- It is needed as a check on calculator answers.

It is of course, easy to make mistakes when using a calculator...For this reason it is essential that pupils should be enabled to acquire good habits in the use of calculators so as to guard against mistakes (Cockcroft, 1982, p.111).

- Students should see it as the easiest way of doing many calculations.

An over-reliance on paper-and-pencil methods can lead to a type of inflexible behaviour that could be described as "calculative momomania" - the tendency to ignore number relationships useful for calculation and, instead, resort to more cumbersome and inappropriate techniques. (Hope, 1986)

- It is an excellent way of learning how numbers work.

Many students enter the secondary school with very inadequate understanding of place value, partly because the formal written computation algorithms are designed to allow one to calculate without understanding. In mental computation on the other hand, the student has to think about the particular numbers involved, and use understanding of number relationships and the operations involved, in order to come up with a mental solution strategy. Thus mental computation demands active thought about numbers and operations rather than unthinking recall of procedures.

- It is a creative and problem-solving approach to numbers.

The essence of problem-solving is to use all one knows about a situation in order to come up with an optimum solution strategy. This makes it apparent that mental computation is a problem-solving activity. For example, faced with the need to calculate 6 x 25 mentally, one may use any of a variety of known relationships to provide a solution strategy, and several different efficient strategies are possible. Thus calculation (arithmetic) approached in this way (from a problem-solving rather than a rule-learning perspective) fits comfortably alongside the rest of the mathematics curriculum.

MENTAL COMPUTATION STRATEGIES

1.6 HOW PEOPLE CALCULATE MENTALLY

When adults are asked to mentally calculate 25 + 89, and then describe their mental solution strategies, commonly the following methods are described:

- Turn the numbers round and start with 89.
- Make the 89 up to 90 and then add 10, then 14.
- Add 25 by first adding 20 and then adding 5.
- Add 25 by first adding 10 then another 10 and then 5 (or 5 and then 20).
- Add 25 by first adding 5 and then 20.
- Add first 80, then 20 to give 100, then add the 5 and 9 to give 14.
- Add the 9 and 5 by thinking '9 and one more is 10 and 4, which makes 14'.
- Think '89 and 25 is the same as 90 and 24'.
- Think '89 and 11 makes 100'.
- Think '25 and 75 makes 100'.
- Picture the calculation as written on paper.

All the above methods are efficient and successful strategies. When asked, many adults state that they are not aware of having specifically learned such methods or procedures in school. The strategies can be seen to relate to an individual's acquired sense of numbers and operations and relationships between numbers, and include the following:

- With addition one can start with either number (commutativity).
- 89 is very near 90, and place value makes 90 an easier number to operate with.
- One can be added to 89 and then subtracted later to compensate.
- 24 can be added by adding 10 then 14.
- 25 can be added as 20 + 5 or 10 + 10 + 5, or 11 +14.
- The tens can be added first or the ones can be added first.
- 80 and 20 are 'compatible' numbers (add to 100).
- 9 and 5 can be added by making 10 and then adding 4 more (bridging ten).

1.7 HOW STUDENTS DEVELOP STRATEGIES

The strategy that most students acquire earliest is the use of counting on and back in ones for addition and subtraction. This is an efficient strategy for small single-digit numbers and students readily become competent and confident with it. Most students then go on to acquire more sophisticated strategies to deal with a wider range of numbers and operations. However, in the absence of specific classroom activities to develop students' mental computation strategies, some students do not develop confidence in further strategies, but continue to use counting-on and back in ones to cope even with addition and subtraction of two digit numbers and single digit multiplication. (Recent research suggests that up to 20% of upper primary students are in this position, and some continue into the secondary school.)

It is extremely important that teachers have a coherent programme for assisting students to become familiar with, and to acquire confidence in using, a flexible range of strategies. The process through which students acquire mental strategies most effectively is:

- first present specific strategies in the form of actions on **objects**,

- these in turn lead to **mental images** and mental models which can be operated on mentally,
- these mental images in turn are often eventually replaced by actions on **symbols** without recourse to images.

Students who are visual thinkers are likely to maintain the link with the visual images; others will progress to the symbols and leave the mental images behind.

1.8 BASIC FACTS AS THE PLATFORM FOR MENTAL COMPUTATION PROFICIENCY

Basic facts are the platform upon which mental computation proficiency builds. Through research, and the analysis of interviews with hundreds of students of primary school age, a list of the range of strategies used by them in the majority of calculations has been compiled. The most common, specific thinking strategies used by students for deriving basic facts for addition, subtraction, multiplication and division are listed in the table below. For lower secondary students it is likely that the range of strategies used for addition/subtraction is similar, however some students may use a wider range of strategies for multiplication and division.

Strategies Used For Basic Facts

addition	Example	Thinking strategy
Commutativity	2 + 6	I'll do 6 + 2 instead
Counting on in ones	3 + 8	Select largest number 8; then count on to 9, 10, 11
Relate to known fact	9 + 3	10 + 3 is 13, so it is one less
Doubles and near doubles	6 + 7	6 + 6 is 12, so it is one more
Bridging 10	8 + 6	8 and 2 is 10, and 4 more makes 14

subtraction	Example	Thinking strategy
Counting back in ones	9 – 2	9, 8, 7
Convert to addition	14 – 8	8 and what makes 14?
Doubles and near doubles	13 – 6	6 + 6 =12, so it is 7

multiplication	Example	Thinking strategy
Commutativity	7 x 3	I can't remember 7 x 3, so I'll do 3 x 7 instead
Skip counting	4 x 5	5, 10, 15, 20
Splitting into known parts	7 x 8	5 x 8 is 40 and 16 more makes 56

division	Example	Thinking strategy
Convert to multiplication	28 ÷ 7	4 x 7 is 28
Repeated subtraction	18 ÷ 3	18: 15, 12, 9, 6, 3, 0

1.9 STRATEGIES USED FOR MENTAL COMPUTATION WITH LARGER NUMBERS

The most common strategies used for addition and subtraction of two - and three - digit numbers are the following:
Initial Strategy
- **Change Subtraction into Addition**: 63 – 58: 58 + ? = 63.

Start with One Number, Process the Other
- **Adding/ Subtracting the second number in parts**: 63 + 15: 63, 73, 78 (sometimes including counting on and back in tens and multiples of ten: 5 x 30: 30, 60, 90, 120, 150).
- **Bridging Tens/Hundreds**: 45 – 8: 45, 40, 37. 85 + 27: 85, 100, 112.

Split One or Both Numbers, Process and Reassemble
- **Working from the left** (tens first): 36 + 28: 50 + 14, 64.
- **Working from the right** (units first): 36 + 28: 14 + 50, 64.
- **Using a mental form of the written algorithm**: 36 + 28: 6 and 8 = 14, put down 4, carry the one, 3 and 2 and 1 make 6: 64.

Further Strategy
- **Using Tens as the Unit**: 120 – 50: 12 (tens) – 5 (tens) = 7 (tens): 70. (This strategy appears to work reliably with addition and subtraction but, unless it is accompanied by understanding, it causes considerable problems with multiplication and division.)

Strategies used for multiplication and division of larger whole numbers are described in Module 4. The range of strategies used for mental computation of fractions, decimals and percentages is only just beginning to be examined, but there is already evidence that it is largely based on, or adapted from, strategies for dealing with whole numbers.

1.10 AN EFFECTIVE APPROACH TO MENTAL COMPUTATION

Traditional approaches to developing mental computation predominantly involved giving students a series of short calculations within strict time limits, and concentrating on speed and accuracy.

Recent research into students' learning suggests that the emphasis should be on helping students to acquire a flexible 'toolkit' of efficient strategies together with the ability to make suitable choices of these strategies for particular calculations. Critical components of this approach are:
- Developing conceptual understanding and number sense on which to base strategy acquisition;
- Developing strategies through actions on objects;
- Concentrating on how students arrive at answers (discussion of strategies used);
- Giving time for students to calculate an answer so that they have space to think about the numbers involved and to try out strategies.
- At an appropriate time, developing speed and accuracy for basic facts through a varied programme of games and memorisation activities.

LEVELS AND ASSESSMENT

1.11 BACKGROUND TO THE LEVELS

In 2001, approximately three thousand students across grades 3 to10 in twelve Tasmanian and ACT schools participated in one of a set of mental computation tests. Items were of two types: ‘Short items’ had three seconds in which to answer and ‘Long items’ had fifteen seconds. These differences were intended to separate items which students might be expected to know instantly from those that they could work out given time. The items were drawn from a bank of two hundred and forty-four items that included only computations with a single step (that is, no items contained more than two terms e.g. 17 – 8). All items were recorded onto a compact disc and supplied to the schools on audiotape. Students in grades 3 and 4 had fifty items presented to them, while those in the other grades attempted sixty-five. Sixteen different test forms were used, four at each of two adjacent grade levels: Grades 3/4, Grades 5/6, Grades 7/8 and Grades 9/10, and these were linked by the use of common items both within grades and across grade levels. Nine ‘link’ items were also included, in which five seconds was provided for students to respond, in order to provide a basis for linking to an earlier study that had identified a developmental scale of mental computation (Callingham & McIntosh, 2001).

The items were analysed using Rasch modelling techniques. This approach to analysis allowed students’ performances and all item difficulties to be estimated using the same measurement scale, so that they are directly comparable. This placed all students and all items in an ordered display from least proficient or, in the case of items, least difficult to most proficient, or most difficult. The underlying variable was then interpreted in terms of the mental computation skills required by each item, which provided a ‘profile’ of students’ mental computation competence. By determining the points at which there was a qualitative change in the demands of the items, eight levels of mental computation competence were identified (Callingham & McIntosh, 2002).

The testing program was repeated in 2002 with the same set of tests, and again in 2003 with new items, particularly for fractions, decimals, and percents. Sufficient items were maintained for the different tests to be linked together, so that all items could be placed on the same scale. The longitudinal testing confirmed the scale identified initially.

Similar items within the levels were clustered and described. Items were also separated out into the following sub-domains for teaching purposes:

- Whole number single digit addition and subtraction
- Whole number single digit multiplication and division
- Two-digit addition and subtraction
- Two-digit multiplication and division
- Decimals addition and subtraction
- Decimals multiplication and division
- Fractions addition and subtraction
- Fractions multiplication and division
- Percentages.

Descriptions of achievement at each level are given below for each of these sub-domains. Items at a given level (for example Level 3) are of approximate equal difficulty across all sub-domains.

1.12 LEVELS OF ACHIEVEMENT IN MENTAL COMPUTATION

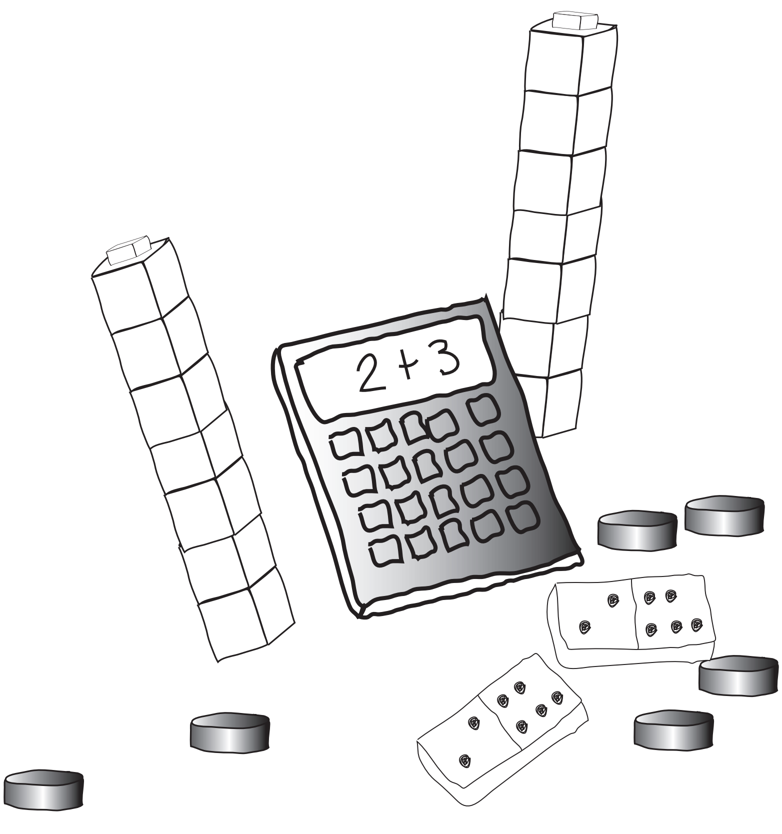
Descriptions of achievements at each level for each sub-domain, together with a listing of the actual items from the tests, are given below. Not all levels are represented in each sub-domain. For example no Whole Number Single Digit Addition and Subtraction items were found above Level 3, whereas even the simplest Fractions Addition and Subtraction item ($\frac{1}{2} + \frac{1}{2}$) appeared at Level 4.

Whole Number Single Digit Addition and Subtraction

	Skills	Items from Tests at this Level
1	Quickly adds from 1, 2 or 3 to a single digit Can calculate mentally sum of two single digits Can quickly add or subtract zero	1 + 6 (S), 6 + 1 (S), 5 + 3 (S), 2 + 8 (S) 7 + 4 (L), 8 + 6 (L), 4 + 7 (L), 6 + 8 (L), 7 + 6 (L), 6 + 7 (L) 4 + 0 (S), 0 + 8 (S), 4 – 0 (S)
2	Knows some single-digit doubles Can quickly subtract 1 or the number itself Knows/can quickly add two single digit numbers	7 + 7 (S) 7 – 1 (S), 8 – 8 (S) 9 + 8 (S), 9 + 8 (lk), 8 + 6 (S), 4 + 7 (S)
3	Can quickly subtract single digit from single digit Knows/ can calculate sums of single digit numbers and their inverses	7 – 6 (S), 8 – 3 (S), 10 – 8 (S) 17 – 7 (S), 17 – 7 (lk), 14 – 8 (L), 11 - 4 (L), 11 – 7 (L), 14 - 6 (S), 17 - 8 (S), 14 - 6 (L), 14 - 7 (L), 13 – 6 (L), 13 – 7 (L), 17 - 8 (lk), 11 - 7 (S)

Whole Number Single Digit Multiplication and Division

Level	Skills	Items from tests at this level
1	Can quickly double a single digit Can quickly multiply single digit by 10	6 x 2 (S), 2 x 10 (L) 4 x 10 (S), 8 x 10 (S), 3 x 10 (S), 5 x 10 (S)
2	Knows multiples of 2 and knows or can quickly calculate some multiples of 3, 4 and 5	7 x 2 (S), 9 x 2 (S), 7 x 10 (L), 4 x 3 (lk), 4 x 3 (L), 5 x 4 (S), 5 x 4 (lk), 5 x 4 (L), 6 x 5 (lk), 6 x 5 (L),
3	Knows/can quickly calculate multiples of 3, 4, 5 Can halve even numbers to 20	6 x 5 (S), 4 x 3 (S), 3 x 6 (lk), 7 x 3 (lk), 3 x 6 (S), 7 x 3 (S), 7 x 3 (L), 7 x 4 (L) Half 18 (L)
4	Can calculate product of single digit numbers Knows or can calculate inverse of first ten multiples of 3, 4 and 5	6 x 9 (L), 8 x 4 (L) 12 ÷ 3 (L), 21 ÷ 3 (S), 12 ÷ 4 (S), 20 ÷ 4 (L), 30 ÷ 5 (S), 30 ÷ 5 (L),
5	Knows most table facts and can calculate the others Knows or can calculate inverse of most table facts	7 x 6 (S), 7 x 6 (lk), 8 x 7 (L), 9 x 8 (S), 9 x 8 (L), 6 x 9 (S), 72 ÷ 8 (L), 70 ÷ 5 (L), 54 ÷ 9 (L), 56 ÷ 7 (L)
6	Knows all table facts and their inverses	8 x 7 (S), 54 ÷ 9 (S), 72 ÷ 8 (S), 56 ÷ 7 (S)



Two Digit Addition and Subtraction

Level	Skills	Items from tests at this level
2	Can add or subtract a single digit to/from a 2-digit number where no exchange is necessary	13 + 6 (S), 72 + 6 (S), 53 + 6 (S), 36 – 5 (L), 25 – 3 (L)
	Can add small single digit number to a 2-digit number	17 + 4 (S), 3 + 28 (L)
3	Can add or subtract a single digit to/from a 2-digit number Can add or subtract a multiple of ten to/from a 2-digit number (answer equals less than 100)	3 + 48 (S), 16 + 8 (S), 4 + 39 (S), 16 + 9 (L), 15 – 3 (S), 27 – 4 (L) 20 + 70 (S), 50 + 30 (S), 36 + 20 (L), 70 – 30 (L), 90 – 70 (S), 80 – 30 (S), 15 + 40 (L)
	Can add some 2-digit numbers with no exchanges and can subtract some special cases	23 + 14 (L), 42 + 16 (L), 16 – 15 (S)
4	Can add two 2-digit number (answer equals less than 100)	27 + 25 (L), 34 + 45 (L)
	Can quickly add and subtract multiples of 10	60 + 80 (S), 70 + 50 (S), 120 – 50 (S), 140 – 60 (S)
	Can subtract a single digit number from a 2-digit number	25 – 8 (L), 36 – 8 (L), 27 – 9 (L)
	Can subtract simple 2-digit number with no exchanges	43 – 12 (L), 35 – 15 (S)
5	Can add and subtract most 2-digit numbers	73 + 40 (L), 25 + 99 (L), 58 + 34 (L), 68 + 19 (L), 37 + 24 (L), 79 + 26 (L), 44 + 67 (L) 41 – 12 (L), 75 – 50 (S), 65 – 35 (S), 57 – 25 (L), 52 – 25 (L) 70 – 34 (L), 74 – 30 (L), 124 – 99 (L), 57 – 18 (L), 80 – 24 (L), 100 - 68 (L)
6	Can subtract a 2-digit number from a 2-digit number and beyond in special cases	92 – 34 (L), 105 – 26 (L), 105 – 97 (L), 264 – 99 (L)
7	Can subtract a 2-digit number from a 2-digit number and beyond	111 - 67

Two Digit Multiplication and Division

Level	Skills	Items form Tests at this Level
2	Can quickly double some multiples of 5	Double 30 (S), Double 15 (L)
3	Can quickly double and halve some multiples of 10 and other simple 2-digit numbers	2 x 40 (S), 40 x 2 (S), Halve 30 (L), Halve 62 (L)
4	Can double 2-digit numbers	15 x 2 (S), 35 x 2 (L), 17 x 2 (L),
	Can multiply some small 2-digit numbers by 10	12 x 10 (S), 14 x 10 (S)
	Can multiply small 2-digit numbers by 3 and 4	23 x 3 (L), 12 x 4 (L)
	Can recognize some simple halves in division	80 ÷ 40
5	Can multiply a 2-digit number by 10	10 x 27 (L), 21 x 10 (S), 27 x 10 (L), 40 x 10 (L)
	Can multiply some extended basic facts	40 x 5 (L), 7 x 200 (S)
	Can multiply some special cases of 2-digit by single digit	14 x 5 (L), 25 x 6 (L)
	Can multiply extended basic facts (up to 3-digit numbers by single digit)	9 x 200 (S), 5 x 40 (S), 60 x 7 (S)
6	Can multiply a 2-digit number by 5 or less	14 x 5 (L), 32 x 4 (L), 23 x 4 (L), 24 x 3, 5 x 24 (L)
	Can halve an even 2-digit number	Halve 76 (L)
	Can divide small multiples of 100 by some small numbers	200 ÷ 5 (L)
7	Can multiply a 2-digit number by a single digit	49 x 3 (L)
	Can divide 2 and some 3-digit numbers by a single digit	150 ÷ 6 (L), 72 ÷ 3 (L), 92 ÷ 4 (L)
	Can multiply some 2-digit numbers by a multiple of 10	12 x 20 (L), 25 x 40 (L), 40 x 80 (L), 14 x 30(L)
8	Can divide a 2 and 3-digit number by a single digit	343 ÷ 7

Decimals Addition and Subtraction

Level	Skills	Items from Tests at this Level
5	Knows/ can calculate 0.25 + 0.25	0.25 + 0.25 (S)
6	Can add and subtract simple decimals (one place only)	0.3 + 0.7 (L), 0.5 + 0.5 (L), 1.3 + 1.7 (L), 0.6 + 1.4, 4.5 – 3 (L), 2 – 0.1 (L), 1 – 0.4 (L)
7	Can add /subtract simple decimals (combining one/two places)	6.2 + 1.9, 0.5 + 0.75, 0.19 + 0.1, 0.19 + 0.01, 0.24 + 0.76, 1.25 – 0.5

Decimals Multiplication and Division

Level	Skills	Items from Tests at this Level
6	Can calculate one half of 0.5	Half of 0.5 (L)
7	Can quickly multiply a decimal (to two places) by 10 and 100 Can multiply a 2-digit number by 0.1 and 0.5 Can multiply decimal tenths by a single digit Can divide a single digit by 0.5 Can quickly calculate 0.1 ÷ 0.1	0.1 x 10 (S), 0.6 x 10 (S), 0.25 x 10 (S), 0.37 x 100 (S) 0.1 x 45, 0.5 x 48 3 x 0.6 3 ÷ 0.5 0.1 ÷ 0.1 (S)
8	Can quickly calculate 0.1 x 0.1	0.1 x 0.1 (S)
	Can quickly divide a single digit by 0.1 Can divide a simple decimal by 5 Can divide some whole numbers by 0.5	2 ÷ 0.1 (S) 0.2 ÷ 5 90 ÷ 0.5

Fractions Addition and Subtraction

Level	Skills	Items from Tests at this Level
4	Knows/ can calculate $\frac{1}{2} + \frac{1}{2}$	$\frac{1}{2} + \frac{1}{2}$ (S),
5	Can add/subtract halves and quarters less than one Can add fractions with common denominators (totals<1)	$\frac{1}{2} + \frac{1}{4}$ (S), $\frac{1}{2} + \frac{1}{4}$ (L), $\frac{3}{4} - \frac{1}{2}$ (L) $\frac{2}{7} + \frac{3}{7}$ (L)
6	Can add and subtract halves (and equivalents) and quarters beyond one Can quickly subtract a simple unit fraction from one	$\frac{1}{2} + \frac{3}{4}$ (L), $\frac{1}{2} + \frac{4}{8}$ (L), $\frac{1}{2} + \frac{5}{10}$ (L), $1\frac{1}{4} - \frac{1}{2}$ $1 - \frac{1}{3}$ (S), $1 - \frac{1}{3}$ (L)
8	Can add one half and one third	$\frac{1}{2} + \frac{1}{3}$

Fractions Multiplication and Division

Level	Skills	Items from Tests at this Level
5	Can calculate $\frac{1}{3}$ of small multiples of 3	$\frac{1}{3}$ of 12
6	Can calculate a half of a half Can calculate a half or quarter of some 2 or 3-digit numbers	$\frac{1}{2}$ of $\frac{1}{2}$ (L) $\frac{1}{4}$ of 120 (L)
7	Can multiply and divide simple unit fractions, and halves and quarters by halves and quarters	$\frac{1}{2}$ of $\frac{1}{3}$, $4 \times \frac{3}{4}$, $3 \div \frac{1}{2}$, $\frac{1}{2} \div \frac{1}{4}$
8	Can calculate $\frac{1}{2} \times \frac{3}{4}$	$\frac{1}{2} \times \frac{3}{4}$

Percentages Multiplication and Division

Level	Skills	Items from Tests at this Level
5	Can calculate 50% and 100% of an even 2-digit number	50% of 48 (S), 100% of 36 (S), 50% of 24 (L)
6	Can calculate 25% of some 2-digit numbers	25% of 80 (S)
7	Can calculate key per cents (eg 10%, 20%, 75%, 150%) of simple 2 and 3-digit numbers	10% of 45, 20% of 15, 75% of 200, 150% of 12
8	Can calculate some harder per cents of some whole numbers	90% of 40, 33⅓% of 600, 30% of 80, 12 ½ % of 24

1.13 LEVELS AND GRADES

The following table shows, for the 2002 testing, the percentage of students in each grade who were at each of the overall levels. Percentages over 20% are highlighted. It is important to note, for example, that while a student in Grade 3 may have an ability level equivalent to Level 7, it would be unlikely that he or she had ever met percentages. The rating means that this child is of equivalent ability to students in later grades who can do percentage items. It is for this reason that we have subdivided the levels into sub-domains, so that teachers can focus on levels associated with content appropriate for their classes.

GRADE	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7	Level 8
3	16.10%	14.41%	35.17%	20.76%	8.90%	3.39%	1.27%	0.00%
4	5.03%	3.52%	17.59%	29.15%	23.62%	18.09%	2.01%	1.01%
5	0.54%	2.15%	3.76%	20.43%	34.41%	27.42%	8.06%	3.23%
6	0.89%	1.33%	4.89%	12.89%	23.56%	29.89%	13.78%	13.78%
7	0.00%	0.44%	3.54%	8.85%	24.34%	42.04%	13.72%	7.08%
8	0.00%	0.50%	0.00%	7.54%	18.59%	31.66%	19.60%	22.11%
9	0.00%	2.00%	1.33%	4.67%	12.67%	30.00%	25.33%	24.00%
10	0.73%	0.73%	0.73%	5.11%	14.60%	30.66%	24.09%	23.36%
Overall	3.54%	3.41%	9.61%	14.48%	20.16%	25.73%	12.45%	10.62%

1.14 ASSESSING LEVELS FOR TEACHING

A look at the relevant descriptions of levels will give teachers a very rough idea of the expected mental computation performance of students in their grade. However, many teachers of Grades 5 upwards have found that their students lack good command of single digit mental calculation, and may not have received any introduction to basic mental strategies. The teacher may therefore, decide to use material from Module 2 and 3. Others may start with Module 4. Whatever the decision, it is better to make consistent use of the development in one Module rather than move quickly from Module to Module.

For a more accurate assessment of student levels, take three to five items from each Module that are at appropriate levels and give these successively to individual students, or to the class. Note at which levels individual students are making more than an occasional error. This is a good indication of where appropriate work can be done. When administering the items, remember the amount of time that was given to students in the original testing:

S: Short items, 3 seconds: the student should know and answer immediately.

Lk: Link items, 5 seconds: the student should know or answer quickly.

L: Long items, 15 seconds: the student should be given time to work the calculation out mentally.

1.15 USING LEVELS TO DETERMINE APPROPRIATE ACTIVITIES

The left hand column of the following tables shows the skills associated with each level for whole numbers (as in the tables above). The next column shows, for Basic Facts (Modules 2 and 3) the strategies that children are likely to use at this level, and for Two-digit Computation (Module 4) the sub-skills associated with the skills in the left-hand column. Finally the right-hand column shows the Module and activity or activities that relate to these skills and strategies. This will provide a good indication of useful activities for students at differing levels.

If in doubt, start earlier in the sequence or at the beginning of the sequence or Module, particularly if work with strategies is new to you or the students.

For fractions, decimals and percentages, because it is conceptual understanding that is almost always lacking, it is best to start at the beginning of the appropriate Modules and judge the appropriate rate of progress by the understanding shown by students.

In any case, alongside work from Modules 2 to 6, use regularly the generic activities at the end of Module 1, which are appropriate for all levels and grades.

Whole Number Single Digit Addition and Subtraction

Level	Skills	Likely Associated Strategies and Sub-Skills	Related Activities MODULE 2
1	Quickly adds from 1, 2 or 3 to a single digit Can calculate mentally sum of two single digits Can quickly add or subtract zero Knows some single-digit doubles Can quickly subtract 1 or the number itself	**Count On 1, 2, 3, 0** **Doubles (0 to 5)** **Tens Facts** **Doubles (6 to 10)**	**2.4** **2.4, 2.6** **2.5** **2.6** **2.6, 2.11**
2	Knows/can quickly add two single digit numbers Can quickly subtract single digit from single digit	**Add 10** **Bridge 10** **Near Doubles** **Count Back 1, 2, 3, 0** **Doubles (subtraction)** **Tens Facts (subtraction)**	**2.7** **2.5** **2.8** **2.12** **2.10** **2.11**
3	Knows/can calculate sums of single digit numbers and their inverses	**Subtract to 10** **Subtract 10** **Bridge 10 (subtraction)** **Near Doubles (subtraction)**	**2.1 to 1.12 revision**

Whole Number Single Digit Multiplication and Division

Level	Skills	Likely Associated Strategies and Sub-Skills	Related Activities MODULE 3
1	Can quickly double a single digit Can quickly multiply single digit by 10	**Doubles** **10 times** **1 times** **0 times**	**3.1** **3.5, 3.12** **3.3, 3.10** **3.4, 3.11**
2	Knows multiples of 2 and knows or can quickly calculate some multiples of 3, 4 and 5	**5 x = half 10 x**	**3.6**
3	Knows/can quickly calculate multiples of 3, 4, 5 Can halve even numbers to 20	**3 x = double and one more lot** **4 x = 2 x 2 x** **Half/double relationship**	**3.2, 3.9** **3.8**
4	Can calculate product of single digit numbers Knows or can calculate inverse of first ten multiples of 3, 4 and 5	 **Multiplication/division relationship 3x, 4x, 5x,**	**3.1 to 1.22 revision and games** **3.9, 3.13, 3.14,**
5	Knows most table facts and can calculate the others Knows or can calculate inverse of all table facts	 **Multiplication/division relationship 6x, 9x,**	**3.1 to 1.22 revision and games** **3.1 to 1.22 revision and games**
6	Knows all table facts and their inverses	**8x = 2x 2x 2x** **7x = 5x + 2x** **Multiplication/division relationship 8x, 7x,**	**3.17, 3.18** **Practice games**

Two Digit Addition and Subtraction

Level	Skills	Likely Associated Strategies and Sub-Skills	Related Activities MODULE 4
2	Can add or subtract a single digit to/from a 2-digit number where no exchange is necessary	**Splitting a 2-digit number 10a + b into 10a and b Adding any 2-digit multiple of ten to any single-digit number Subtracting units digit from a 2-digit number Subtracting tens digit from a 2-digit number Naming the next multiple of ten for any 2-digit number**	**4.1**
	Can add small single digit number to a 2-digit number		**4.1**
3	Can add or subtract a single digit to/from a 2-digit number	**Adding a single digit to any 2-digit number**	**4.1**
	Can add or subtract a multiple of ten to/ from a 2-digit number (answer equals less than 100)	**Counting on in tens from a multiple of ten Subtracting a single digit from any 2-digit multiple of ten Subtracting any 2-digit multiple of ten from any 2-digit multiple of ten Doubling any 2-digit multiple of ten**	**4.1**
	Can add some 2-digit numbers with no exchanges and can subtract some special cases	**Saying what needs to be added to any 2-digit number to make the next multiple of ten**	**4.2, 4.3**
4	Can add two 2-digit number (answer equals less than 100)	**Adding a 2-digit multiple of ten to any 2-digit number**	**4.2, 4.3 Practice and revision games**
	Can quickly add and subtract multiples of 10	**Adding any two 2-digit multiples of ten Counting back in tens from a multiple of ten Counting back in tens from any 2-digit number**	**4.5**
	Can subtract a single from a 2-digit number	**Subtracting any 2-digit number from the next multiple of ten Subtracting a single digit from any 2-digit number**	**4.5**
	Can subtract simple 2-digit number with no exchanges	**Subtracting any 2-digit multiple of ten from its double Subtracting any 2-digit multiple of ten from any multiple of ten less than 200**	**4.6**
5	Can add and subtract most two digit numbers	**Splitting a 2-digit number into any tens and ones Subtracting a 2-digit multiple of ten from any 2-digit number For any 2 digit number, give its complement to 100**	**4.6, 4.7 Practice and revision games**
6	Can subtract a 2-digit number from a 2-digit number and beyond in special cases	**Can convert subtraction to addition**	**4.6, 4.7 Practice and revision games**
7	Can subtract a 2-digit number from a 2-digit number and beyond	**Flexible and competent use of strategies**	**4.6, 4.7 Practice and revision games**

USEFUL RESOURCES

1.16 List of printed resources

The following material has been found useful to provide additional activities that share the same general approach as the current series.

Author	Year	Title	Publisher (ISBN)	Summary
James Burnett	1998	Teaching Number Facts using a Number Sense Approach: Addition and Subtraction	Prime Education 0 9585 582 0 5	Plenty of activities for teaching Basic Fact strategies. Includes Black Line Masters.
James Burnett		Teaching Number Facts using a Number Sense Approach: Multiplication and Division	Prime Education	
James Burnett	2001	Mastering Mental Maths: Extending the Addition Facts	Prime Education 1 876842 00 8	Five units of 5 activities, each comprising page of teacher notes and an OHT page, rehearsing strategies for adding 2-digit numbers
James Burnett	2001	Mastering Mental Maths: Extending the Subtraction Facts	Prime Education	
Bev Dunbar	2000	Exploring 0 – 50 Numeration	Blake Education 1-86509-143-X	Collection of activities, most with associated Black Line Masters. Three pages of records/ programs at the end.
Bev Dunbar	2000	Exploring 0 – 50 Operations	Blake Education 1-86509-144-8	Collection of activities, most with associated Black Line Masters. Three pages of records/ programs at the end.
Bev Dunbar	2000	Exploring 0 – 100 Numeration	Blake Education 1-86509-145-6	Collection of activities, most with associated Black Line Masters. Three pages of records/ programs at the end.
Bev Dunbar	2000	Exploring 0 – 100 Operations	Blake Education 1-86509-146-4	Collection of activities, most with associated Black Line Masters. Three pages of records/ programs at the end.
Calvin Irons	1999	Numeracy 2: 160 Mental Maths Discussions	Mimosa Publications 0 7327 2473 2	Small spiral bound booklet giving daily MC ‘discussion’ activity
Alistair McIntosh, Ellita De Nardi, Paul Swan	1994	Think Mathematically!	Longman: Pearson Education 0 582 80314 4	Framework and background to teaching mental computation. Includes seven flexible key formats for sessions, support activities and background material on strategies
Len Vincent	2000	Number Squares, Frames & Tablecloths: Book 1: Years 3-4	Len Vincent Publications,	Collection of blackline masters for practicing four operations. Some are interesting
Len Vincent	2000	Number Squares, Frames & Tablecloths: Book 2: Years 5-6	Len Vincent Publications,	Collection of blackline masters for practicing four operations. Some are interesting
Various	1996	Maths Focus	Scholastic Australia	A comprehensive maths series, with some relevant individual booklet titles: Give and Take; Reinforcing Skills and Concepts.

GENERIC ACTIVITIES TO DEVELOP MENTAL COMPUTATION

1.17 INTRODUCTION TO GENERIC ACTIVITIES

The activities in Modules 2 to 6 mainly consist of organised sequences of activities designed to introduce and consolidate the use of specific strategies for particular kinds of calculations.

The four generic activities are as follows

1. My method (Students describing their strategies)
2. How else? (Devising calculations which give the same answer)
3. How will I calculate? (Selecting suitable calculation methods)
4. What's related? (Recognising related calculations)

These activities can be adapted for use at any time with any ability or grade of students. They give students the opportunity to use and discuss their own methods for calculating. We suggest that Activity 1.1 in particular be used regularly and frequently with any group of students. A session may consist of no more than one or two calculations.

Other generic activities are included in the book *Think Mathematically!* (see the Resources section in this Module)

ACTIVITY 1.1 MY METHOD

OVERVIEW

Students explain their strategies for a given mental calculation. This activity gives students the opportunity to articulate their thinking and to hear the range of strategies used by other students.

THE ACTIVITY

1. Teacher gives a calculation orally or writes it on the board horizontally. The calculation should be within the capability of all students. At least 15 seconds time is given so that all students have time to perform the calculation.
2. Students perform the calculation mentally.
3. Students discuss their methods of solution in groups or teacher asks three or four students to explain their computation strategy. Who else did it the same way? Who did it a different way? Students' strategies can be written informally on the board, for example, for 37 + 25:

- 30 + 20 = 50, 7 + 5 = 12, 50 + 12 = 62
- 37 + 3 = 40, 40 + 20 = 60, 60 + 2 = 62
- 25 + 25 + 12

4. Discuss the variety of methods used. Ask students which method they prefer, with reasons.

COMMENTS ON THE ACTIVITY

- No attempt should be made to indicate a 'preferred method' by the teacher. Students should feel free to use any strategy that works efficiently for them.

ASSESSING PROGRESS

- Students become more comfortable articulating their strategies
- Students follow and recognise strategies described by others
- Students critically discuss and compare strategies

PRACTICE EXAMPLES

At the end of a session students can be asked to record different ways of performing the given calculation.

ACTIVITY 1.2 HOW ELSE ?

OVERVIEW

Students devise mentally a variety of calculations which give the same answer. The activity allows all students to perform some mental calculations at their own level of confidence.

THE ACTIVITY

1. Write a number between 5 and 30 on the board.
2. Ask students for calculations to which this number is the answer.
3. Indicate that you will accept any calculation unless someone says that this does not produce the right answer.
4. Accept the first 8 - 10 correct answers without comment, placing them in one of five columns on the board: Additions (of two numbers), Subtractions, Multiplications, Divisions, Any other calculation.
5. Make restrictions on the calculations allowed, for example:

- Additions only
- Additions of three numbers
- Use an addition AND a multiplication
- Use a fraction

COMMENTS ON THE ACTIVITY

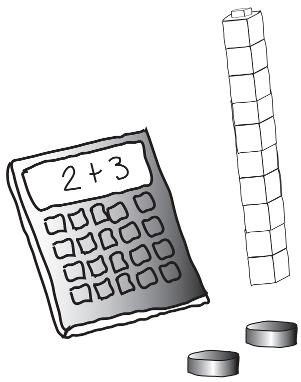
- The ‘Any other calculation’ column allows more confident students to provide complicated calculations without undermining the confidence of other students.
- It is important that all calculations are accepted as equally valid.

ASSESSING PROGRESS

- Students extend the range of calculations they can provide for a given number.
- Students recognise patterns, for example that 13 – 8, 14 – 9, 15 – 10 all equal 5.

PRACTICE EXAMPLES

At the end of the session students can be given a few minutes to record individually as many calculations as they can which have the given answer.



ACTIVITY 1.3 HOW WILL I CALCULATE?

OVERVIEW

Students are shown a variety of calculations and are asked to decide, with reasons, how they would calculate each – mentally, with paper-and-pencil, or with a calculator. The aim is to encourage students to make sensible choices based on a realistic awareness of their strengths and weaknesses.

THE ACTIVITY

1. Write 10 calculations of varying degrees of difficulty on the board and ask students to decide how they would calculate each: mentally, with paper-and-pencil, or using a calculator.
2. Take each calculation in turn and ask some students for their decisions with reasons.
3. Discuss what makes some calculations easy to perform mentally.

COMMENTS ON THE ACTIVITY

- It is important that students do not gain the impression that there is a 'right' method for all; rather they should gain awareness of their strengths and weaknesses and also of why some calculations are easier than others.
- The activity can be useful in revealing conceptual misunderstandings: for example some students will say $\frac{1}{3} + \frac{3}{5}$ is easy because the answer is $\frac{4}{8}$.

ASSESSING PROGRESS

- Students make appropriate choices of method based on sound reasons.
- Students articulate their thinking with confidence

PRACTICE EXAMPLES

At the end of the session students can record their final decisions with reasons.

VARIATIONS /EXTENSIONS

As an alternative this can be made competitive, with students first determining how they will do each calculation and then, when they have recorded this, actually performing the calculation using their preferred calculation method. Students then gain a score on the basis of something like:

Incorrect answer:	0
Calculator correct:	1
Written correct:	3
Mental correct:	5

ACTIVITY 1.4 WHAT'S RELATED?

OVERVIEW

Students are given a calculation and are asked to provide related calculations. The aim of this activity is to strengthen and extend students' awareness that computations are related to each other and that, if they know one relationship, for example 3 + 2 = 5, other relationships spring from it, for example 3 + 3 = 6 or 30 + 20 = 50 or 6 + 4 = 10.

THE ACTIVITY

1. Write a calculation on the board, for example 5 + 3 = 8. Ask for students to suggest, with reasons, other calculations that are related to or are connected with this: for example:
   5 + 3 = 8:
   50 + 30 = 80, 0.5 + 0.3 = 0.8, 6 + 3 = 9, 4 + 4 = 8, 25 + 3 = 28, 3 + 5 = 8…
2. Write each reasoned correct suggestion on the board, clustering those that are similar in some way. (Students may be asked to suggest which are similar)
3. Students can be asked for particular connections. For example

- start with 5 +... (e.g. 5 + 4 = 9)
- use multiples of 10...
  (e.g. 500 + 300 = 800)
- include a fraction...
  e.g. $\frac{5}{9} + \frac{3}{9} = \frac{8}{9}$)
- include a decimal…
  (e.g. 0.5 + 0.3 = 0.8)
- use subtraction… (e.g. 5 + 3 – 2 = 6)
- use multiplication…
  (e.g. 5 x 4 + 3 x 4 = 8 x 4)
- have three numbers on the left-hand side of the equation…
  (e.g. 5 + 3 + 40 = 48)

COMMENTS ON THE ACTIVITY

- When calculating mentally we constantly use relationships between numbers and operations to make the calculation easier, and the aim of the activity is to make explicit the range and nature of such relationships.
- It is important that students articulate appropriately the relationship between calculations.

ASSESSING PROGRESS

- Students provide a range of related calculations.
- Students articulate the nature of the relationship.

PRACTICE EXAMPLES

At the end of the session students can be asked to record individually as many related calculations as they can: older students can be asked to record their reasons.

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References

Australian Education Council (1991). *A national statement on mathematics for Australian schools*. Melbourne: Curriculum Corporation.

Callingham, R & McIntosh, A.J. (2002). Mental computation competence across years 3-10. In B. Barton, K. C. Irwin, M. Pfannkuch & M. O. J. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia Incorporated, Vol I, pp 155 - 162). Auckland, NZ: MERGA.

Cockcroft, W.H. (1982). *Mathematics counts. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the Chairmanship of Dr W.H. Cockcroft*. London: Her Majesty's Stationery Office.

Hope, J. A. (1986). Mental calculation: Anachronism or basic skill? In H. L. Schoen (Ed.), *Estimation and mental computation: 1986 Yearbook of the National Council of Teachers of Mathematics* (pp. 45-54). Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Northcote, M., & McIntosh, A.J. (1999). What mathematics do adults really do in everyday life? *Australian Primary Mathematics Classroom,* 4(1). 19 – 21

Reys, R.E. (1984). Mental computation and estimation: Past, present and future. *Elementary School Journal, 84* (5), 546-557.

Wandt, E. & Brown, G. W. (1957). Non-Occupational uses of mathematics: mental and written - approximate and exact. *Arithmetic Teacher, 4*(4), 151-154