The Improving Mathematics Education in Schools (TIMES) Project

FRACTIONS AND THE INDEX LAWS IN ALGEBRA

A guide for teachers - Years 8–9

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FRACTIONS AND THE INDEX LAWS IN ALGEBRA

ASSUMED KNOWLEDGE

The arithmetic of integers and of fractions.

- Basic methods of mental arithmetic.

- The HCF and LCM in arithmetic.

- Algebraic expressions with whole numbers and negatives.

- The index laws in arithmetic with nonzero whole numbers indices.

- The three index laws involving products in algebra.

- Linear equations

MOTIVATION

The various uses of algebra require systematic skills in manipulating algebraic expressions. This module is the third of four modules that provide a systematic introduction to basic algebraic skills.

In the module, *Algebraic Expressions*, we introduced algebra using only whole numbers and the occasional fraction for the pronumerals. Then in the module *Negatives and the Index Laws in Algebra* we extended the methods of that module to expressions involving negatives.

The present module extends the methods of algebra so that all negative fractions and decimals can also be substituted into algebraic expressions, and can appear as the solutions of algebraic equations.

Although earlier modules occasionally used negative fractions, this module provides the first systematic account of them, and begins by presenting the four operations of arithmetic, and powers, in the context of negative fractions and decimals. The resulting number system is called the *rational numbers*. This system is sufficient for all the normal calculations of every life, because adding, subtracting, multiplying and dividing rational numbers (apart from division by zero), and taking whole number powers of rational numbers, always produces another rational numbers. In particular, scientific calculations often involve manipulation of equations with decimals.

The module also begins the discussion of algebraic fractions, which routinely cause problems for students. In this module, the discussion is quickly restricted by excluding numerators and denominators with more than one term. These more complicated algebraic fractions are then covered in more detail in the fourth module, *Special Expansions and Algebraic Fractions*.

Once fractions have been introduced into algebra, the two index laws dealing with quotients can be stated in a more systematic form, and then integrated with algebra. This completes the discussion of the index laws in arithmetic and algebra as far as nonzero whole number indices are concerned.

CONTENT

THE RATIONAL NUMBERS

Our arithmetic began with the whole numbers

  0, 1, 2, 3, 4,…

We then constructed extensions of the whole numbers in two different directions. First, in the module, *Fractions* we added the positive fractions, such as $\frac{2}{3}$ and $\frac{9}{2} = 4\frac{1}{2}$, which can be written as the ratio of a whole number and a non–zero whole number, and we showed how to add, subtract, multiply and divide fractions. This gave a system of numbers consisting of zero and all positive fractions. Within this system, every non–zero number has a reciprocal, and division without remainder is always possible (expect by zero). Only zero has an opposite, however, and a subtraction $a - b$ is only possible when $b \le a$.

Secondly, in the module, *Integers* we started again with the whole numbers and added the opposites of the whole numbers. This gave a system of numbers called the *integers*, consisting of all whole numbers and their opposites,

  …, −3, −2, −1, 0, 1, 2, 3,…

and we showed how to add, subtract, multiply and divide integers. Within this system of integers, every number has an opposite, and subtraction is always possible. Only 1 and −1 have reciprocals, however, and division without remainder is only possible in situations such as 21 ÷ 7 when the dividend is an integer multiple of the divisor.

Arithmetic needs to be done in a system without such restrictions – we want both subtraction and division to be possible all the time (except for division by zero). To achieve this, we start with the integers and positive fractions that we have already. Then we add the opposites of the positive fractions, so that our number system now contain all numbers such as

$$-\frac{2}{3} \text{ and } -\frac{9}{2} = -4\frac{1}{2}$$

All these numbers together are called the **rational numbers** – the word 'rational' is the adjective from 'ratio'. We have actually been using negative fractions in these notes for some time. These remarks are intended to formalise the situation. The usual laws of signs apply to multiplying and dividing in this larger system, so that a fraction with negative integers in the numerator, or in the denominator, or in both, can be written as either a positive or a negative fraction,

$$\frac{-2}{3} \text{ and } \frac{2}{-3} = -\frac{2}{3} \text{ and } \frac{-2}{-3} = \frac{2}{3}$$

Because of these identities, it is usual to define a rational number as the ratio of an integer and a non–zero integer:

'A **rational number** is a number that can be written as a fraction, where a is an integer and b is a non–zero integer'.

Thus positive and negative mixed numerals, terminating decimals, and recurring decimals are rational numbers because they can be written as fractions of integers. For example,

$$9\frac{1}{5} = \frac{46}{5}, \qquad -23.917 = \frac{-23\,917}{1000}, \qquad -1.\dot{0}\dot{9} = \frac{-12}{11}$$

In later modules on topics such as surds, circles, trigonometry and logarithms, we will need to introduce further numbers, called irrational numbers, that cannot be written as fractions. Some examples of such numbers are

$$\sqrt{2}, \ \pi, \ \sin 22°, \ \log_2 5.$$

The module, *The Real Numbers* will discuss the resulting more general system of arithmetic.

EXAMPLE

Show that each number below is rational by writing it as a fraction $\frac{a}{b}$, where a and b are integers with $b \neq 0$.

a 5   b $-\frac{3}{7}$ c $5\frac{1}{3}$

d 0.002   e -4.7 f $1.\dot{3}$

SOLUTION

a $5 = \frac{5}{1}$ b $-\frac{3}{7} = \frac{-3}{7}$ c $5\frac{1}{3} = \frac{16}{3}$

d $0.002 = \frac{2}{1000}$ e $-4.7 = \frac{-47}{10}$ f $1.\dot{3} = \frac{4}{3}$

Constructing the rational numbers

There are two ways to construct the rational numbers. The first way is to construct the positive fractions first and then the negatives, which is roughly what happened historically. This involves a three–step procedure:

1 First construct the whole numbers 0, 1, 2, 3, 4,…

2 Then construct the non–negative fractions, such as $\frac{3}{4}$ and $\frac{9}{4}$, as the ratio of two whole numbers, where the second is non–zero. This process involves identifying a fraction such as $\frac{2}{6}$, where the two whole numbers have a common factor, with its simplest form $\frac{1}{3}$, and identifying a fraction such as $\frac{5}{1}$, where the denominator is 1, with the whole number 5.

3 Lastly construct the rational numbers by constructing the opposites of every positive fraction and whole number.

The other method is to construct the integers first and then the fractions. The three steps are now:

1 First construct the whole numbers 0, 1, 2, 3, 4,..

2 Then construct the integers by constructing the opposites …, −4, −3, −2, −1 of all the nonzero whole numbers.

3 Lastly, construct the rational numbers by constructing all the fractions such as $\frac{3}{4}$ and $\frac{9}{4}$ the ratio of two integers, where the second is non–zero. This process again involves identifying equivalent fractions, and identifying fractions with denominator 1 as integers.

These two procedures are equally acceptable mathematically, and their final results are identical in structure. Students learn about rational numbers, however, by working with examples of them in arithmetic and algebra, and there is little to be gained by such explanations.

ADDING AND SUBTRACTING NEGATIVE FRACTIONS AND DECIMALS

When adding and subtracting fractions and decimals that may be positive or negative, the rules are a combination of the rules for integers and the rules for adding and subtracting positive fractions and decimals:

● To subtract a negative number, add its opposite.

● Perform addition and subtraction from left to right.

● When dealing with mixed numerals, it is usually easier to separate the integer part and the fraction.

The following examples demonstrate such calculations first in arithmetical examples, then in substitutions and equations.

EXAMPLE

Simplify:

a $\frac{5}{8} + \left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)$

b $32.6 + (-23.6) - (-26.4)$

SOLUTION

a $\frac{5}{8} + \left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right) = \frac{5}{8} - \frac{6}{8} + \frac{2}{8}$

$= \frac{1}{8}.$

b $32.6 + (-23.6) - (-26.4) = 32.6 - 23.6 + 26.4$

$= 35.4.$

EXAMPLE

Evaluate $-x - y + z$ when

a $x = -20\frac{2}{3}, y = 25\frac{1}{2}$ and $z = -35\frac{3}{4}$.

b $x = -3.6, y = 5.6$ and $z = -2.07$.

SOLUTION

a $-x - y + z = -\left(-20\frac{2}{3}\right) - 25\frac{1}{2} + \left(-35\frac{3}{4}\right)$

$= 20 + \frac{8}{12} - 25 - \frac{6}{12} - 35 - \frac{9}{12}$

$= -40 - \frac{7}{12}$

$= -40\frac{7}{12}$

Beware that in notation $-40\frac{7}{12}$, the negative sign applies to the whole number $40\frac{7}{12}$, so that $-40\frac{7}{12}$ means $-\left(40 + \frac{7}{12}\right) = -40 - \frac{7}{12}$.

b $-x - y + z = -(-3.6) - 5.6 + (-2.07)$

$= 3.6 - 5.6 - 2.07$

$= -4.07$

EXERCISE 1

Simplify $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$.

MULTIPLYING NEGATIVE FRACTIONS AND DECIMALS

As with addition and subtraction, we need to combine the rules already discussed:

- *First determine the sign of the product, then deal with the numbers. The product of two positives or two negatives is positive. The product of a positive and a negative is negative.*

- *In the absence of brackets, the order of operations is:*

    1   *Powers,*

    2   *Multiplication and division from left to right,*

    3   *Addition and subtraction from left to right.*

- *When multiplying fractions, first convert them to improper fractions.*

- *When multiplying decimals, convert them to fractions whose denominators are powers of 10.*

    (Some people prefer to use the rule, 'Before discarding trailing zeroes, the number of decimal places in the answer equals the total number decimals places in the question.')

The following examples demonstrate these procedures first in arithmetical examples, then with substitutions and equations.

EXAMPLE

Simplify:

a $2\frac{1}{2} \times \left(-3\frac{1}{3}\right)^2 \times \left(-\frac{3}{5}\right)^3$ **b** $-1.2 \times (-0.2)^3$

SOLUTION

a   The product is negative because there are five minus signs in the product.

$$2\frac{1}{2} \times \left(-3\frac{1}{3}\right)^2 \times \left(-\frac{3}{5}\right)^3 = -\frac{5}{2} \times \frac{10}{3} \times \frac{10}{3} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$$
$$= -6$$

b   The product is positive because there are four minus signs in the product.

$$-1.2 \times (-0.2)^3 = \frac{12}{10} \times \frac{2^3}{10^3}$$
$$= \frac{96}{10^4}$$
$$= 0.0096$$

EXAMPLE

Simplify $-3x^2 - yz$ when:

a $x = -1\frac{1}{2}$, $y = -2\frac{1}{2}$ and $z = 4\frac{1}{5}$ **b** $x = 1.1$, $y = -0.4$ and $z = -12.1$

SOLUTION

a $-3x^2 - yz = -3 \times \frac{3}{2} \times \frac{3}{2} + \frac{5}{2} \times \frac{21}{5}$ (The 3 is not squared – indices come first)

$\qquad = -\frac{27}{4} + \frac{21}{2}$

$\qquad = -6\frac{3}{4} + 10\frac{1}{2}$

$\qquad = -6 - \frac{3}{4} + 10 + \frac{2}{4}$ (Care with mixed numerals: $-6\frac{3}{4}$ means $-6 - \frac{3}{4}$).

$\qquad = 4 - \frac{1}{4}$

$\qquad = 3\frac{3}{4}$

b $-3x^2 - yz = \frac{-3 \times 11^2}{100} - \frac{4}{10} \times \frac{121}{10}$

$\qquad = -\frac{363}{100} - \frac{484}{100}$

$\qquad = -1.21$

EXAMPLE

Solve each equation:

a $-\frac{x}{3} + 4\frac{3}{4} = -5\frac{3}{4}$ **b** $\frac{x}{-1.2} + 2.01 = 2$

SOLUTION

a $-\frac{x}{3} + 4\frac{3}{4} = -5\frac{3}{4}$

$\qquad -\frac{x}{3} = -10\frac{1}{2}$ $\boxed{-4\frac{3}{4}}$

$\qquad x = 31\frac{1}{2}$ $\boxed{\times (-3)}$

In the last line, it was easier to calculate $3 \times 10\frac{1}{2} = 30 + 1\frac{1}{2}$ than to use improper fractions

b $\frac{x}{-1.2} + 2.01 = 2$

$\qquad \frac{x}{-1.2} = -0.01$ $\boxed{-2.01}$

$\qquad x = \frac{1}{100} \times \frac{12}{10}$ $\boxed{\times -1.2}$

$\qquad = 0.012$

EXERCISE 2

How many factors must be taken in the product $(-0.8) \times 0.9 \times (-1.0) \times \ldots$ for the product to exceed 1?

DIVIDING NEGATIVE FRACTIONS AND DECIMALS

We have already reviewed the order of operations. The other necessary rules are:

- *First determine the sign of the quotient, then deal with the numbers.*

  The quotient of two positives or two negatives is positive.

  The quotient of a positive and a negative is negative.

- *When dividing by a fraction, multiply by its reciprocal. An improper fraction must first be converted to an improper fraction.*

- *To divide by a decimal, write the calculation as a fraction, then multiply top and bottom by a power of 10 large enough to make the divisor a whole number.*

- *(Some people prefer to change both decimals to fractions and then use fraction methods.)*

EXAMPLE

Simplify:

a $1\frac{1}{2} \div \left(-2\frac{1}{2}\right)$ **b** $0.9 \times (-0.08) \div (-0.06)^2$

SOLUTION

a $1\frac{1}{2} \div \left(-2\frac{1}{2}\right) = -\frac{3}{2} \times \frac{2}{5}$

$\qquad = -\frac{3}{5}.$

b $0.9 \times (-0.08) \div (-0.06)^2 = \frac{-0.072}{0.0036}$

$\qquad\qquad\qquad\qquad = \frac{-720}{36}$ (Multiply top and bottom by 10000.)

$\qquad\qquad\qquad\qquad = -20.$

EXAMPLE

Simplify $\frac{3a}{b} - c$ when:

a $a = \frac{8}{9}, b = -1\frac{1}{10}$ and $c = -2$.   **b** $a = 0.09, b = -0.027$ and $c = -0.1$.

SOLUTION

a $\frac{3a}{b} - c = -\frac{3}{1} \times \frac{8}{9} \times \frac{10}{11} + 2$ **b** $\frac{3a}{b} - c = \frac{3 \times 0.09}{-0.027} - (-0.1)$

$\qquad = -\frac{80}{33} + 2$ $\qquad\qquad = -\frac{3 \times 90}{27} + 0.1$

$\qquad = -\frac{14}{33}$ $\qquad\qquad = -10 + 0.1$

$\qquad\qquad\qquad\qquad = -9.9$

The reciprocal

We have seen that the reciprocal of a positive fraction such as $\frac{2}{5}$ is the fraction $\frac{5}{2}$ formed by interchanging the numerator and denominator, because

$$\frac{2}{5} \times \frac{5}{2} = 1$$

Similarly, the reciprocal of a negative fraction such as $-\frac{2}{5}$ is $-\frac{5}{2}$, because

$$\left(-\frac{2}{5}\right) \times \left(-\frac{5}{2}\right) = 1$$

● *To form the reciprocal, interchange numerator and denominator, and do not change the sign.*

Reciprocals of positive and negative fractions are important in various later applications, particularly perpendicular gradients.

EXAMPLE

Find, as a mixed numeral when appropriate, the reciprocals of:

a $-\frac{4}{13}$ **b** $\frac{20}{107}$ **c** $5\frac{1}{6}$ **d** $-7\frac{7}{10}$

SOLUTION

a The reciprocal of $-\frac{4}{13}$ is $-\frac{13}{4} = -3\frac{1}{6}$.

b The reciprocal of $\frac{20}{107}$ is $\frac{107}{20} = 5\frac{7}{20}$.

c The reciprocal of $5\frac{1}{6} = \frac{31}{6}$ is $\frac{6}{31}$.

d The reciprocal of $-7\frac{7}{10} = -\frac{77}{10}$ is $-\frac{10}{77}$.

Compound fractions

There is a very straightforward way to simplify a compound fraction such as $\dfrac{\frac{2}{3} + \frac{3}{4}}{2\frac{1}{2} + \frac{1}{6}}$

● *To simplify a compound fraction, multiply top and bottom by the lowest common denominator of all the fractions.*

In the example above, the lowest common denominator of all the fractions is 12, so multiplying top and bottom by 12

$$\frac{\frac{2}{3} + \frac{3}{4}}{2\frac{1}{2} + \frac{1}{6}} = \frac{8 + 9}{30 + 2} = \frac{17}{32}$$

It is also possible to rewrite the fraction as a division $\left(\frac{2}{3} + \frac{3}{4}\right) \div \left(2\frac{1}{2} + \frac{1}{6}\right)$, but such an approach requires more steps.

EXAMPLE

a Simplify $\dfrac{-1\frac{1}{3} + 1\frac{1}{2}}{-1\frac{1}{3} - 1\frac{1}{2}}$

b Evaluate $\dfrac{x + 2a}{x - a}$ when $x = -3\frac{1}{3}$ and $a = \frac{3}{5}$

SOLUTION

a Multiply top and bottom by 6.

$$\frac{-1\frac{1}{3} + 1\frac{1}{2}}{-1\frac{1}{3} - 1\frac{1}{2}} = \frac{-8 + 9}{-8 - 9} = -\frac{1}{17}$$

b Multiply top and bottom by 15.

$$\frac{x + 2a}{x - a} = \frac{-3\frac{1}{3} + \frac{6}{5}}{-3\frac{1}{3} - \frac{3}{5}}$$

$$= \frac{-50 + 18}{-50 - 9}$$

$$= \frac{-32}{-59} = \frac{39}{59}$$

Solving equations with fractions

An equation with several fractions can be solved by the standard method, 'Move every term in x to one side, and all the constants to the other.' The resulting fraction calculations, however, can be quite elaborate. There is a far quicker approach that gets rid of all the fractions in one single step,

- *If an equation involves fractions, multiply by the lowest common denominator.*

- *Similarly, if an equation involves decimals, multiply through by a suitable power of 10.*

The following example uses first this quicker approach, then the standard method, to solve two equations. Compare the approaches and decide for yourself.

EXAMPLE

Use the quicker approach above, and then the standard approach, to solve these equations.
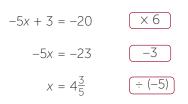
Solve:

a $-\frac{5}{6}x + \frac{1}{2} = -3\frac{1}{3}$

b $-0.024x + 0.1 = 0.04$

SOLUTION

a $-\frac{5}{6}x + \frac{1}{2} = -3\frac{1}{3}$

Multiply by 6 to remove all fractions.

$-5x + 3 = -20$ ⬚ × 6

$-5x = -23$ ⬚ −3

$x = 4\frac{3}{5}$ ⬚ ÷ (−5)

or

$$-\tfrac{5}{6}x + \tfrac{1}{2} = -3\tfrac{1}{3}$$

$$-\tfrac{5}{6}x = -3 - \tfrac{2}{6} - \tfrac{3}{6} \qquad \boxed{-\tfrac{1}{2}}$$

$$-\tfrac{5}{6}x = -3\tfrac{5}{6}$$

$$x = \tfrac{6}{5} \times \tfrac{23}{6} \qquad \boxed{\times\left(-\tfrac{6}{5}\right)}$$

$$= 4\tfrac{3}{5}$$

b $-0.024x + 0.1 = 0.04$

Multiply by 1000 to remove all decimals.

$$-24x + 100 = 40 \qquad \boxed{\times 1000}$$

$$-24x = -60 \qquad \boxed{-100}$$

$$x = 2.5 \qquad \boxed{\div (-24)}$$

or

$$-0.024x + 0.1 = 0.04$$

$$-0.024x = -0.06 \qquad \boxed{-0.1}$$

$$= \tfrac{-0.06}{-0.024} \qquad \boxed{\div (-0.024)}$$

$$= \tfrac{60}{24}$$

$$= 2.5$$

EXERCISE 3

Use an efficient method to find the mean of the numbers 1, 0.9, 0.8, ..., −2.

The system of rational numbers

We have now introduced four number systems in turn.

First, we introduced the *whole numbers* 0, 1, 2, 3, 4,... The set of whole numbers is **closed** under addition and multiplication, which means that when we add or multiply two whole numbers, we obtain another whole number. The whole numbers are not closed under either subtraction or division, however, because, for example,

7 − 10 is not a whole number   and   7 ÷ 10 is not a whole number.

Next, we added the *positive fractions* to the whole numbers, so that our number system now contained all non–negative rational numbers, including numbers such as 4, $\tfrac{2}{3}$, and $5\tfrac{5}{7}$. This system is closed under division (except by zero), but is still not closed under subtraction.

Then we introduced the *integers* …, −3, −2, −1, 0, 1, 2, 3, … consisting of all whole numbers and their opposites. This system is closed under subtraction, but not under division.

Now that we are working in the *rational numbers* – the integers together with all positive and negative fractions – we finally have a system that is closed under all four operations of addition, subtraction, multiplication and division (except by zero). They are also closed under the operation of taking whole number powers.

This set of rational numbers is therefore a very satisfactory system for doing arithmetic, and is quite sufficient for all everyday needs. When we move to the even larger system of *real numbers*, every real number can be approximated as close as we like by rational numbers (indeed by terminating decimals), so that the rational numbers will remain extremely important.

CANCELLING, MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS

An **algebraic fraction** is a fraction involving pronumerals,

$$\frac{x}{3}, \quad \frac{5}{x}, \quad \frac{a+b}{c}, \quad -\frac{3r}{s-t}.$$

The pronumerals in the fraction represent numbers, so we deal with algebraic fractions using exactly the same procedures that we use in arithmetic.

Substitution into algebraic fractions

We have seen in arithmetic that the denominator in a fraction cannot be zero. This means that:

In the expression $\frac{5}{x}$, we cannot substitute $x = 0$.

In the expression $\frac{a+b}{c}$, we cannot substitute $c = 0$.

In the expression $-\frac{3r}{s-t}$, we cannot substitute the same value for s and t.

Apart from this qualification, substitution into algebraic fractions works exactly the same as substitution into any algebraic expression. For example, if $r = 5$, $s = 7$ and $t = 4$, then

$$-\frac{3r}{s-t} = \frac{15}{7-4} = 5$$

Substitution of negative numbers requires the usual care with signs. If $r = -2$, $s = -7$ and $t = 5$,

$$-\frac{3r}{s-t} = \frac{-6}{-7-5} = \frac{-6}{-12} = \frac{1}{2}$$

EXAMPLE

a What value of a cannot be substituted into $\frac{a+3}{a-3}$?

b What value of a makes $\frac{a+3}{a-3}$ zero?

c Evaluate $\frac{a+3}{a-3}$ when $a = -2$.

SOLUTION

a When $a = 3$, the denominator is zero, and the expression cannot be evaluated.

b When $a = -3$, the numerator is zero but the denominator is not zero, so the expression is zero.

c When $a = -2$,
$$\frac{a+3}{a-3} = \frac{-2+3}{-2-3}$$
$$= \frac{1}{-5}$$
$$= -\frac{1}{5}$$

The rest of this module will exclude denominators, such as $s - t$ and $a - 3$, that have two or more terms. Such expressions will be considered in the later module, *Special Expansions and Algebraic Fractions*.

Cancelling algebraic fractions

Because pronumerals are just numbers, we can cancel both pronumerals and numbers in an algebraic fraction. For example,

$$\frac{3xy}{6x} = \frac{y}{2} \quad \text{and} \quad \frac{a^2 b}{ab} = a.$$

We cannot cancel zero, so strictly we should exclude zero values of the pronumeral when we cancel. This would mean writing

$$\frac{3xy}{6x} = \frac{y}{2} \text{ provided } x \neq 0 \quad \text{and} \quad \frac{a^2 b}{ab} = a \text{ provided } a \neq 0 \text{ and } b \neq 0.$$

Such qualifications become important much later in calculus. At this stage, however, it is not appropriate to bring the matter up, unless students insist.

EXAMPLE

Simplify each algebraic fraction by cancelling any common factors:

a $\frac{5}{10b}$ **b** $\frac{9ab}{12bc}$ **c** $\frac{4b^2}{-6ab}$ **d** $\frac{-12xy^2 z}{-12x^2 yz}$

SOLUTION

a $\frac{5}{10b} = \frac{1}{2b}$ **b** $\frac{9ab}{12bc} = \frac{3a}{4c}$ **c** $\frac{4b^2}{-6ab} = -\frac{2b}{3a}$ **d** $\frac{-12xy^2 z}{-12x^2 yz} = \frac{y}{x}$

Dividing algebraic expressions

When we divide, the laws of signs are the same as the laws for multiplying:

- *The quotient of two negatives or two positives is positive.*
- *The quotient of a positive and a negative is negative.*

The following examples illustrate all the possibilities:

$$\frac{6x^2}{2x} = 3x \qquad\qquad \frac{-6x^2}{2x} = -3x \qquad\qquad \frac{6x^2}{-2x} = -3x \qquad\qquad \frac{-6x^2}{-2x} = 3x$$

As with multiplication, there are three steps in handling more complicated expressions:
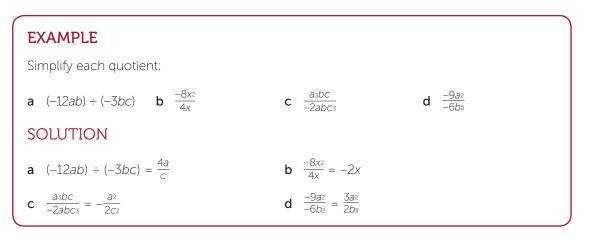
- *First deal with the signs.*
- *Then deal with the numerals.*
- *Then deal with each pronumeral in turn.*

$$\frac{-3xy}{-6x^2} = \frac{y}{2x} \qquad\qquad \text{and} \qquad (21abc^2) \div (-7b^2c) = -\frac{3ac}{b}$$

The last answer can also be written as $\frac{-3ac}{b}$ or even as $\frac{3ac}{-b}$, but the best way to write the answer $-\frac{3ac}{b}$, with the negative sign out the front of the fraction.

Fraction notation is usually used for quotients, but the division sign can also be used, as in the second example above.

EXAMPLE

Simplify each quotient:

a $(-12ab) \div (-3bc)$ **b** $\frac{-8x^2}{4x}$ **c** $\frac{a^3bc}{-2abc^3}$ **d** $\frac{-9a^2}{-6b^2}$

SOLUTION

a $(-12ab) \div (-3bc) = \frac{4a}{c}$ **b** $\frac{-8x^2}{4x} = -2x$

c $\frac{a^3bc}{-2abc^3} = -\frac{a^2}{2c^2}$ **d** $\frac{-9a^2}{-6b^2} = \frac{3a^2}{2b^2}$
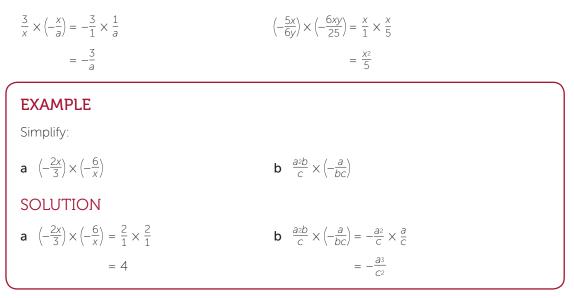

Multiplying algebraic fractions

We have seen how to multiply fractions in arithmetic,

$$\frac{15}{5} \times \left(-\frac{10}{21}\right) = -\frac{5}{7} \times \frac{10}{7}$$

$$= -\frac{50}{49}$$

- *First deal with the signs.*
- *Then cancel any common factors from a numerator and a denominator.*
- *Then multiply the numerators, and multiply the denominators.*

Exactly the same method applies to multiplying algebraic fractions. For example,

$$\frac{3}{x} \times \left(-\frac{x}{a}\right) = -\frac{3}{1} \times \frac{1}{a}$$

$$= -\frac{3}{a}$$

$$\left(-\frac{5x}{6y}\right) \times \left(-\frac{6xy}{25}\right) = \frac{x}{1} \times \frac{x}{5}$$

$$= \frac{x^2}{5}$$

EXAMPLE

Simplify:

a $\left(-\frac{2x}{3}\right) \times \left(-\frac{6}{x}\right)$

b $\frac{a^2b}{c} \times \left(-\frac{a}{bc}\right)$

SOLUTION

a $\left(-\frac{2x}{3}\right) \times \left(-\frac{6}{x}\right) = \frac{2}{1} \times \frac{2}{1}$

$= 4$

b $\frac{a^2b}{c} \times \left(-\frac{a}{bc}\right) = -\frac{a^2}{c} \times \frac{a}{c}$

$= -\frac{a^3}{c^2}$

Dividing by an algebraic fraction

We know the law of signs for division, and we know how to divide by a fraction. To divide by an algebraic fraction, we apply these same laws in succession

1 *First determine the sign of the quotient.*

2 *To divide by a fraction, multiply by its reciprocal*

For example,

$$\frac{4}{x} \div \left(-\frac{6}{x}\right) = -\frac{4}{x} \times \frac{x}{6}$$

$$= -\frac{2}{1} \times \frac{1}{3}$$

$$= -\frac{2}{3}$$

$$\frac{3a^2b}{c} \div \frac{3ab^2}{c} = \frac{3a^2b}{c} \times \frac{c}{3ab^2}$$

$$= \frac{a}{1} \times \frac{1}{b}$$

$$= \frac{a}{b}$$

EXAMPLE

Simplify each quotient:

a $\frac{12}{x} \div \frac{3}{x}$

b $\frac{6x^2}{y} \div \frac{4x}{y^2}$

c $\frac{3ab}{c} \div (-6abc)$

d $\left(-\frac{2a^2}{c}\right) \div \left(-\frac{2a}{c^2}\right)$

SOLUTION

a $\frac{12}{x} \div \frac{3}{x} = \frac{12}{x} \times \frac{x}{3}$

$= \frac{4}{1} \times \frac{1}{1}$

$= 4$

b $\frac{6x^2}{y} \div \frac{4x}{y^2} = \frac{6x^2}{y} \times \frac{y^2}{4v}$

$= \frac{6x^2}{y} \times \frac{y^2}{4v}$

$= \frac{3xy}{2}$

c $\frac{3ab}{c} \div (-6abc) = \frac{3ab}{c} \times \left(\frac{-1}{6abc}\right)$

$= -\frac{1}{c} \times \frac{1}{2c}$

$= -\frac{1}{2c^2}$

d $\left(-\frac{2a^2}{c}\right) \div \left(-\frac{2a}{c^2}\right) = \frac{2a^2}{c} \times \frac{c^2}{2a}$

$= \frac{a}{1} \times \frac{c}{1}$

$= ac$

The reciprocal of an algebraic fraction

Reciprocals of algebraic fractions are formed in exactly the same way as was done in arithmetic.

The reciprocal of $\frac{2x}{y}$ is $\frac{y}{2x}$, because $\frac{2x}{y} \times \frac{y}{2x} = 1$.

The reciprocal of $-a^3$ is $-\frac{1}{a^3}$, because $(-a^3) \times \left(-\frac{1}{a^3}\right) = 1$.

The reciprocal of $-\frac{7xy}{3t}$ is $-\frac{3t}{7xy}$, because $\left(-\frac{7xy}{3t}\right) \times \left(-\frac{3t}{7xy}\right) = 1$.

- *To form the reciprocal, interchange numerator and denominator, and do not change the sign.*

EXAMPLE

Write down the reciprocal of each algebraic fraction:

a $\frac{2a}{b}$ **b** $-\frac{2a}{b}$ **c** $\frac{6x^2}{7yz}$ **d** $-\frac{6x^2}{7yz}$

SOLUTION

a $\frac{b}{2a}$ **b** $-\frac{b}{2a}$ **c** $\frac{7yz}{6x^2}$ **d** $-\frac{7yz}{6x^2}$

THE INDEX LAWS FOR FRACTIONS

The previous algebra module *Negatives and the Index Laws in Algebra* dealt with the index laws for a product of powers of the same base, for a power of a power, and for a power of a product:

- *To multiply powers of the same base, add the indices:*

- $a^m \times a^n = a^{m+n}$.

- *To raise a power to a power, add the indices:*

- $(a^m)^n = a^{mn}$.

- *The power of a product os the product of the powers:*

- $(ab)^n = a^n b^n$.

In these identities, *a* and *b* are any numbers, and *m* and *n* are any nonzero whole numbers.

We now turn to the remaining two index laws, which involve quotients.

The index law for the power of a quotient

To see what happens when we take a power of a fraction, we simply write out the power.

$$\left(\frac{2}{5}\right)^3 = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$
$$= \frac{2^3}{5^3}$$

$$\left(\frac{a}{b}\right)^5 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$$
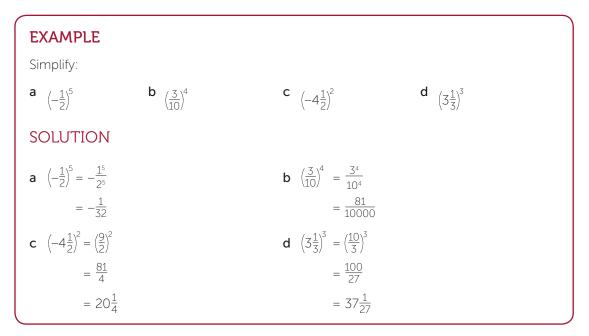$$= \frac{a^5}{b^5}$$

The general situation can thus be expressed as:

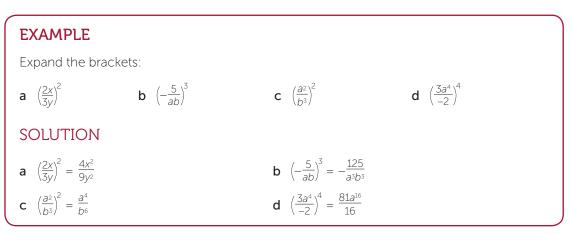- *The power of a quotient is the quotient of the powers:*

    $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ *where a and b ≠ 0 are numbers, and n is a nonzero whole number.*

The following examples illustrate the use of the law in arithmetic and algebra.

EXAMPLE

Simplify:

a $\left(-\dfrac{1}{2}\right)^5$ **b** $\left(\dfrac{3}{10}\right)^4$ **c** $\left(-4\dfrac{1}{2}\right)^2$ **d** $\left(3\dfrac{1}{3}\right)^3$

SOLUTION

a $\left(-\dfrac{1}{2}\right)^5 = -\dfrac{1^5}{2^5}$ **b** $\left(\dfrac{3}{10}\right)^4 = \dfrac{3^4}{10^4}$

     $= -\dfrac{1}{32}$ $= \dfrac{81}{10000}$

c $\left(-4\dfrac{1}{2}\right)^2 = \left(\dfrac{9}{2}\right)^2$ **d** $\left(3\dfrac{1}{3}\right)^3 = \left(\dfrac{10}{3}\right)^3$

     $= \dfrac{81}{4}$ $= \dfrac{100}{27}$

     $= 20\dfrac{1}{4}$ $= 37\dfrac{1}{27}$

When applied to an algebraic fraction, the laws previously introduced for the power of a product and for the power of a power, are often required as well. For example,

$$\left(\dfrac{2x}{3y}\right)^3 = \dfrac{8x^3}{27y^3} \qquad \text{and} \qquad \left(\dfrac{1}{5a^3}\right)^2 = \dfrac{1}{25a^6}$$

EXAMPLE

Expand the brackets:

a $\left(\dfrac{2x}{3y}\right)^2$ **b** $\left(-\dfrac{5}{ab}\right)^3$ **c** $\left(\dfrac{a^2}{b^3}\right)^2$ **d** $\left(\dfrac{3a^4}{-2}\right)^4$

SOLUTION

a $\left(\dfrac{2x}{3y}\right)^2 = \dfrac{4x^2}{9y^2}$ **b** $\left(-\dfrac{5}{ab}\right)^3 = -\dfrac{125}{a^3b^3}$

c $\left(\dfrac{a^2}{b^3}\right)^2 = \dfrac{a^4}{b^6}$ **d** $\left(\dfrac{3a^4}{-2}\right)^4 = \dfrac{81a^{16}}{16}$

The index law for the quotient of powers of the same base

The remaining index law from arithmetic deals with dividing powers of the same base. The module *Multiples, Factors and Powers* developed the law in the form, 'To divide powers of the same base, subtract the indices'. For example,

$$7^5 \div 7^3 = (7 \times 7 \times 7 \times 7 \times 7) \div (7 \times 7 \times 7) = 7^2.$$

Fractions, however, had not been introduced at that stage, so the divisor could not be a

higher power than the dividend. Now that we have fractions, we can express the law in fractional notation and do away with the restriction:

$$\frac{7^5}{7^3} = \frac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7 \times 7} \qquad\qquad \frac{7^4}{7^4} = \frac{7 \times 7 \times 7 \times 7}{7 \times 7 \times 7 \times 7} \qquad\qquad \frac{7^3}{7^5} = \frac{7 \times 7 \times 7}{7 \times 7 \times 7 \times 7 \times 7}$$

$$= 7^2 \qquad\qquad\qquad\qquad = 1 \qquad\qquad\qquad\qquad = \frac{1}{7^2}$$

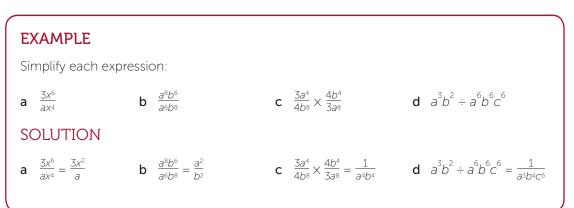The base can also be a pronumeral, and the law can now be written as:

- *To divide powers of the same base, subtract the indices.*

For example,

$$\frac{a^5}{a^3} = a^2 \qquad \text{and} \qquad \frac{a^4}{a^4} = 1 \qquad \text{and} \qquad \frac{a^3}{a^5} = \frac{1}{a^2}.$$

Later, when zero and negative indices have been introduced in the module The Index Laws, we will be able to state this law in its usual, more concise form:

$$\frac{a^m}{a^n} = a^{m-n}.$$

EXAMPLE

Simplify each expression:

a $\dfrac{3x^6}{ax^4}$ **b** $\dfrac{a^8 b^6}{a^6 b^8}$ **c** $\dfrac{3a^4}{4b^8} \times \dfrac{4b^4}{3a^8}$ **d** $a^3 b^2 \div a^6 b^6 c^6$

SOLUTION

a $\dfrac{3x^6}{ax^4} = \dfrac{3x^2}{a}$ **b** $\dfrac{a^8 b^6}{a^6 b^8} = \dfrac{a^2}{b^2}$ **c** $\dfrac{3a^4}{4b^8} \times \dfrac{4b^4}{3a^8} = \dfrac{1}{a^4 b^4}$ **d** $a^3 b^2 \div a^6 b^6 c^6 = \dfrac{1}{a^3 b^4 c^6}$

ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

When a set of algebraic fractions have a common denominator, it is straightforward to add them. As always, we collect like terms,

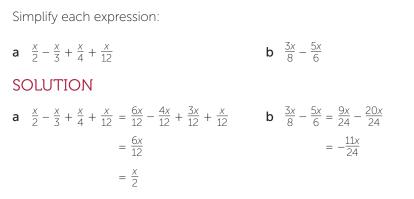$$\frac{3}{x} + \frac{a}{x} - \frac{7}{x} + \frac{4a}{x} = \frac{5a}{x} - \frac{4}{x} \quad \text{(which is perhaps written as } \frac{5a-4}{x}\text{)}$$

EXAMPLE

Simplify each expression:

a $\dfrac{5}{x} - \dfrac{a}{x} - \dfrac{4}{x}$ **b** $\dfrac{7}{2x} - \dfrac{11}{2x} - \dfrac{20}{2x}$

SOLUTION

a $\dfrac{5}{x} - \dfrac{a}{x} - \dfrac{4}{x} = \dfrac{1}{x} - \dfrac{a}{x}$, or $-\dfrac{1-a}{x}$ **b** $\dfrac{7}{2x} - \dfrac{11}{2x} - \dfrac{20}{2x} = -\dfrac{24}{2x} = -\dfrac{12}{x}$

Using a common denominator

When the fractions do not have a common denominator, we need to find the *lowest common denominator*, just as we did for arithmetic fractions. We shall restrict the discussion to numerical denominators at this stage, and leave denominators with pronumerals to the module *Special Expansions and Algebraic Fractions*.

$$\frac{x}{4} - \frac{x}{6} = \frac{3x}{12} - \frac{2x}{12}$$

$$= \frac{x}{12}.$$

EXAMPLE

Simplify each expression:

a $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} + \frac{x}{12}$ **b** $\frac{3x}{8} - \frac{5x}{6}$

SOLUTION

a $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} + \frac{x}{12} = \frac{6x}{12} - \frac{4x}{12} + \frac{3x}{12} + \frac{x}{12}$ **b** $\frac{3x}{8} - \frac{5x}{6} = \frac{9x}{24} - \frac{20x}{24}$

$$= \frac{6x}{12}$$ $$= -\frac{11x}{24}$$

$$= \frac{x}{2}$$

LINKS FORWARD

As mentioned in the Motivation section, this module is the third of four modules dealing with basic manipulation of algebraic expressions. The four modules are:

- *Algebraic Expressions*
- *Negatives and the Index Laws in Algebra*
- *Fractions and the Index Laws in Algebra (the present module)*
- *Special Expansions and Algebraic Fractions*

Algebraic fractions have been introduced in this module, but the algebra of algebraic fractions can cause considerable difficulty if it is introduced in full strength before more fundamental skills have been assimilated. Algebraic fractions will therefore be considered in more details in the fourth module *Special Expansions and Algebraic Fractions*.

The index laws will be greatly expanded in scope once negative and fractional indices are introduced in the module Indices. One piece of clumsiness in the present module concerned the quotient of the difference of powers, where we needed to write down three different examples

$$\frac{x^5}{x^3} = x^2 \qquad \text{and} \qquad \frac{x^4}{x^4} = 1 \quad \text{and} \qquad \frac{x^3}{x^5} = \frac{1}{x^2}.$$

Once the zero index and negative indices are brought into algebra, all this can be replaced

with the single law

$\frac{x^m}{x^n} = x^{m-n}$ where m and n are integers.

Square roots and cube roots can also be represented using fractional indices – for example, $\sqrt{5}$ is written as $5^{\frac{1}{2}}$ and $\sqrt[3]{5}$ is written as $5^{\frac{1}{3}}$ – and yet the five index laws still hold even when the index is any rational number. Negative and fractional indices are introduced in the module, *Indices and Logarithms*. Later in calculus, an index can be any real number.

ANSWERS TO EXERCISES

EXERCISE 1

The lowest common denominator is 60. Grouping the numbers in pairs before adding them all reduces the size of the numerator.

$$\begin{aligned}
\text{Sum} &= \left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{6}\right) \\
&= -\frac{1}{2} - \frac{1}{12} - \frac{1}{30} \\
&= -\frac{30}{60} - \frac{5}{60} - \frac{2}{60} \\
&= -\frac{37}{60}
\end{aligned}$$

EXERCISE 2

We can ignore all negative products. Since $(-0.8) \times 0.9 \times (-1.0) \times 1.1$ is less than 1, the first

product greater than 1 is $(-0.8) \times 0.9 \times (-1.0) \times 1.1 \times (-1.2) \times 1.3 \times (-1.4)$.

EXERCISE 3

One approach is to notice that when the 31 numbers are added, the first 21 numbers cancel out. The remaining 10 numbers can then be added by grouping them in pairs:

$1 + 0.9 + 0.8 + \ldots + (-2) = -1.1 - 1.2 - 1.3 - \ldots -2.$

$$\begin{aligned}
&= (-1.1 - 2) + (-1.2 - 1.9) + (-1.3 - 1.8) + (-1.4 - 1.7) + (-1.5 - 1.6) \\
&= -3.1 \times 5 \\
&= -15.5.
\end{aligned}$$

Hence mean $= \frac{-15.5}{31} = \frac{-31}{62} = -0.5.$

A better approach is realise that -0.5 is the middle number, and that the other 30 numbers fall into pairs equally spaced one to the left and one to the right of -0.5. Hence the mean is -0.5.

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- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

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