AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

The Improving Mathematics Education in Schools (TIMES) Project

# AREA, VOLUME AND SURFACE AREA

## A guide for teachers - Years 8–10

MEASUREMENT AND GEOMETRY : Module 11

June 2011

YEARS 8 10

Area, Volume and Surface Area

(Measurement and Geometry: Module 11)

For teachers of Primary and Secondary Mathematics

510

Cover design, Layout design and Typesetting by Claire Ho

The Improving Mathematics Education in Schools (TIMES) Project 2009-2011 was funded by the Australian Government Department of Education, Employment and Workplace Relations.

The Improving Mathematics Education in Schools (TIMES) Project

# AREA, VOLUME AND SURFACE AREA

## A guide for teachers - Years 8–10

MEASUREMENT AND GEOMETRY : Module 11

June 2011

Peter Brown
Michael Evans
David Hunt
Janine McIntosh
Bill Pender
Jacqui Ramagge

YEARS 8 10

# AREA, VOLUME AND SURFACE AREA

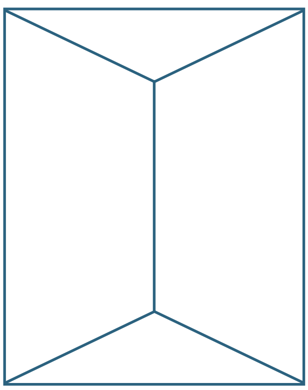
## ASSUMED KNOWLEDGE

- Knowledge of the areas of rectangles, triangles, circles and composite figures.
- The definitions of a parallelogram and a rhombus.
- Familiarity with the basic properties of parallel lines.
- Familiarity with the volume of a rectangular prism.
- Basic knowledge of congruence and similarity.
- Since some formulas will be involved, the students will need some experience with substitution and also with the distributive law.

## MOTIVATION

The area of a plane figure is a measure of the amount of space inside it. Calculating areas is an important skill used by many people in their daily work. Builders and tradespeople often need to work out the areas and dimensions of the structures they are building, and so do architects, designers and engineers.

While rectangles, squares and triangles appear commonly in the world around us, other shapes such as the parallelogram, the rhombus and the trapezium are also found. Consider, for example, this aerial view of a roof.



The view consists of two trapezia and two triangles.

Similarly, solids other than the rectangular prism frequently occur. The Toblerone © packet (with the base at the end) is an example of a triangular prism, while an oil drum has the shape of a cylinder. It is important to be able to find the volume of such solids.

Medical specialists measure such things as blood flow rate (which is done using the velocity of the fluid and the area of the cross-section of flow) as well as the size of tumours and growths.

In physics the area under a velocity-time graph gives the distance travelled.

In this module we will use simple ideas to produce a number of fundamental formulas for areas and volumes. Students should understand why the formulas are true and commit them to memory.

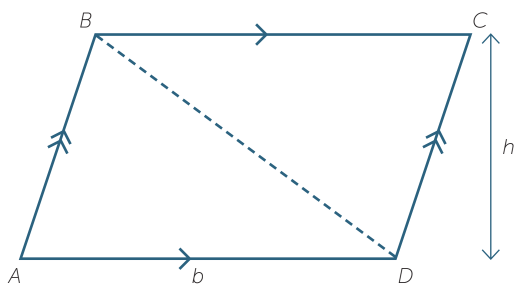
# CONTENT

## AREA OF A PARALLELOGRAM

A parallelogram is a quadrilateral with opposite sides equal and parallel.

We can easily find the area of a parallelogram, given its base $b$ and its height $h$.
In the diagram below, we draw in the diagonal $BD$ and divide the figure into two triangles, each with base length $b$ and height $h$. Since the area of each triangle is $\frac{1}{2}bh$ the total area $A$ is given by

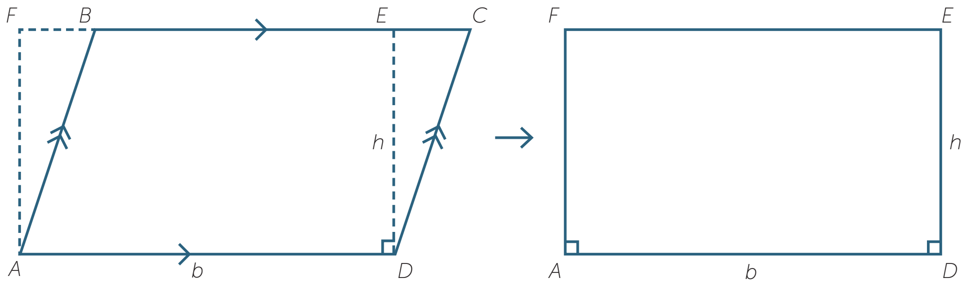
$$A = bh.$$



Note that the two triangles in the diagram not only have the same area, they are actually congruent triangles.

Some teachers may prefer to establish the area formula for a parallelogram **without** using the area of a triangle formula so that they can develop the area of a triangle using the area formula for a parallelogram.

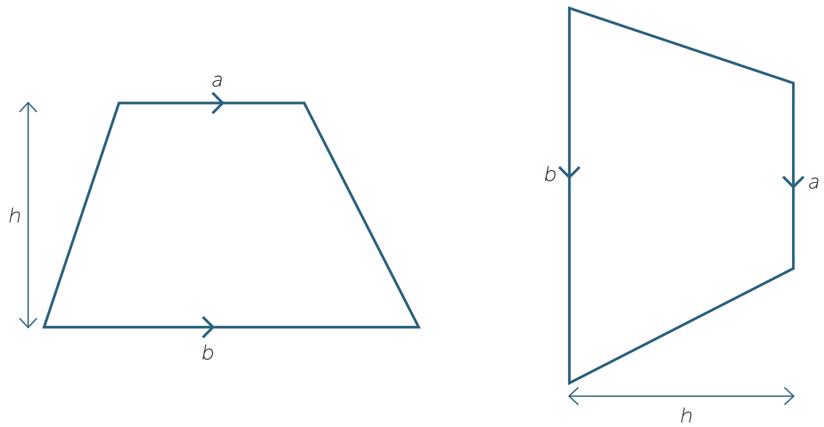
This can be done by showing that the triangle on the right in the left hand diagram below can be positioned on the left to form a rectangle whose base and height are the same as those of the parallelogram, so again, the area is equal to *bh*.



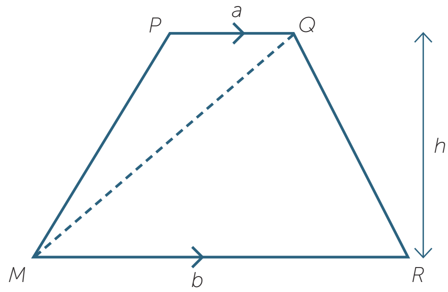

## AREA OF A TRAPEZIUM

A trapezium is a quadrilateral that has one pair of opposite sides parallel. (The name comes from the Greek word for *table*.)

We can find the area of a trapezium if we know the lengths of the two parallel sides and the perpendicular distance between these two sides.




As we did with the parallelogram, we draw one of the diagonals. We then have two triangles, both with height $h$, and one with base $a$, one with base $b$.




Thus the area $A$ of the trapezium is

$$A = \tfrac{1}{2}ah + \tfrac{1}{2}bh$$

$$= \tfrac{1}{2}h(a + b).$$

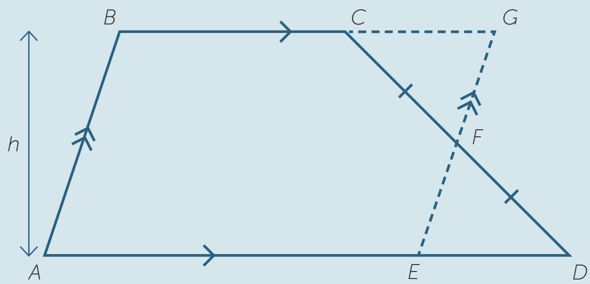
So the formula for the area of a trapezium with parallel sides $a$ and $b$ and the perpendicular distance $h$ between them is

$$A = \tfrac{1}{2}h(a + b).$$

This can be thought of as 'the height times the average of the parallel sides'.

## EXERCISE 1

Here is another derivation of the area formula for a trapezium. Suppose *ABCD* is a trapezium.




Take $F$ to be the midpoint of $CD$ and draw through it the line $EG$ parallel to $AB$.

**a** Explain why triangles $CFG$ and $DFE$ are congruent.

**b** What does this tell us about $CG$ and $ED$?

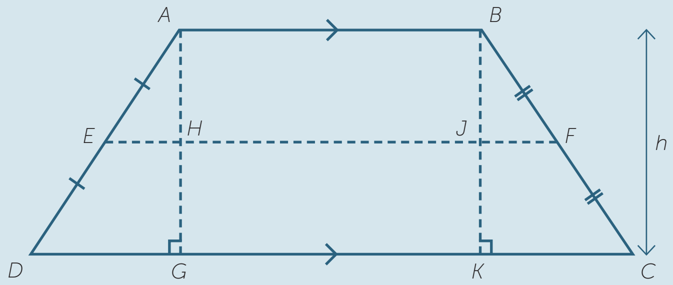
**c** Explain why $AE = \tfrac{1}{2}(BC + AD)$.

**d** Use the formula for area of a parallelogram to derive the formula for the area of the trapezium.

## EXERCISE 2

(This exercise involves the use of similar triangles).

In the diagram, *ABCD* is a trapezium with *AB* parallel to *DC* and distance $h$ between them. The points *E* and *F* are the midpoints of *AD* and *BC* respectively. *AG* is perpendicular to *DC* at *G* and meets *EF* at *H*. Let $a = AB$, $b = DC$ and $\ell = EF$.




**a** Show that *EF* is parallel to *DC*.

**b** By considering triangles *AEH* and *ADG* show that $AH = HG = \frac{h}{2}$.

**c** By comparing the areas of the three trapezia thus formed, or otherwise, show that the area of the trapezium *ABCD* is equal to $h\ell$.

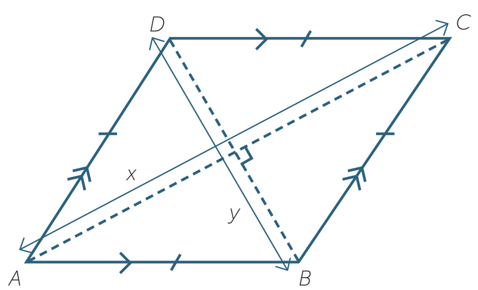
## AREA OF A RHOMBUS AND A KITE

A rhombus is a quadrilateral with all sides equal. In the module, *Rhombuses, Kites and Trapezia* using simple geometric arguments, we showed

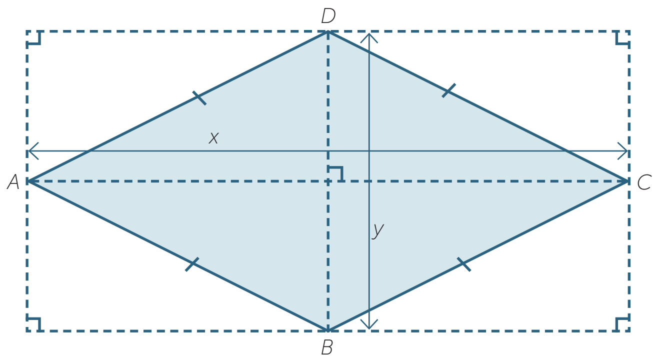
- the opposite sides are parallel
- the diagonals bisect each other at right angles

Thus a rhombus is a parallelogram and we can calculate the area of a rhombus using the formula for the area of a parallelogram.

Now take a rhombus with diagonals of length $x$ and $y$.



Standing the rhombus on one corner, we see that the two diagonals cut the rhombus into four right-angled triangles, which can be completed to form four rectangles inside a larger rectangle.




Since the eight triangles have the same area, (indeed, they are all congruent), the area of the rhombus is one half the area of the large rectangle, which is $xy$.

Hence, if $x$ and $y$ are the lengths of the diagonals of a rhombus, then

$$\text{Area of a rhombus} = \frac{1}{2}xy.$$

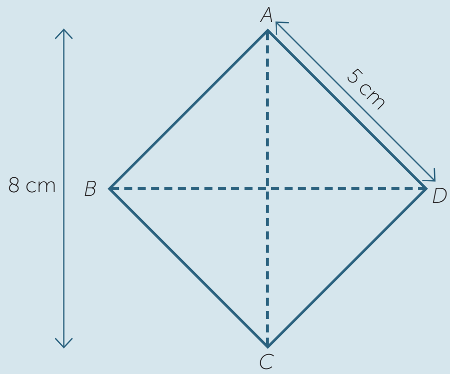
The area of a rhombus is half the product of the lengths of the diagonals.

## EXERCISE 3

Suppose *ABCD* is a rhombus with one diagonal 8 cm and one side 5 cm as shown.

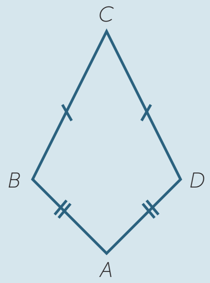
**a** Use Pythagoras' theorem to find the length of the other diagonal.

**b** Hence find the area of the rhombus.



## EXERCISE 4

A kite is a quadrilateral that has two pairs of adjacent sides equal.

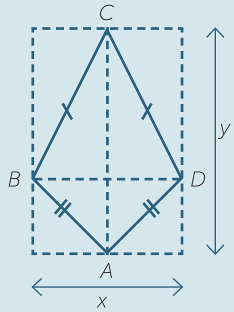



a Use congruence and the two isoceles triangles to show that the diagonals of a kite are perpendicular.

b Clearly we can complete the kite to form a rectangle whose area is twice that of the kite, so

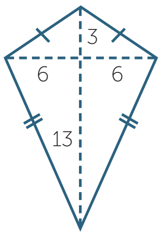
Area of a kite = $\frac{1}{2}xy$,

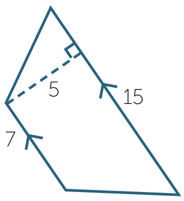
where $x$ and $y$ are the lengths of diagonals of the kite.

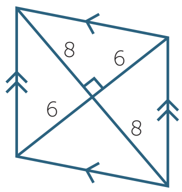



### EXAMPLE

Find the area of each figure: (All measurements are in centimetres.)

a



b



c



### SOLUTIONS

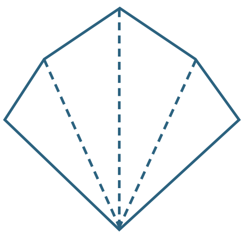
a Area = $\frac{1}{2} \times (13 + 3) \times (6 + 6) = 96$ cm$^2$.

b Area = $\frac{1}{2} \times 5 \times (7 + 15) = 55$ cm$^2$.

c Area = $\frac{1}{2} \times (8 + 8) \times (6 + 6) = 96$ cm$^2$.

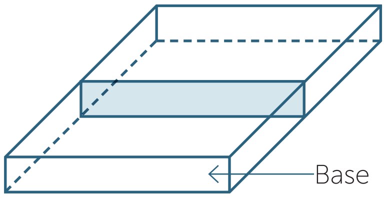
## AREA OF POLYGONS

Any polygon may be dissected into triangles.
Hence the area of any polygon is defined and can be calculated by calculating the area of each triangle.



## VOLUME OF A PRISM

A **polyhedron** is a solid bounded by polygons. A **right prism** is a polyhedron that has two congruent and parallel faces (called the base and top) and all its remaining faces are rectangles. This means that when a right prism is stood on its base, all the walls are vertical rectangles. We will generally say 'prism' when we really mean 'right prism'. A prism has **uniform cross-section**. This means that when you take slices through the solid parallel to the base you get polygons congruent to the base. So the area of each slice is always the same. In a rectangular prism, the cross-section is always a rectangle.

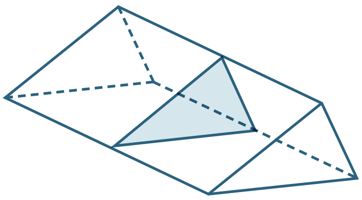



In the module *Introduction to Measurement* we saw that the volume of a rectangular prism is given by the area of the base times the height, or

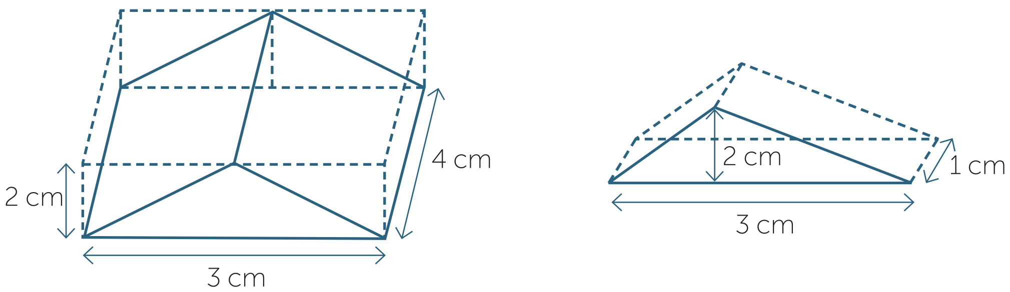
Volume = $lwh$, where $l$ and $w$ are the length and width of the prism and $h$ is the height.

## TRIANGULAR PRISMS

In a triangular prism, each cross-section parallel to the triangular base is a triangle congruent to the base.



Suppose we have a triangular prism whose length is 4 cm as shown in the diagram. We can cut the prism into layers, each of length of 1 cm.




We saw earlier that we can complete an acute-angled triangle to form a rectangle with twice the area.

Similarly we can complete the triangular prism to form a rectangular prism. The volume of each of the 1 cm layers is half the volume of the corresponding rectangular prism, i.e.

$$\text{Volume of each layer} = \tfrac{1}{2} \times 3 \times 2 \text{ cm}^3.$$

Hence the volume of the triangular prism $= \tfrac{1}{2} \times 3 \times 2 \times 4$

$= 12 \text{ cm}^3$.

Thus the volume of a triangular prism is given by

$$\text{Volume} = \text{area of triangular cross-section} \times \text{perpendicular height} = Ah.$$

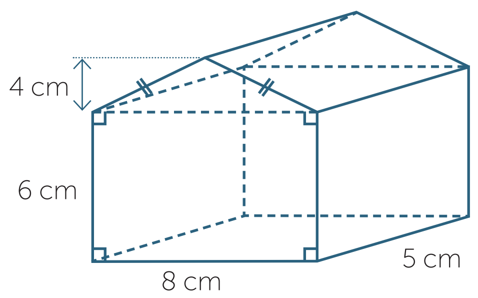
Since any polygon can be dissected into triangles, the volume of any prism with polygonal base is the area $A$ of the polygonal base times the height $h$, that is

$$\text{Volume} = Ah$$

where $A$ is the area of the polygonal base and $h$ is the height when the prism is sitting on its base.

## EXAMPLE

Find the volume of the prism shown in the diagram.




## SOLUTION

The cross-section is the front face of the prism, and consists of a triangle and a rectangle.

$$A = \left(\tfrac{1}{2} \times 8 \times 4\right) + (8 \times 6)$$

$$= 64 \text{ cm}^2.$$

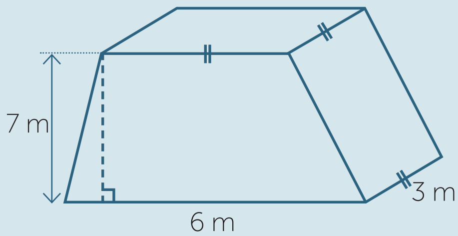
$$\text{Volume} = Ah$$

$$= 64 \times 5$$

$$= 320 \text{ cm}^3.$$

## EXERCISE 5

A large pedestal is in the shape of a prism whose front face is a trapezium.



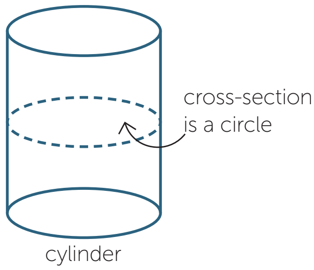

a Find the area of the front face.

b Find the volume of the pedestal.

## VOLUME OF A CYLINDER

Cylinders are ubiquitous in everyday life. For example tinned food normally comes in a can whose shape is a cylinder.

If we slice a cylinder parallel to its base, then each cross-section is a circle of the same size as the base.




Thus a cylinder has the same basic property as a prism and we will take the formula for the volume of a cylinder to be the area of the circular base times the height. We cannot prove this formula rigorously at this stage, because the proof involves constructing the cylinder as a limit of prisms.

If the base circle of the cylinder has radius $r$, then we know that the area of the circle is $A = \pi r^2$. If the height of the cylinder is $h$, then its volume is

$$\text{Volume} = \pi r^2 \times h = \pi r^2 h.$$

### EXAMPLE

For a cylinder with radius 7 cm and height 3 cm, find:

a the exact volume, in terms of $\pi$.

b an approximate value for the volume, using $\pi \approx \frac{22}{7}$.

### SOLUTION

a $V = \pi r^2 h$

$= \pi \times 49 \times 3$

$= 147\pi$ cm$^3$

a $V = \pi r^2 h$

$\approx \frac{22}{7} \times 49 \times 3$

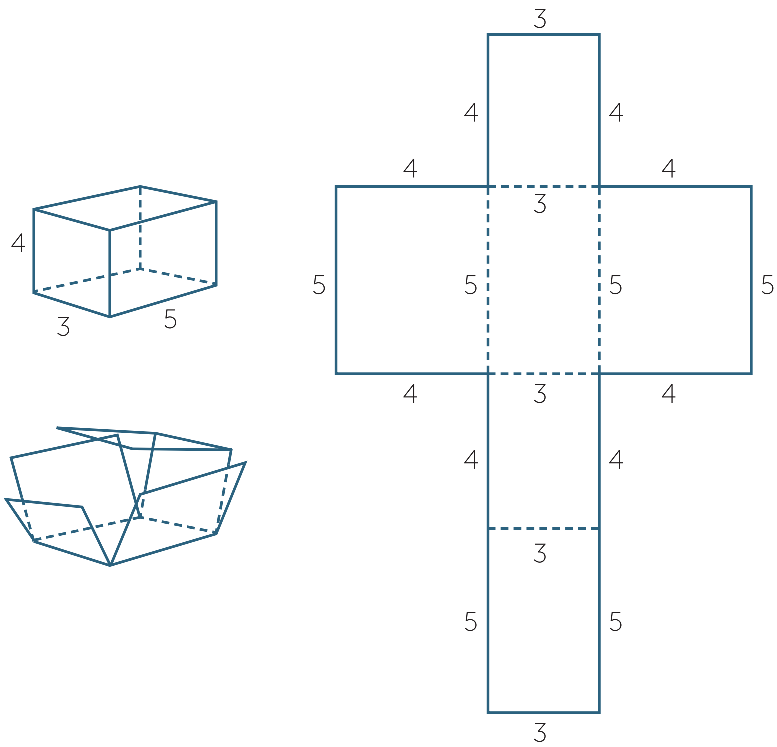
$= 462$ cm$^3$.

## EXERCISE 6

A thermos flask of height 30 cm is in the shape of two cylinders, one inside the other. It has an inner radius of 8 cm and an outer radius of 10 cm. What is the volume between the two cylinders?

## SURFACE AREA OF A PRISM

Suppose we take a rectangular prism whose dimensions are 3 cm by 4 cm by 5 cm and open it out as shown below.




We can find the area of the flattened rectangular prism by adding up the areas of the six rectangles. There are three pairs of equal rectangles, so the total area is

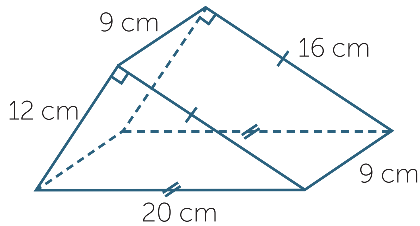
$A = 2 \times (3 \times 4 + 3 \times 5 + 4 \times 5) = 94\ \text{cm}^2$.

This is called the **surface area** of the prism.

Thus the surface area of a prism is the sum of the areas of its faces. Indeed, the surface area of a polyhedron is also the sum of the areas of all its faces.

## EXAMPLE

Find the surface area of the triangular prism shown opposite.




## SOLUTION

Area of front = $\frac{1}{2} \times 12 \times 16 = 96$ cm$^2$.

Area of back= 96 cm$^2$.

Area of the three rectanglar faces = $(9 \times 20) + (9 \times 12) + (9 \times 16)$

= 432 cm$^2$.

Total surface area = 96 + 96 + 432

= 624 cm$^2$.

## EDGE LENGTH

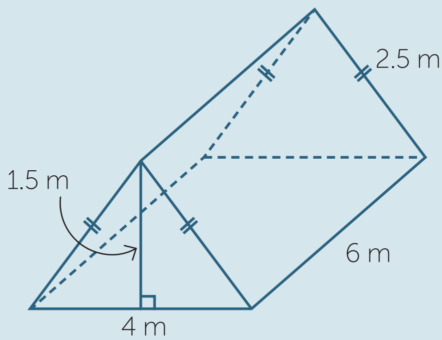
The **edge length** of a prism is the sum of the lengths of all its edges.

## EXERCISE 7

Find the total edge length of the prism in the above example.

## EXERCISE 8

A tent made from calico, including the ground sheet, is in the shape of a triangular prism, with dimensions as shown.
How much calico is needed to make the tent?




# LINKS FORWARD

## AREAS

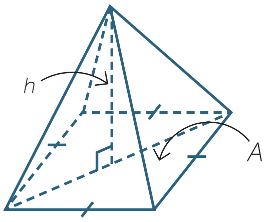
We can now find the areas of the basic figures of geometry. We have also seen, in the module on circles, that the area of a circle is given by $A = \pi r^2$, where $r$ is the radius. To make sense of the area of a figure that is not bounded either by straight lines or circular arcs, we need integral calculus. While these ideas go back to Archimedes and Eudoxus, the systematic development of integral calculus is due to Newton and Leibniz.

We can use trigonometry, to find the areas of various figures given enough information about their sides and angles.

## VOLUMES: PYRAMIDS AND PRISMS

It can be shown that the volume of a square pyramid is one third of the volume of the corresponding right prism with the same height and base.

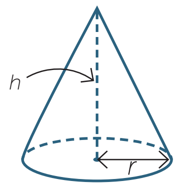
Volume of a pyramid = $\frac{1}{3}Ah$,
where $A$ is the area of the base and $h$ is the perpendicular height measured from the base.




This formula holds for pyramids with a polygonal base with area $A$.

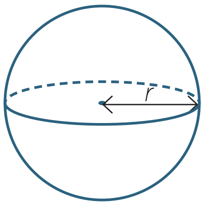
The cross-sections of a cone (or sphere) are circles but the radii of the cross-sections differ. The volume of a cone is one third of the volume of the corresponding cylinder with the same height and radius.

Volume of a cone = $\frac{1}{3}\pi r^2 h$,
where $r$ is the radius of the base and $h$ is the height.




Finally, the volume of a sphere is given by

Volume of a sphere = $\frac{4}{3}\pi r^3$,
where $r$ is the radius of the sphere.




This completes the volume formulas for the basic solids. Solids with irregular boundaries can be dealt with using integral calculus. These are all treated in the module, *Cones, Pyramids and Spheres*.

## SURFACE AREA

In the same way that we 'cut open' a prism to find the surface area, we can 'cut open' a cylinder of radius $r$ and height $h$ to show that the area of the curved surface is $2\pi rh$. Adding in the two circular ends, we arrive at the formula $A = 2\pi rh + 2\pi r^2$ for the total surface area of a cylinder. The surface area formula for a cone is $A = \pi r^2 + \pi rl$, where $r$ is the radius and $l$ is the slant height. Finally, the surface area of a sphere is given by $A = 4\pi r^2$, where $r$ is the radius of the sphere.

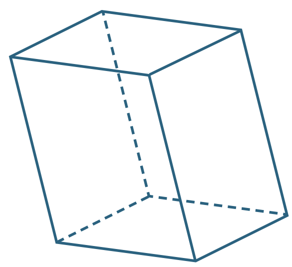
# HISTORY AND APPLICATIONS

Many of the names of the figures and solids whose area and volume we have found come from the Greek. For example, *trapezium* (despite the Latin ending) comes from the Greek word for *table,* while *prism* is derived from a Greek word meaning *to saw* (since the cross-sections, or cuts, are congruent), also the word *cylinder* is from a Greek word meaning *to roll*. The ancient Greeks were the first to systematically investigate the areas and volumes of plane figures and solids.

During the Hellenistic Period, the great mathematician Archimedes (c. 287 – 212 BC) approximated the area of a circle using inscribed polygons and found very good approximations to $\pi$. He also derived the formulas for the volume and surface area of the sphere. Archimedes developed a technique to find areas and volumes called 'the method of exhaustion' that came close to the ideas used in modern calculus.

Prior to the development of the integral calculus, which took areas and volumes to a new level of abstraction, the Italian mathematician Bonaventura Francesco Cavalieri (1598-1647) developed a result known as *Cavalieri's Principle* which states that two objects have the same volume if the areas of their corresponding cross-sections are equal in all cases. (The same principle had been previously discovered by Zu Gengzhi (480–525) in China.) A clever use of this method shows that the volume of a hemisphere radius $r$ is the same as the volume of the solid obtained by removing a cone of radius $r$ and height $r$ from a cylinder of the same height and radius, thus showing that the volume of the hemisphere is $\frac{2}{3}\pi r^3$.

Cavalieri's principle can be used to find the volume of oblique solids (as opposed to right solids). Thus, an oblique prism has parallel horizontal base and top but the sides are not vertical. Such a solid is called a **parallelepiped** (another Greek word meaning *parallel planes.*)



Using Cavalieri's principle, it can shown that the volume formula is the same as that for a prism, namely:

$$\text{Volume} = \text{area of base} \times \text{perpendicular height}.$$

The next big advance came with integral calculus, when sense could be made of the concept of *area under a curve* using the ideas of a limit. Although much progress had been made on this by Fermat and Descartes, it was the (independent) work of Newton and Leibniz that led to the modern theory of integration.

There are approximate methods for finding the area of a figure with an irregular boundary. One quite accurate one is called *Simpson's Rule*, which was, in fact, known by Cavalieri, rediscovered by Gregory (1638-1675), and attributed to Thomas Simpson (1710-1761). This rule enables us to find an approximate value of the area of an irregular figure by taking measurements across the figure at various points along some axis. It is used today by cardiologists in measuring, for example, the right ventricular (**RV**) volume relating to blood flow in the heart.

# ANSWERS TO EXERCISES

## EXERCISE 1

a $CF = DF$ (F is the midpoint of CD)

$\angle CFG = \angle DFE$ (vertically opposite angles)

$\angle GCF = \angle EDF$ (alternate angles)

Triangle *CFG* is congruent to triangle *DFE* (SAS)

b $CG = ED$ (matching sides of congruent triangles)

c $2AE = AE + BG = AD - ED + CG + BG$

$= AD + BG$

$AE = \frac{1}{2}(AD + BG)$

d Area of trapezium = area of parallelogram

$= AE \times h$

$= \frac{1}{2}(AD + BG) \times h$

## EXERCISE 2

a Slide triangles *ADG* and *BCK* together to form triangle ACD (B and A are coincident). *E* and *F* are midpoints of *AC* and *AD* respectively. Triangle *AFE* is similar to triangle *ACD* and thus *EF* is parallel to *DC* (corresponding angles are equal).

b Triangle AEH is similar to triangle *ADG* (AAA)

$AH = HG = \frac{h}{2}$

c Area = $\frac{h}{2}(AB + CD)$

$= \frac{h}{2}(2(HJ + EH + JF))$

$= hl$

## EXERCISE 3

a 6 cm

b 24 $cm^2$

## EXERCISE 4

a Triangle *CBA* is congruent to *CDA* (SSS)

Triangle BCE is congruent to triangle *DCE* (SAS)

$\angle CEB = \angle CED = 90°$

b Area of rectangle = $xy$.

## EXERCISE 5

a 31.5 $m^2$

b 94.5 $m^3$

## EXERCISE 6

$1080\pi$ $cm^3$

## EXERCISE 7

123 cm

## EXERCISE 8

60 $m^2$

INTERNATIONAL CENTRE
OF EXCELLENCE FOR
EDUCATION IN
MATHEMATICS

The aim of the International Centre of Excellence for Education in Mathematics (ICE-EM) is to strengthen education in the mathematical sciences at all levels-from school to advanced research and contemporary applications in industry and commerce.

ICE-EM is the education division of the Australian Mathematical Sciences Institute, a consortium of 27 university mathematics departments, CSIRO Mathematical and Information Sciences, the Australian Bureau of Statistics, the Australian Mathematical Society and the Australian Mathematics Trust.

AUSTRALIAN MATHEMATICS TRUST

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

The ICE-EM modules are part of *The Improving Mathematics Education in Schools* (TIMES) *Project.*

The modules are organised under the strand titles of the Australian Curriculum:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

www.amsi.org.au