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The Improving Mathematics Education in Schools (TIMES) Project

QUADRATIC EQUATIONS

A guide for teachers - Years 9–10

NUMBER AND ALGEBRA ■
Module 34

June 2011

YEARS
9
10

Quadratic Equations

(Number and Algebra : Module 34)

For teachers of Primary and Secondary Mathematics

510

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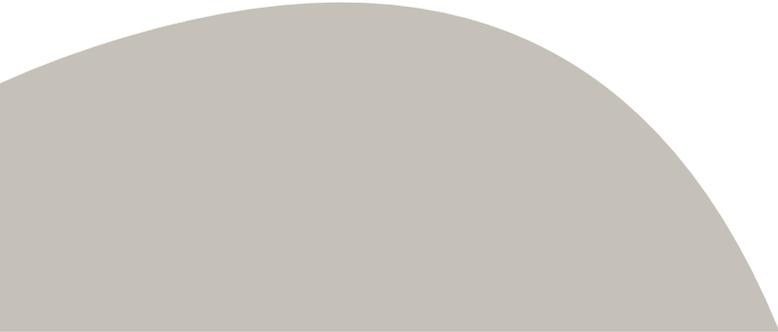
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NUMBER AND ALGEBRA •
Module 34

June 2011

Peter Brown
Michael Evans
David Hunt
Janine McIntosh
Bill Pender
Jacqui Ramagge

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QUADRATIC EQUATIONS

ASSUMED KNOWLEDGE

- Facility with solving linear equations
- All of the content of the module, *Factorisation*.
- Facility with arithmetic of positive and negative numbers

MOTIVATION

In the module, *Linear equations* we saw how to solve various types of linear equations. Such equations arise very naturally when solving elementary everyday problems.

A **linear equation** involves the unknown quantity occurring to the first power, thus, for example,

$$2x - 7 = 9$$

$$3(x + 2) - 5(x - 8) = 16$$

$$\frac{5x}{3} = 8$$

are all examples of linear equations.

Roughly speaking, quadratic equations involve the square of the unknown. Thus, for example, $2x^2 - 3 = 9$, $x^2 - 5x + 6 = 0$, and $\frac{6x^2}{5} - 4x = 2x - 1$ are all examples of quadratic equations. The equation $\frac{x+1}{x+5} = \frac{2x+5}{3x+7}$ is also a quadratic equation.

The essential idea for solving a linear equation is to isolate the unknown. We keep rearranging the equation so that all the terms involving the unknown are on one side of the equation and all the other terms to the other side. The rearrangements we used for linear equations are helpful but they are not sufficient to solve a quadratic equation. In this module we will develop a number of methods of dealing with these important types of equations.

While quadratic equations do not arise so obviously in everyday life, they are equally important and will frequently turn up in many areas of mathematics when more sophisticated problems are encountered. Both in senior mathematics and in tertiary and engineering mathematics, students will need to be able to solve quadratic equations with confidence and speed. Surprisingly, when mathematics is employed to solve complicated and important real world problems, quadratic equations very often make an appearance as part of the overall solution.

The history of quadratics will be further explored in the History section, but we note here that these types of equations were solved by both the Babylonians and Egyptians at a very early stage of world history. The techniques of solution were further refined by the Greeks, the Arabs and Indians, and finally a complete and coherent treatment was completed once the notion of complex numbers was understood. Thus quadratic equations have been central to the history and applications of mathematics for a very long time.

CONTENT

QUADRATIC EQUATIONS

A **quadratic** is an expression of the form $ax^2 + bx + c$, where a , b and c are given numbers and $a \neq 0$.

The **standard form** of a **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are given numbers and } a \neq 0.$$

We seek to find the value(s) of which make the statement true, or to show that there are no such values.

Thus, for example, the values $x = 3$ and $x = 2$ satisfy the equation, $x^2 - 5x + 6 = 0$. This is easily checked by substitution.

These values are called the **solutions** of the equation. Linear equations that are written in the standard form, $ax + b = 0$, $a \neq 0$, have one solution. Quadratic equations may have no solutions, one solution, or, as in the above example, two solutions.

There are two special types of quadratic equations, that are best dealt with separately.

Quadratic equations with no term in x

When there is no term in x we can move the constant to the other side.

EXAMPLE

Solve $x^2 - 9 = 0$.

SOLUTION

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = 3 \text{ or } x = -3.$$

(Note that this equation can also be solved by factoring using the difference of squares identity. While this is a valid approach, it makes a simple problem appear complicated, which is, in general, not a good way to do mathematics.)

Quadratic equations with no constant term

EXAMPLE

Solve $x^2 - 9x = 0$.

SOLUTION

In this case, we can write

$$x^2 - 9x = 0$$

$$x(x - 9) = 0$$

Since the product of the two factors is 0, one or both of the factors is zero, $x(x - 9) = 0$.

$$\text{so } x = 0 \quad \text{or} \quad x - 9 = 0$$

Hence the two solutions are $x = 0$, or $x = 9$.

These two methods work just as well when the coefficient of x^2 is not one.

The two previous examples were relatively easy since in the first case it was easy to isolate the unknown while in the second, a common factor enabled the left-hand side to be easily factored.

SOLVING QUADRATIC EQUATIONS WITH THREE TERMS

We will now deal with the equation $ax^2 + bx + c = 0$ in which neither a nor b nor c are zero.

There are three basic methods of solving such quadratic equations:

- by factoring
- by completing the square
- by the quadratic formula

Each method is important and needs to be mastered. Different situations will require different approaches, and while the last two methods always work, the method of factoring is very quick and accurate, provided the equation has rational solutions.

Solving quadratic equations by factoring

The method of solving quadratic equations by factoring rests on the simple fact, used in example (2) above, that if we obtain zero as the product of two numbers then at least one of the numbers must be zero.

That is, if $AB = 0$ then $A = 0$ or $B = 0$

In the module, *Factorisation*, we first saw how to factor monic quadratics, then we learnt how to factorise non-monic quadratics.

To factor $x^2 + bx + c$ we try to find two numbers whose sum is b and whose product is c . We now apply this idea to solving quadratic equations.

EXAMPLE

Solve $x^2 - 7x + 12 = 0$.

SOLUTION

We factor the left-hand side by finding two numbers whose product is 12 and whose sum is -7 . Clearly, -4 , -3 are the desired numbers. We can then factor as:

$$x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

Since the product of the two factors is zero, one of the factors is zero.

Thus $x - 4 = 0$, or $x - 3 = 0$

so $x = 4$, or $x = 3$

The same method can also be applied to non-monic quadratic equations. A **non-monic quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a and c are given numbers, and $a \neq 1$ or 0 . This is the general case.

Thus $2x^2 + 5x + 3 = 0$ is an example of a non-monic quadratic equation.

EXAMPLE

Solve the equation $2x^2 + 5x + 3 = 0$.

SOLUTION

Using the factoring method from the module *Factorisation*, we multiply 2 and 3 to give 6 and find two numbers that multiply to give 6 and add to give 5. The desired numbers are 2 and 3. We use these numbers to split the middle term and factor in pairs.

$$2x^2 + 5x + 3 = 0$$

$$2x^2 + 2x + 3x + 3 = 0 \quad (\text{split the middle term})$$

$$2x(x + 1) + 3(x + 1) = 0$$

$$(x + 1)(2x + 3) = 0$$

We can now equate each factor to zero and obtain

$$x + 1 = 0, \text{ or } 2x + 3 = 0$$

$$x = -1, \text{ or } x = -\frac{3}{2}.$$

As was pointed out in the module, *Factorisation*, the order in which the middle terms are written does not affect the final factorisation, and hence does not effect the solutions of the quadratic.

EXERCISE 1

Solve the equations.

a $4x^2 - 20 = 0$ **b** $x^2 - x - 12 = 0$ **c** $3x^2 + 2x - 8 = 0$

Note: While the values of which satisfy $2x^2 + 5x + 3 = 0$ are $x = -1$ or $x = -\frac{3}{2}$, we often say that the solution of $2x^2 + 5x + 3 = 0$ are $x = -1$ and $x = -\frac{3}{2}$.

Common simplifications of quadratics

It is often convenient to simplify a quadratic equation before any method of solution is applied.

- If the coefficient of x^2 is negative multiply through by -1 .

$$-x^2 + 5x - 6 = 0 \text{ becomes } x^2 - 5x + 6 = 0$$

- 'multiply out fractions'

$$\frac{x^2}{2} - \frac{5x}{2} + 3 \text{ becomes } x^2 - 5x + 6 = 0$$

- If there is a common factor divide through by it.

$$3x^2 - 15x + 18 = 0 \text{ becomes } x^2 - 5x + 6 = 0$$

Equations that can be rearranged to be a quadratic equation in standard form

The **standard form** for a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

We may however, be given a quadratic equation that is not in this form and so our first step is to re-write the equation into this standard form.

EXAMPLE

Solve $\frac{x-2}{3} = \frac{5}{x}$.

SOLUTION

$$\frac{x-2}{3} = \frac{5}{x}$$

$$x(x-2) = 3 \times 5 \quad (\text{cross-multiplication})$$

$$x^2 - 2x = 15$$

$$x^2 - 2x - 15 = 0 \quad (\text{Rearrange})$$

$$(x+3)(x-5) = 0$$

$$x+3 = 0 \text{ or } x-5 = 0$$

$$x = -3 \text{ or } x = 5$$

EXERCISE 2

Solve $\frac{1}{3-x} - \frac{4}{5} = \frac{1}{9-2x}$.

Applications**EXAMPLE**

A rectangle has one side 3cm longer than the other. The rectangle has area 28cm^2 . What is the length of the shorter side?

SOLUTION

Let x cm be the length of the shorter side. The other side has length $(x+3)$ cm.

$$\text{Area} = x(x+3) = 28\text{cm}^2$$

$$x^2 + 3x - 28 = 0$$

$$(x-4)(x+7) = 0$$

$$x-4 = 0 \text{ or } x+7 = 0$$

$$x = 4 \text{ or } x = -7$$

Since length must be positive, the solution to the problem is $x = 4$. The shorter side has length 4cm.

EXERCISE 3

Each number in the sequence 5, 9, 13, 17, ... is obtained by adding 4 to the previous number. The sum S of the first n numbers in the sequence is given by $S = 2n^2 + 3n$.

How many numbers must be added to make the sum equal to 152?

Completing the square

The quadratic equations encountered so far, had one or two solutions that were rational. There are many quadratics that have irrational solutions, or in some cases no real solutions at all.

For example, it is not easy at all to see how to factor the quadratic $x^2 - 5x - 3 = 0$. Indeed it has no rational solutions. We will see shortly that the solutions are $x = \frac{5 + \sqrt{37}}{2}$ and $x = \frac{5 - \sqrt{37}}{2}$.

To deal with more general quadratics, we employ a technique known as *completing the square*. Historically, this was the most commonly used method of solution.

The technique of completing the square is used not only for solving quadratic equations, but also in further mathematics for such things as:

- finding the centre and radius of a circle – given its algebraic equation,
- finding the maximum or minimum of a quadratic function,
- finding the axis of symmetry of a parabola,
- putting integrals into standard form in calculus.

This is an important technique that will appear in other settings and so is a basic skill that students who intend to proceed to senior mathematics need to master.

In the early stages, students will need to be told when to apply which method. With experience, they will use completing the square whenever they cannot see how to apply the factor method shown above. The method of completing the square works in every case, including the situation in which the factor method applies.

In earlier modules we have seen the two identities referred to as **perfect squares**:

$$a^2 + 2ab + b^2 = (a + b)^2 \text{ and } a^2 - 2ab + b^2 = (a - b)^2.$$

Thus, for example, $x^2 + 6x + 9 = (x + 3)^2$ and $x^2 - 4x + 4 = (x - 2)^2$.

Notice that in the quadratics above, the constant term in each case is the square of **half** the coefficient of x . The method of completing the square simply involves adding in a number make a given quadratic expression into a perfect square.

Completing the square in an expression

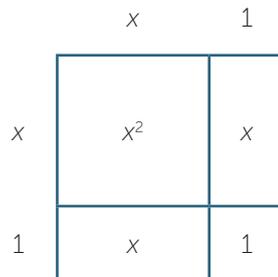
We begin by completing the square on the quadratic expression $x^2 + 2x - 6$.

We focus on $x^2 + 2x$ and ask: What number must be added to $x^2 + 2x$ to make the expression into a perfect square?

The key step is to take half the coefficient of x and square it. The same rule applies when the coefficient of x is negative.

In this case, the answer is 1, since $x^2 + 2x + 1 = (x + 1)^2$.

This can be seen diagrammatically, where a square is added to 'complete the square'.



We can then write:

$$\begin{aligned} x^2 + 2x - 6 &= (x^2 + 2x + 1) - 1 - 6 && \text{(add and subtract 1)} \\ &= (x + 1)^2 - 7. \end{aligned}$$

In the case when the coefficient of x is odd, we will need to use fractions. For example, to complete the square on $x^2 - 3x + 1$, we note that half of -3 is $-\frac{3}{2}$ and $\frac{9}{4}$. Hence we have $x^2 - 3x + 1 = (x^2 - 3x + \frac{9}{4}) - \frac{9}{4} + 1 = (x - \frac{3}{2})^2 - \frac{5}{4}$.

Solving quadratic equations by completing the square

We can now apply the method of completing the square to solve quadratic equations. To complete the square for an equation, we will add in a factor on each side to produce a square.

EXAMPLE

Solve $x^2 + 2x - 6 = 0$.

SOLUTION

It is easiest to move the constant term onto the other side first and then complete the square.

$$x^2 + 2x - 6 = 0$$

$$x^2 + 2x = 6$$

$$x^2 + 2x + 1 = 7 \quad \text{(Add 1 to both sides to produce a square)}$$

$$(x + 1)^2 = 7$$

We can now take the positive and negative square roots to obtain

$$x + 1 = \sqrt{7} \text{ or } x + 1 = -\sqrt{7},$$

so, $x = -1 + \sqrt{7} \text{ or } x = -1 - \sqrt{7}.$

Notice that the solutions are irrational, and so this equation could not be easily solved using the factoring method.

EXAMPLE

Solve $x^2 - 6x - 2 = 0$.

SOLUTION

$$x^2 - 6x - 2 = 0$$

$$x^2 - 6x + 9 - 9 - 2 = 0 \quad (\text{Complete the square.})$$

$$(x - 3)^2 = 11$$

$$x - 3 = \sqrt{11} \text{ or } x - 3 = -\sqrt{11}$$

Hence $x = 3 + \sqrt{11}$ or $x = 3 - \sqrt{11}$.

There are, of course, quadratic equations which cannot be solved using real numbers. For example, if we apply the method to the equation $x^2 - 6x + 12 = 0$, we obtain $(x - 3)^2 = -3$ and $(x - 3)^2 \geq -3$ since the equation cannot be solved.

EXERCISE 4

Solve $x^2 - 5x - 3 = 0$ by completing the square and also show that $x^2 - 5x + 7 = 0$ has no solutions.

Non-monic quadratics

You will have noticed that we have not solved any non-monic quadratics by completing the square. This is generally rather tricky for students to complete and non-monic quadratics that cannot be solved by factoring can always be solved using the quadratic formula.

To solve a non-monic quadratic by completing the square, it is easiest to divide the equation by the leading coefficient and so make the quadratic monic. This will lead to fractions as the following example shows.

EXAMPLE

Solve $3x^2 - 5x + 1 = 0$.

SOLUTION

Divide the equation by 3 and shift the constant term to the other side.

$$3x^2 - 5x + 1 = 0$$

$$x^2 - \frac{5}{3}x = -\frac{1}{3}$$

$$x^2 - \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} - \frac{1}{3}$$

$$\left(x - \frac{5}{3}\right)^2 = \frac{13}{36}$$

So, $x = \frac{5 + \sqrt{13}}{6}$ or $x = \frac{5 - \sqrt{13}}{6}$.

As mentioned above, these solutions can also be found using the quadratic formula.

The quadratic formula

The method of completing the square always works. By applying it to the general quadratic equation $ax^2 + bx + c = 0$ we obtain the well-known **quadratic formula**.

To derive the formula, we will begin by multiplying the equation through by $4a$, which although not the usual first step in completing the square, will make the algebra much easier.

$$ax^2 + bx + c = 0$$

$$4a^2x^2 + 4abx + 4ac = 0$$

We now note that $(2ax + b)^2 = 4a^2x^2 + 4abx + b^2$ so adding b^2 will produce a square.

$$4a^2x^2 + 4abx = -4ac$$

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$(2ax + b)^2 = b^2 - 4ac.$$

We pause at this stage to note that if $b^2 - 4ac$ is negative, then there is no solution. If $b^2 - 4ac$ is positive, we then proceed to take the positive and negative square roots to solve for x . If $b^2 - 4ac$ is equal to 0, then there will only be 1 solution. We suppose then that $b^2 - 4ac$ is positive and proceed to find the solutions.

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \sqrt{b^2 - 4ac} \text{ or } 2ax + b = -\sqrt{b^2 - 4ac}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This last formula is called the quadratic formula, sometimes written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If the quantity $b^2 - 4ac = 0$ then there will only be one solution, $x = -\frac{b}{2a}$.

In this case, the quadratic will be a perfect square. The quantity $b^2 - 4ac$ plays an important role in the theory of quadratic equations and is called the *discriminant*.

Thus, in summary, when solving $ax^2 + bx + c = 0$, first calculate the discriminant $b^2 - 4ac$. Then,

- if $b^2 - 4ac$ is negative, then there is no solution.
- if $b^2 - 4ac$ is positive, then the solutions are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.
- if $b^2 - 4ac$ is zero, then there is only one solution $x = -\frac{b}{2a}$.

While students do not need to learn the derivation of the formula, they do need to remember the formula itself.

Note: If $b^2 - 4ac$ is zero, then the quadratic is a perfect square.

EXAMPLE

Solve $x^2 - 10x - 3 = 0$ by

- a** using the formula. **b** completing the square.

SOLUTION

- a** Here $a = 1$, $b = -10$, $c = -3$,

$$\begin{aligned} \text{so } b^2 - 4ac &= 100 + 12 \\ &= 112. \end{aligned}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{10 + \sqrt{112}}{2} \quad \text{or} \quad x = \frac{10 - \sqrt{112}}{2}$$

$$x = \frac{10 + 4\sqrt{7}}{2} \quad \text{or} \quad x = \frac{10 - 4\sqrt{7}}{2} \quad (\text{Simplify the surd.})$$

$$x = \frac{2(5 + 2\sqrt{7})}{2} \quad \text{or} \quad x = \frac{2(5 - 2\sqrt{7})}{2}$$

$$x = 5 + 2\sqrt{7} \quad \text{or} \quad x = 5 - 2\sqrt{7}$$

- b** $x^2 - 10x - 3 = 0$

$$x^2 - 10x + 25 - 25 - 3 = 0$$

$$(x - 5)^2 = 28$$

$$x - 5 = \sqrt{28} \quad \text{or} \quad x - 5 = -\sqrt{28}$$

$$x = 5 + 2\sqrt{7} \quad \text{or} \quad x = 5 - 2\sqrt{7}$$

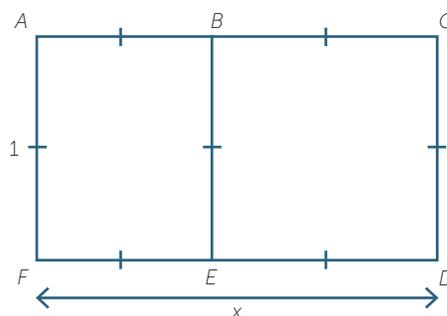
EXERCISE 5

Re-solve the quadratic equation $3x^2 - 5x + 1 = 0$ using the quadratic formula.

A further application

One very interesting application involves a number known to the Greeks as the **golden ratio**.

A **golden rectangle** is a rectangle such as $ACDF$ drawn below, with sides of length 1 and x , and with the property that if a 1×1 square ($BCDE$) is removed, the resulting rectangle ($ABEF$) is similar to the original one. That is, $ACDF$ is an enlargement of $ABEF$.



The Greeks regarded the relative dimensions of the rectangle $ABEF$ as 'most pleasing to the eye'. The front facade of the Parthenon has its sides in this ratio.

Since the rectangles are similar, their sides are in proportion. Now $EF = x - ED = x - 1$, and $\frac{EF}{AF} = \frac{AF}{AC}$. Thus, $\frac{x-1}{1} = \frac{1}{x}$.

We can multiply both sides by and re-arrange to form a quadratic equation.

$$\frac{x-1}{1} = \frac{1}{x}$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0.$$

Applying the quadratic formula, with $a = 1$, $b = -1$, $c = -1$ and $b^2 - 4ac = 5$, we have

$$x = \frac{1 + \sqrt{5}}{2} \text{ or } x = \frac{1 - \sqrt{5}}{2}.$$

The second of these numbers is negative and so cannot be the solution of the given problem. Hence the value of x is $\frac{1 + \sqrt{5}}{2}$ which is approximately, correct to three decimal places. This number is called the **golden ratio** and arises in several places in mathematics, some of the them quite unexpected.

Equations reducible to quadratics

We conclude our discussion by mentioning equations that are not strictly quadratic, but can be reduced to a quadratic equation by a simple substitution.

EXAMPLE

Solve $2^{2x} - 5 \times 2^x - 24 = 0$.

SOLUTION

Put $u = 2^x$ then the equation becomes $u^2 - 5u - 24 = 0$.

This equation factors as $(u - 8)(u + 3) = 0$

and so $u = 8$ or $u = -3$.

Replacing u , we have $2^x = 8$, $2^x = -3$.

The second equation has no solution, since $2^x > 0$, while the first equation has solution $x = 3$.

EXERCISE 6

Solve the equation $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$.

LINKS FORWARD

In the module, *The quadratic function*, we will look in detail at the graphs of the quadratic function $y = ax^2 + bx + c$, which represents a **parabola**. The technique of completing the square that we have gone through in this chapter will be used to find the axis of symmetry of the parabola.

Quadratic inequalities

Replacing the sign $=$ with an inequality sign produces a quadratic inequality. These have many applications including finding the domain and range of a given function.

The method of solution is similar to that for solving quadratic equations.

EXAMPLE

Solve $x^2 - x - 2 < 0$.

SOLUTION

Factoring the equation, we have $(x - 2)(x + 1) < 0$

Now if the product of two numbers is negative, then the numbers must have opposite sign. Since $(x - 2)$ is smaller than $(x + 1)$, we have $(x - 2) < 0$ and $(x + 1) > 0$ giving $x < 2$ and $x > -1$.

We can combine these to write, $-1 < x < 2$.

Cubic equations

Cubic equations are generally not covered in detail in the school syllabus, but arise as a natural generalization of quadratic equations.

For example, $x^3 - x + 2 = 0$ is an example of a cubic equation.

In the module, *Polynomials*, a factoring method will be developed to solve cubic equations that have rational roots.

Just as there is quadratic formula for solving quadratic equations, there is also a cubic formula for solving cubic equations. There is a simple procedure for taking a general cubic equation and removing the term involving x . Thus, we may assume that our cubic is in the form $x^3 - px + q = 0$. (This is sometimes called a *depressed cubic*, or a cubic in *Weierstrass form*).

In this case, a clever procedure, going back to the 15th century, enables us to solve the cubic and write the solution as

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{24}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{24}}}.$$

The quantity $\frac{q^2}{4} + \frac{p^3}{24}$ under the square root sign above is called the discriminant of the cubic. Cubic equations may have either 1, 2 or 3 real roots. The above formula only produces 1 real root.

EXERCISE 7

Apply the formula to find a real root of $x^3 - x - 1 = 0$. (Using calculus, it can be shown that this equation has only 1 real root.)

HISTORY

Perhaps surprisingly, quadratic equations are known from quite early on in the history of mathematics. The Babylonians were solving quadratic equations as early as 2000 BC. Their method of solution was different from ours and was expressed verbally as a series of steps (with no proof.) They also solved non-linear simultaneous equations that lead in standard algebra to quadratics. For example, $x + y = 10$, $xy = 5$.

THE BABYLONIAN METHOD OF SOLUTION

It should be emphasised that the following method, although interesting, is not recommended for the classroom.

We illustrate using the equation $x^2 - 2x - 8 = 0$.

Step 1 Take the constant term on the other side and factor the left-hand side.

$$x(x - 2) = 8.$$

Step 2 Put a equal to the average of these terms, that is $a = x - 1$.

$$\text{Then } x = a + 1, (x - 2) = a - 1.$$

Step 3 Substitute and solve for using the difference of squares identity.

$$x(x - 2) = 8$$

$$(a + 1)(a - 1) = 8$$

$$a^2 - 1 = 8$$

$$a^2 = 9.$$

Now the Babylonians did not use negative numbers, so for them the only solution is $a = 3$, and so $x = 4$.

This method can also be used to find irrational solutions.

EXERCISE 8

Use the Babylonian method to solve $x^2 + 2x - 6 = 0$.

THE EGYPTIANS

The first known occurrence of a quadratic equation in Egyptian mathematics occurs in the Berlin Papyrus, dating from the Middle Kingdom in Egypt (c.2160–1700). The problem is:

To divide 100 square measures into two squares such that the side of one of the squares shall be three fourths the side of the other.

Translated into modern notation, this problem requires us to solve the simultaneous equations

$$x^2 + y^2 = 100, \text{ and } y = \frac{3}{4}x.$$

The Egyptians gave the solution as a sequence of unexplained steps which basically use ideas of proportion.

THE GREEKS

The Greeks also solved quadratic equations, but used graphical/geometric methods to do so.

Euclid (Book 2, Proposition 11) solved the quadratic $x^2 + ax = a^2$ geometrically. The method used was basically a form of completing the square.

In later books in the Elements (e.g. Proposition 11 in Book IV), Euclid gives geometric constructions equivalent to solving a general quadratic equation. There is no algebraic solution in Euclid.

THE INDIANS

Throughout antiquity various rules were given for special cases and types of quadratics.

The so-called *Hindu Rule* was first given by Sridhara in about 1025. It said:

Multiply both sides of the equation by a number equal to four times the [co-efficient] of the square, and add to them a number equal to the square of the original [co-efficient] of the unknown quantity. [Then extract the root].

That is, given $ax^2 + bx = c$ multiply by $4a$ and add to b^2 obtain $4a^2x^2 + 4abx + b^2 = b^2 + 4ac$, whence $2ax + b = \pm\sqrt{b^2 + 4ac}$, and so the equation can be solved. Notice that is essentially our derivation of the quadratic formula.

THE ARABS

Al-Khwarizmi (9th century A.D.), in the first Arabic text-book on algebra, solves quadratics of the form $x^2 + ax = b$ by completing the square.

He adds $\frac{1}{4}a^2$ to both sides and obtains $x + \frac{1}{2}a = \sqrt{\frac{1}{4}a^2 + b}$, from which he can extract x .

LATER WRITERS

Many other general methods also have appeared.

Interestingly the first written appearance of the method using factorisation did not occur until 1631, and an explicit form of the quadratic formula does not appear until Vieta c.1580.

Although particular examples of cubic equations arose in antiquity, the general cubic equation was not solved until the 15th century, as was the quartic equation. Attempts to find formula for the solution of the quintic (equations of degree 5) and higher degree equations, in the same spirit as the quadratic formula, were shown in the 19th century to be impossible. That means that there is no general formula for the solution of equations of degree 5 or higher in terms of radicals, that is, using combinations of square, cube, fourth or higher roots alone.

ANSWERS TO EXERCISES

EXERCISE 1

a $x = \sqrt{5}$ or $x = -\sqrt{5}$

b $x = 4$ or $x = -3$

c $x = -2$ or $x = \frac{4}{3}$

EXERCISE 2

$$x = 2 \text{ or } x = \frac{39}{8}$$

EXERCISE 3

$$n = 8$$

EXERCISE 4

$$x = \frac{5 - \sqrt{37}}{2} \text{ or } x = \frac{5 + \sqrt{37}}{2}$$

$$x^2 - 5x + 7 = \left(x - \frac{25}{3}\right)^2 + \frac{3}{4}$$

EXERCISE 5

$$x = \frac{5 - \sqrt{13}}{6} \text{ or } x = \frac{5 + \sqrt{13}}{6}$$

EXERCISE 6

$$x = -2 \text{ or } x = -1 \text{ or } x = 3 \text{ or } x = 4$$

EXERCISE 8

$$x = -1 + \sqrt{7} \text{ or } x = -1 - \sqrt{7}$$



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