Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Functions: Module 7

Trigonometric functions and circular measure

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Trigonometric functions and circular measure – A guide for teachers (Years 11–12)

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Trigonometric functions and circular measure

Assumed knowledge

The content of the modules:

- *Introductory trigonometry* (Years 9–10)
- *Further trigonometry* (Year 10)
- *Trigonometric functions* (Year 10).

Motivation

The Greeks developed a precursor to trigonometry, by studying chords in a circle. However, the systematic tabulation of trigonometric ratios was conducted later by the Indian and Arabic mathematicians. It was not until about the 15th century that a 'modern' approach to trigonometry appeared in Europe.

Two of the original motivations for the study of the sides and angles of a triangle were astronomy and navigation. These were of fundamental importance at the time and are still very relevant today.

With the advent of coordinate geometry, it became apparent that the standard trigonometric ratios could be graphed for angles from 0° to 90°. Using the coordinates of points on a circle, the trigonometric ratios were extended beyond the angles found in a right-angled triangle. It was then discovered that the trigonometric functions are periodic and so can be used to model periodic behaviour in nature and in science generally.

More recently, the discovery and exploitation of electricity and electromagnetic waves introduced exciting and very powerful new applications of the trigonometric functions. Indeed, of all the applications of classical mathematics, this is possibly one of the most profound and world changing. The trigonometric functions give the key to understanding and using wave motion and manipulating signals in communications. Decomposing signals into combinations of trigonometric functions is known as Fourier analysis.

In this module, we will revise the basics of triangle trigonometry, including the sine and cosine rules, and angles of any magnitude.

We will then look at trigonometric expansions, which will be very important in the later module *The calculus of trigonometric functions*. All this is best done, initially, using angles measured in degrees, since most students are more comfortable with these units.

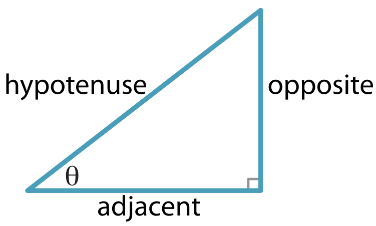
We then introduce radian measure, which is a natural way of measuring angles using arc length. This is absolutely essential as a prerequisite for the calculus of the trigonometric functions and also gives us simple formulas for the area of a sector, the area of a segment and arc length of a circle.

Finally, we conclude with a section on graphing the trigonometric functions, which illustrates their wave-like properties and their periodicity.

Content

The trigonometric ratios

If we fix an acute angle θ, then all right-angled triangles that have θ as one of their angles are similar. So, in all such triangles, corresponding pairs of sides are in the same ratio.



The side opposite the right angle is called the **hypotenuse**. We label the side opposite θ as the **opposite** and the remaining side as the **adjacent**. Using these names we can list the following standard ratios:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Allied to these are the three reciprocal ratios, **cosecant**, **secant** and **cotangent**:

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}, \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

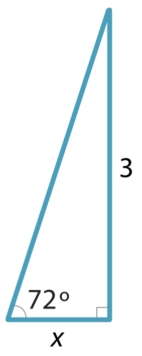
These are called the reciprocal ratios as

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

The trigonometric ratios can be used to find lengths and angles in right-angled triangles.

Example

Find the value of x in the following triangle.



Solution

We have

$$\frac{x}{3} = \cot 72^\circ = \frac{1}{\tan 72^\circ}.$$

Hence

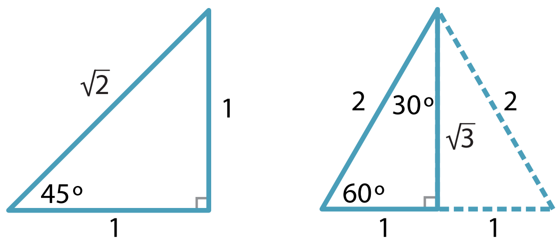
$$x = \frac{3}{\tan 72^\circ} \approx 0.97, \quad \text{correct to two decimal places.}$$

Special angles

The trigonometric ratios of the angles 30°, 45° and 60° can be easily expressed as fractions or surds, and students should commit these to memory.

Trigonometric ratios of special angles

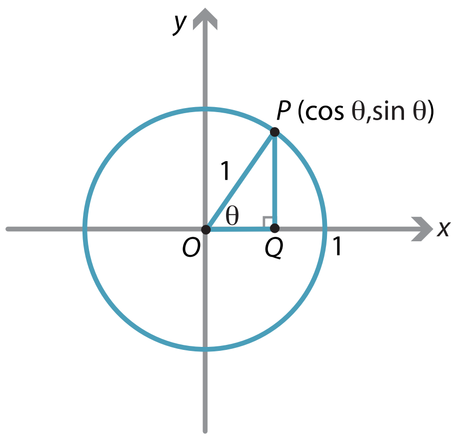
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$



Triangles for the trigonometric ratios of special angles.

Extending the angles

In the module *Further trigonometry* (Year 10), we showed how to redefine the trigonometric functions in terms of the coordinates of points on the unit circle. This allows the definition of the trigonometric functions to be extended to the second quadrant.



If the angle θ belongs to the first quadrant, then the coordinates of the point P on the unit circle shown in the diagram are simply $(\cos\theta, \sin\theta)$.

Thus, if θ is the angle between OP and the positive x-axis:

- the **cosine** of θ is defined to be the x-coordinate of the point P on the unit circle
- the **sine** of θ is defined to be the y-coordinate of the point P on the unit circle.

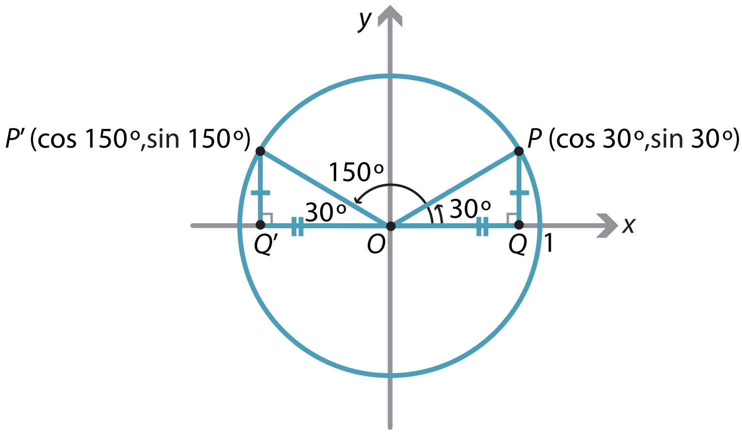
We can apply this definition to any angle.

The **tangent** ratio is then defined by

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

provided $\cos\theta$ is not zero.

As an example, let us take θ to be $30°$, so P has coordinates $(\cos 30°, \sin 30°)$. Now move the point P around the circle to P', so that OP' makes an angle of $150°$ with the positive x-axis. Note that $30°$ and $150°$ are supplementary angles. The coordinates of P' are $(\cos 150°, \sin 150°)$.



But we can see that the triangles OPQ and $OP'Q'$ are congruent, so the y-coordinates of P and P' are the same. Thus, $\sin 150° = \sin 30°$. Also, the x-coordinates of P and P' have the same magnitude but opposite sign, so $\cos 150° = -\cos 30°$.

From this typical example, we see that if θ is any obtuse angle, then its supplement $180° - \theta$ is acute, and the sine of θ is given by

$$\sin\theta = \sin(180° - \theta), \qquad \text{where } 90° < \theta < 180°.$$

Similarly, if θ is any obtuse angle, then the cosine of θ is given by

$$\cos\theta = -\cos(180° - \theta), \qquad \text{where } 90° < \theta < 180°.$$

In words this says:

- the sine of an obtuse angle equals the sine of its supplement
- the cosine of an obtuse angle equals minus the cosine of its supplement.

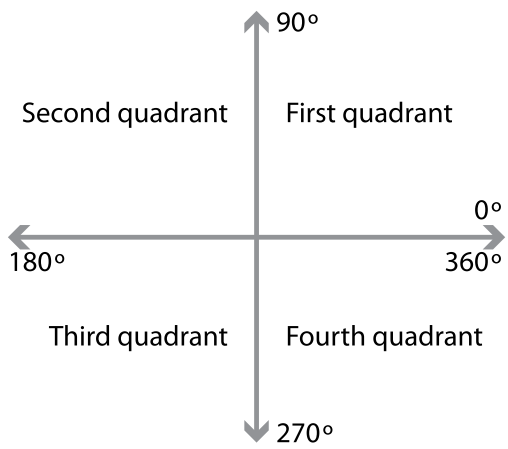
Exercise 1

On the unit circle, place the point P corresponding to each of the angles $0°$, $90°$, $180°$, $270°$ and $360°$. By considering the coordinates of each of these points, complete the following table of trigonometric ratios.

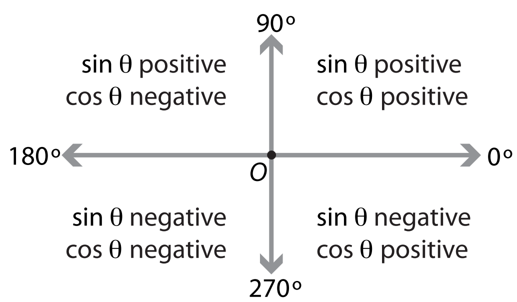
Coordinates of P	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
	$0°$			
	$90°$			undefined
	$180°$			
	$270°$			undefined
	$360°$			

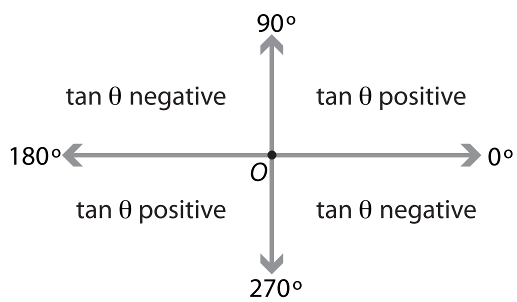
The four quadrants

The coordinate axes divide the plane into four quadrants, labelled *first, second, third* and *fourth* as shown. Angles in the third quadrant, for example, lie between 180° and 270°.



By considering the x- and y-coordinates of the point P as it lies in each of the four quadrants, we can identify the sign of each of the trigonometric ratios in a given quadrant. These are summarised in the following diagrams.



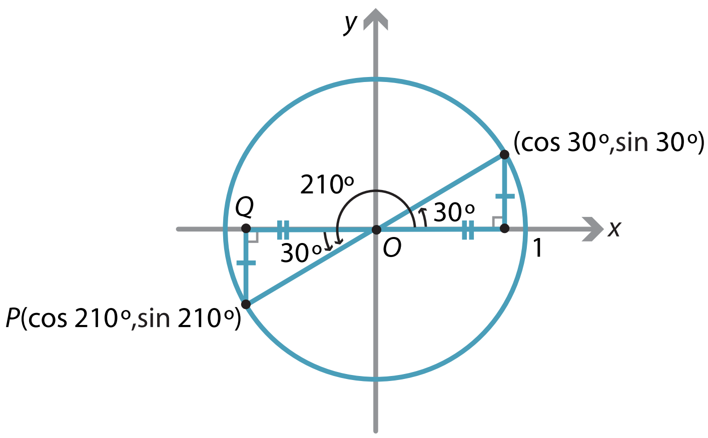


Related angles

In the module *Further trigonometry* (Year 10), we saw that we could relate the sine and cosine of an angle in the second, third or fourth quadrant to that of a related angle in the first quadrant. The method is very similar to that outlined in the previous section for angles in the second quadrant.

We will find the trigonometric ratios for the angle 210°, which lies in the third quadrant. In this quadrant, the sine and cosine ratios are negative and the tangent ratio is positive.

To find the sine and cosine of 210°, we locate the corresponding point P in the third quadrant. The coordinates of P are $(\cos 210^\circ, \sin 210^\circ)$. The angle POQ is 30° and is called the **related angle** for 210°.



So,

$$\cos 210^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}.$$

Hence

$$\tan 210^\circ = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

In general, if θ lies in the third quadrant, then the acute angle $\theta - 180^\circ$ is called the related angle for θ.

Exercise 2

a Use the method illustrated above to find the trigonometric ratios of 330°.

b Write down the related angle for θ, if θ lies in the fourth quadrant.

The basic principle for finding the related angle for a given angle θ is to subtract 180° from θ or to subtract θ from 180° or 360°, in order to obtain an acute angle. In the case when the related angle is one of the special angles 30°, 45° or 60°, we can simply write down the exact values for the trigonometric ratios.

In summary, to find the trigonometric ratio of an angle between 0° and 360°:

- find the related angle
- obtain the sign of the ratio by noting the quadrant
- evaluate the trigonometric ratio of the related angle and attach the appropriate sign.

Example

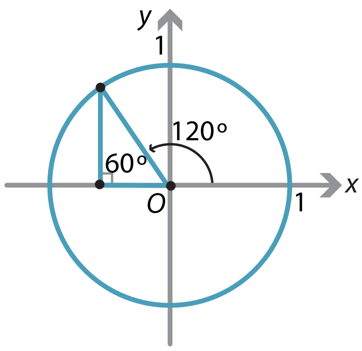
Use the related angle to find the exact value of

1 $\sin 120°$ **2** $\cos 150°$ **3** $\tan 300°$ **4** $\cos 240°$.

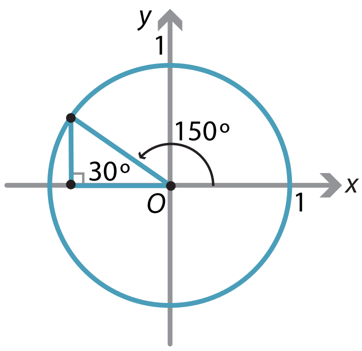
Solution

1 The angle 120° is in the second quadrant, and its related angle is 60°.

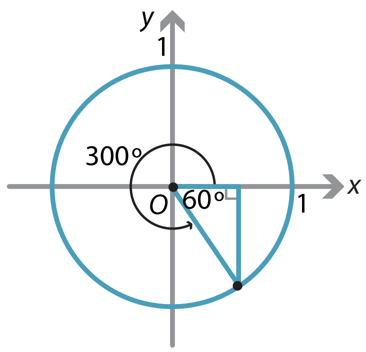
So $\sin 120° = \sin 60° = \dfrac{\sqrt{3}}{2}$.



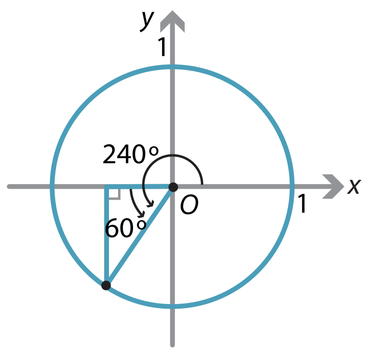
2 $\cos 150° = -\cos 30° = -\dfrac{\sqrt{3}}{2}$



3 $\tan 300° = -\tan 60° = -\sqrt{3}$



4 $\cos 240° = -\cos 60° = -\frac{1}{2}$



Exercise 3

Find the exact value of

a $\sin 210°$ **b** $\cos 315°$ **c** $\tan 150°$.

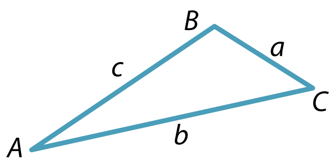
The sine and cosine rules

The sine rule

In the module *Further trigonometry* (Year 10), we introduced and proved the **sine rule**, which is used to find sides and angles in non-right-angled triangles.

In the triangle ABC, labelled as shown, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



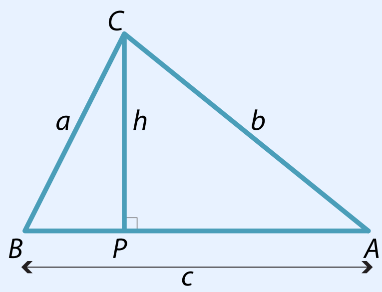
Clearly, we may also write this as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

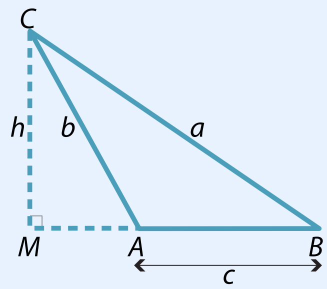
In general, one of the three angles may be obtuse. The formula still holds true, although the geometric proof is slightly different.

Exercise 4

a Find two expressions for h in the diagram below, and hence deduce the sine rule.

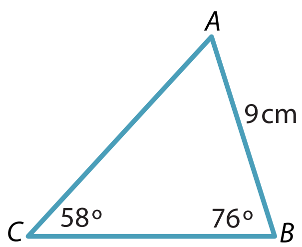


b Repeat the method of part (a), using the following diagram, to show that the sine rule holds in obtuse-angled triangles.



Example

The triangle ABC has $AB = 9$ cm, $\angle ABC = 76°$ and $\angle ACB = 58°$.



Find, correct to two decimal places,

1 AC **2** BC.

Solution

1 Applying the sine rule gives

$$\frac{AC}{\sin 76^\circ} = \frac{9}{\sin 58^\circ}$$

and so

$$AC = \frac{9 \sin 76^\circ}{\sin 58^\circ}$$
$$\approx 10.30 \text{ cm} \qquad \text{(to two decimal places).}$$

2 To find BC, we first find the angle $\angle CAB$ opposite it.

$$\angle CAB = 180^\circ - 58^\circ - 76^\circ$$
$$= 46^\circ.$$

Thus, by the sine rule,

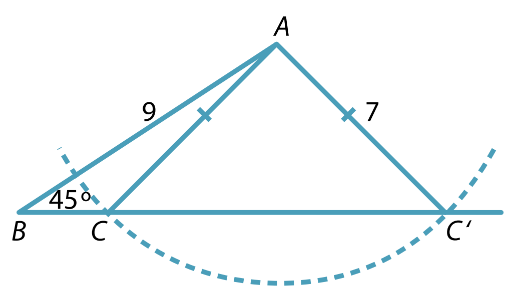
$$\frac{BC}{\sin 46^\circ} = \frac{9}{\sin 58^\circ}$$

and so

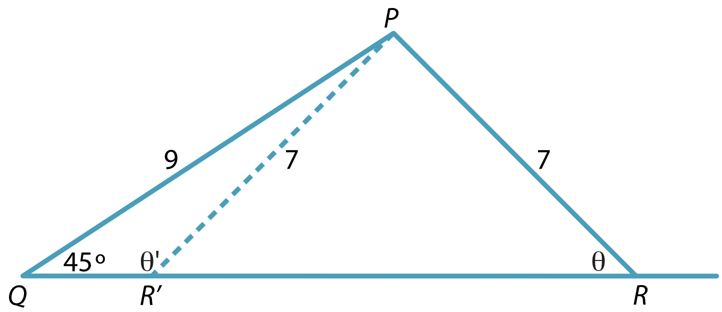
$$BC \approx 7.63 \text{ cm} \qquad \text{(to two decimal places).}$$

The ambiguous case

In the module *Congruence* (Year 8), it was emphasised that, when applying the SAS congruence test, the angle in question has to be the angle included between the two sides. For example, the following diagram shows two non-congruent triangles ABC and ABC' having two pairs of matching sides and sharing a common (non-included) angle.



Suppose we are told that a triangle PQR has $PQ = 9$, $\angle PQR = 45^\circ$ and $PR = 7$. Then the angle opposite PQ is not uniquely determined. There are two non-congruent triangles that satisfy the given data.



Applying the sine rule to the triangle, we have

$$\frac{\sin\theta}{9} = \frac{\sin 45^\circ}{7}$$

and so

$$\begin{aligned}\sin\theta &= \frac{9\sin 45^\circ}{7}\\ &\approx 0.9091.\end{aligned}$$

Thus $\theta \approx 65^\circ$, assuming that θ is acute. But the supplementary angle is $\theta' \approx 115^\circ$. The triangle PQR' also satisfies the given data. This situation is sometimes referred to as the *ambiguous case.*

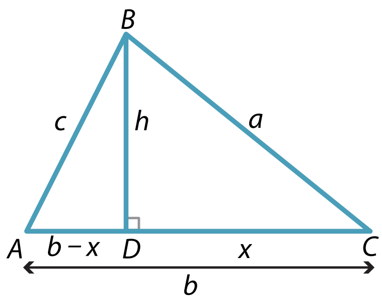
Since the angle sum of a triangle is 180°, in some circumstances only one of the two angles calculated is geometrically valid.

The cosine rule

We know from the SAS congruence test that a triangle is completely determined if we are given two sides and the included angle. However, if we know two sides and the included angle in a triangle, the sine rule does not help us determine the remaining side or the remaining angles.

The second important formula for general triangles is the cosine rule.

Suppose ABC is a triangle and that the angles A and C are acute. Drop a perpendicular from B to the line interval AC and mark the lengths as shown in the following diagram.



In the triangle ABD, applying Pythagoras' theorem gives

$$c^2 = h^2 + (b - x)^2.$$

Also, in the triangle BCD, another application of Pythagoras' theorem gives

$$h^2 = a^2 - x^2.$$

Substituting this expression for h^2 into the first equation and expanding,

$$\begin{aligned} c^2 &= a^2 - x^2 + (b - x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx. \end{aligned}$$

Finally, from triangle BCD, we have $x = a \cos C$ and so

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

This last formula is known as the **cosine rule**.

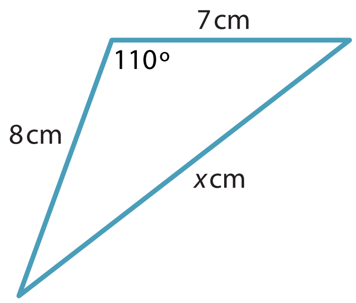
Notice that, if $C = 90°$, then since $\cos C = 0$ we obtain Pythagoras' theorem, and so we can regard the cosine rule as Pythagoras' theorem with a correction term.

The cosine rule is also true when the angle C is obtuse. But note that, in this case, the final term in the formula will produce a positive number, because the cosine of an obtuse angle is negative. Some care must be taken in this instance.

By relabelling the sides and angles of the triangle, we can also write the cosine rule as $a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = a^2 + c^2 - 2ac \cos B$.

Example

Find the value of x to one decimal place.



Solution

Applying the cosine rule gives

$$
\begin{aligned}
x^2 &= 7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 110^\circ \\
&= 113 + 112 \cos 70^\circ \\
&\approx 151.306,
\end{aligned}
$$

so $x \approx 12.3$ (to one decimal place).

Finding angles

If the three sides of a triangle are known, then the three angles are uniquely determined. (This is the SSS congruence test.) Again, the sine rule is of no help in finding the three angles, since it requires the knowledge of (at least) one angle, but we can use the cosine rule instead.

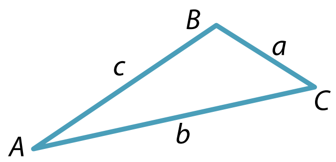
We can substitute the three side lengths a, b, c into the formula $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle opposite the side c, and then rearrange to find $\cos C$ and hence C.

Alternatively, we can rearrange the formula to obtain

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

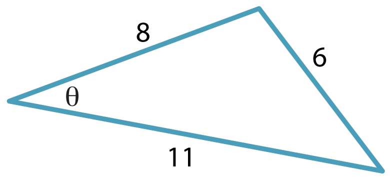
and then substitute. Students may choose to rearrange the cosine rule or to learn this further formula. Using this form of the cosine rule often reduces arithmetical errors.

Recall that, in any triangle ABC labelled as shown, if $a < b$, then angle $A <$ angle B.



Example

A triangle has side lengths 6 cm, 8 cm and 11 cm. Find the smallest angle in the triangle.



Solution

The smallest angle in the triangle is opposite the smallest side.

Applying the cosine rule:

$$6^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos\theta$$

$$\cos\theta = \frac{8^2 + 11^2 - 6^2}{2 \times 8 \times 11} = \frac{149}{176}.$$

So $\theta \approx 32.2°$ (correct to one decimal place).

The area of a triangle

We saw in the module *Introductory trigonometry* (Years 9–10) that, if we take any triangle with two given sides a and b about a given (acute) angle θ, then the area of the triangle is

$$\text{Area} = \frac{1}{2}ab\sin\theta.$$

This formula also holds when θ is obtuse.

Exercise 5

A triangle has two sides of length 5 cm and 4 cm containing an angle θ. Its area is 5 cm^2. Find the two possible (exact) values of θ and draw the two triangles that satisfy the given information.

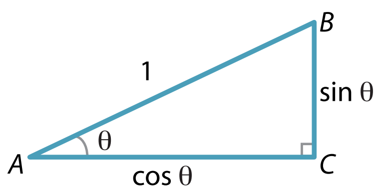
Exercise 6

Write down two different expressions for the area of a triangle ABC and derive the sine rule from them.

Trigonometric identities

The Pythagorean identity

There are many important relationships between the trigonometric functions which are of great use, especially in calculus. The most fundamental of these is the **Pythagorean identity**. For acute angles, this is easily proven from the following triangle ABC with hypotenuse of unit length.



With $\angle BAC = \theta$, we see that $AC = \cos\theta$ and $BC = \sin\theta$. Hence Pythagoras' theorem tells us that

$$\cos^2\theta + \sin^2\theta = 1.$$

This formula holds for all angles, since every angle can be related to an angle in the first quadrant whose sines and cosines differ only by a sign, which is dealt with by squaring. Dividing this equation respectively by $\cos^2\theta$ and by $\sin^2\theta$, we obtain

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{and} \qquad \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

From these standard identities, we can prove a variety of results.

Example

Prove the following identities:

1 $(1-\sin\theta)(1+\sin\theta)=\cos^2\theta$

2 $\dfrac{2\cos^3\theta-\cos\theta}{\sin\theta\cos^2\theta-\sin^3\theta}=\cot\theta.$

Solution

1
$$\begin{aligned}\text{LHS} &= (1-\sin\theta)(1+\sin\theta)\\ &= 1-\sin^2\theta && \text{(difference of two squares)}\\ &= \cos^2\theta && \text{(Pythagorean identity)}\\ &= \text{RHS}\end{aligned}$$

2
$$\begin{aligned}\text{LHS} &= \frac{2\cos^3\theta-\cos\theta}{\sin\theta\cos^2\theta-\sin^3\theta}\\ &= \frac{\cos\theta\left(2\cos^2\theta-1\right)}{\sin\theta\left(\cos^2\theta-\sin^2\theta\right)}\\ &= \frac{\cos\theta\left(2\cos^2\theta-1\right)}{\sin\theta\left(\cos^2\theta-(1-\cos^2\theta)\right)}\\ &= \frac{\cos\theta\left(2\cos^2\theta-1\right)}{\sin\theta\left(2\cos^2\theta-1\right)}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \cot\theta\\ &= \text{RHS}\end{aligned}$$

Exercise 7

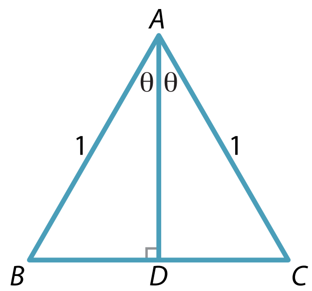
Prove that

$$\frac{1}{\sec\theta+\tan\theta}=\frac{\cos\theta}{1+\sin\theta}.$$

Double angle formulas

The sine and cosine functions are not linear. For example, $\cos(A+B) \neq \cos A + \cos B$. The correct formula for $\cos(A+B)$ is given in the next subsection. In the special case when $A = B = \theta$, we would like to obtain a simple formula for $\cos(2\theta) = \cos 2\theta$, called the *double angle formula*. It is particularly useful in applications and in calculus and is quite easy to derive.

Consider the following isosceles triangle ABC, with sides AB and AC both of length 1 and apex angle 2θ. We will need to assume, for the present, that $0° < \theta < 90°$.



Let AD be the perpendicular bisector of the interval BC. Then, using basic properties of an isosceles triangle, we know that AD bisects $\angle BAC$ and so $BD = DC = \sin\theta$. Applying the cosine rule to the triangle ABC, we immediately have

$$\begin{aligned}\cos 2\theta &= \frac{1^2 + 1^2 - 4\sin^2\theta}{2 \times 1 \times 1} \\ &= \frac{2 - 4\sin^2\theta}{2} \\ &= 1 - 2\sin^2\theta.\end{aligned}$$

Replacing 1 by $\cos^2\theta + \sin^2\theta$, we arrive at the double angle formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

This may also be written as $\cos 2\theta = 2\cos^2\theta - 1$ or $\cos 2\theta = 1 - 2\sin^2\theta$, but is best learnt in the form given above.

We derived this formula under the assumption that θ was between 0° and 90°, but the formula in fact holds for all values of θ. We shall see this in the next subsection, where we prove the general expansion formula for $\cos(A+B)$.

Example

Find $\cos 22\frac{1}{2}^\circ$ in surd form.

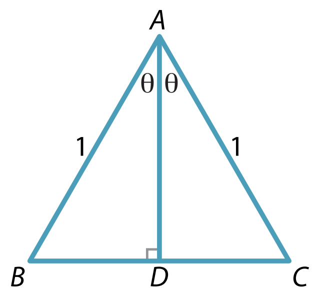
Solution

Putting $\theta = 22\frac{1}{2}^\circ$ into the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$ and writing x for $\cos 22\frac{1}{2}^\circ$, we obtain

$$\begin{aligned}
\cos 45^\circ &= 2x^2 - 1 \\
\frac{1}{\sqrt{2}} &= 2x^2 - 1 \\
2x^2 &= \frac{1+\sqrt{2}}{\sqrt{2}} \\
x &= \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}},
\end{aligned}$$

which simplifies to $\frac{1}{2}\sqrt{2+\sqrt{2}}$.

We can also find a double angle formula for sine using the same diagram.



In this case, we write down formulas for the area of $\triangle ABC$ in two ways:

- On the one hand, the area is given by $\frac{1}{2}AB \cdot AC\ \sin(\angle BAC) = \frac{1}{2}\sin 2\theta$.
- Since $AD = \cos\theta$, we can alternatively split the triangle into two right-angled triangles and write the area as $2 \times \frac{1}{2}\sin\theta\ \cos\theta = \sin\theta\ \cos\theta$.

Equating these two expressions for the area, we obtain

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

As before, we assumed that θ lies between 0° and 90°, but the formula is valid for all values of θ.

Exercise 8

Find $\sin 15^\circ$ in surd form.

From the Pythagorean identity and the double angle formula for cosine, we have

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta.$$

Adding these equations and dividing by 2 we obtain $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$, while subtracting them and dividing by 2 we obtain $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. These formulas are very important in integral calculus, as discussed in the module *The calculus of trigonometric functions*.

Exercise 9

Use the double angle formulas for sine and cosine to show that

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}, \qquad \text{for } \tan\theta \neq \pm 1.$$

Summary of double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}, \qquad \text{for } \tan\theta \neq \pm 1$$

Trigonometric functions of compound angles

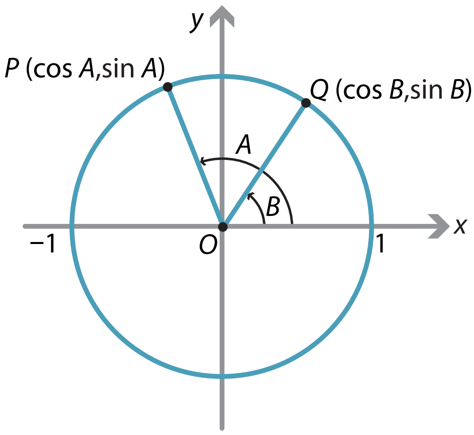
In the previous subsection, we derived formulas for the trigonometric functions of double angles. That derivation, which used triangles, was only valid for a limited range of angles, although the formulas remain true for all angles.

In this subsection we find the expansion formulas for $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$ and $\cos(A-B)$, which are valid for all A and B. The double angle formulas can be recovered by putting $A = B = \theta$.

These formulas are quite central in trigonometry. In the module *The calculus of trigonometric functions*, they are used to find, among other things, the derivative of the sine function.

To prove the $\cos(A-B)$ formula, from which we can obtain the other expansions, we return to the circle definition of the trigonometric functions.

Consider two points $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ on the unit circle, making angles A and B respectively with the positive x-axis.



We will calculate the distance PQ in two ways and then equate the results. First we apply the cosine rule to the triangle OPQ. Note that, in the diagram above, $\angle POQ = A-B$. In general, it is always the case that $\cos(\angle POQ) = \cos(A-B)$. So the cosine rule gives

$$PQ^2 = 1^2 + 1^2 - 2\times 1\times 1\times \cos(A-B) = 2 - 2\cos(A-B).$$

On the other hand, using the square of the distance formula from coordinate geometry,

$$\begin{aligned} PQ^2 &= (\cos B - \cos A)^2 + (\sin B - \sin A)^2 \\ &= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B \\ &= 2 - 2(\cos A\cos B + \sin A\sin B). \end{aligned}$$

Equating the two expressions for PQ^2, we have

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

We can easily obtain the formula for $\cos(A+B)$ by replacing B with $-B$ in the formula for $\cos(A-B)$ and recalling that $\cos(-\theta) = \cos\theta$ (the cosine function is an even function) and $\sin(-\theta) = -\sin\theta$ (the sine function is an odd function).

Hence

$$\begin{aligned} \cos(A+B) &= \cos A\cos(-B) + \sin A\sin(-B) \\ &= \cos A\cos B - \sin A\sin B. \end{aligned}$$

Using the identity $\sin\theta = \cos(90° - \theta)$, we can show that

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B.$$

These four compound angle formulas are important and the student should remember them. Most other trigonometric identities can be derived from these and the standard Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$.

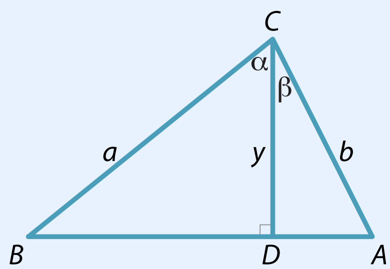
Exercise 10

Use the identity $\sin\theta = \cos(90° - \theta)$ to derive the sine expansions.

The following exercise gives a simple geometric derivation of the sine expansion.

Exercise 11

Fix acute angles α and β. Construct a triangle ABC as shown in the following diagram. (Start by drawing the line interval CD. Then construct the right-angled triangles BCD and ACD.)



a Prove that $y = a\cos\alpha$ and $y = b\cos\beta$.

b By comparing areas, show that

$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ay\sin\alpha + \frac{1}{2}by\sin\beta.$$

c Deduce the expansion formula for $\sin(\alpha+\beta)$.

Using the compound angle formulas, we can extend the range of angles for which we can obtain exact values for the trigonometric functions.

Example

Find the exact value of

1 $\cos 75°$ **2** $\sin 75°$ **3** $\cos 105°$.

Solution

1 $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{1}{4}(\sqrt{6} - \sqrt{2})$$

2 $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

3 $\cos 105^\circ = \cos(45^\circ + 60^\circ)$

$$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$$

$$= \frac{1}{\sqrt{2}}\frac{1}{2} - \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}(\sqrt{2} - \sqrt{6})$$

(Note that we could obtain $\cos 105^\circ$ directly from $\cos 75^\circ$, since the two angles are supplementary.)

We can also find expansions for $\tan(A+B)$ and $\tan(A-B)$. Recalling that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we can write

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Dividing the numerator and denominator by $\cos A\cos B$, we obtain

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Since $\tan(-B) = -\tan B$, we have

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

Note carefully the pattern with the signs.

Exercise 12

Find the exact value of $\tan 15°$.

Putting $A = B = \theta$ in the expansion formula for $\tan(A+B)$, we obtain

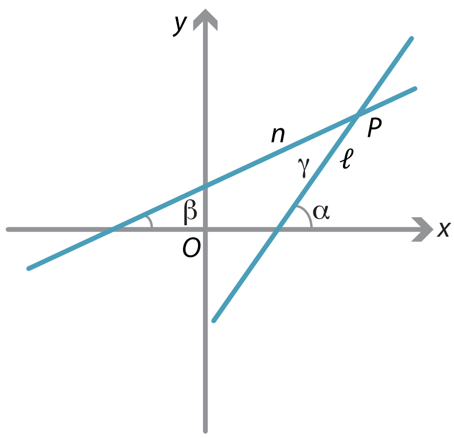
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Exercise 13

Show that $t = \tan 67\frac{1}{2}°$ satisfies the quadratic equation $t^2 - 2t - 1 = 0$ and hence find its exact value.

The angle between two lines

The tangent expansion formula can be used to find the angle, or rather the tangent of the angle, between two lines.



Suppose two lines ℓ and n with gradients m_1 and m_2, respectively, meet at the point P. The gradient of a line is the tangent of the angle it makes with the positive x-axis. So, if ℓ and n make angles α and β, respectively, with the positive x-axis, then $\tan\alpha = m_1$ and $\tan\beta = m_2$. We will assume for the moment that $\alpha > \beta$, as in the diagram above.

Now, if γ is the angle between the lines (as shown), then $\gamma = \alpha - \beta$. Hence

$$\tan\gamma = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\,\tan\beta} = \frac{m_1 - m_2}{1+m_1 m_2},$$

provided $m_1 m_2 \neq -1$. If $m_1 m_2 = -1$, the two lines are perpendicular and $\tan\gamma$ is not defined.

In general, the above formula may give us a negative number, since it may be the tangent of the obtuse angle between the two lines.

Hence, if we are interested only in the acute angle, since $\tan(180° - \theta) = -\tan\theta$, we can take the absolute value and say that, if γ is the acute angle between the two lines, then

$$\tan\gamma = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

provided the lines are not perpendicular.

Example

Find, to the nearest degree, the acute angle between the lines $y = 2x - 1$ and $y = 3x + 4$.

Solution

Here $m_1 = 2$ and $m_2 = 3$. So, if θ is the acute angle between the two lines, we have

$$\tan\theta = \left|\frac{2-3}{1+6}\right| = \frac{1}{7}$$

and therefore $\theta \approx 8°$.

Exercise 14

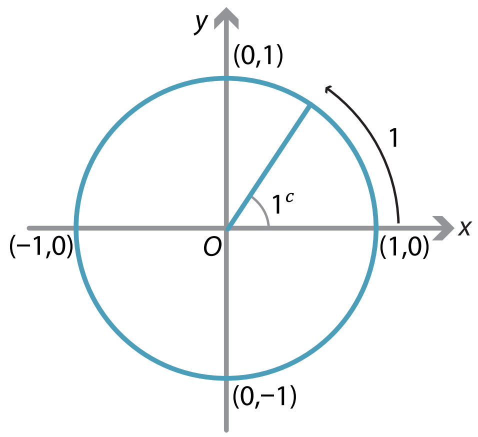
Find the two values of m such that the angle between the lines $y = mx$ and $y = 2x$ is $45°$. What is the relationship between the two lines you obtain?

If we define the angle between two curves at a point of intersection to be the angle between their tangents at that point, then the above formula — along with some differential calculus — can be used to find that angle.

Radian measure

The measurement of angles in degrees goes back to antiquity. It may have arisen from the idea that there were roughly 360 days in a year, or it may have arisen from the Babylonian penchant for base 60 numerals. In any event, both the Greeks and the Indians divided the angle in a circle into 360 equal parts, which we now call degrees. They further divided each degree into 60 equal parts called minutes and divided each minute into 60 seconds. An example would be $15°22'16''$. This way of measuring angles is very inconvenient and it was realised in the 16th century (or even before) that it is better to measure angles via arc length.

We define one radian, written as 1^c (where the c refers to *circular measure*), to be the angle subtended at the centre of a unit circle by a unit arc length on the circumference.



Definition of one radian.

Since the full circumference of a unit circle is 2π units, we have the conversion formula

$$360^\circ = 2\pi \text{ radians}$$

or, equivalently,

$$180^\circ = \pi \text{ radians.}$$

So one radian is equal to $\dfrac{180}{\pi}$ degrees, which is approximately 57.3°.

Since many angles in degrees can be expressed as simple fractions of 180, we use π as a basic unit in radians and often express angles as fractions of π. The commonly occurring angles 30°, 45° and 60° are expressed in radians respectively as $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$.

Example

Express in radians:

1 135° **2** 270° **3** 100°.

Solution

1 $135^\circ = 3 \times 45^\circ = \dfrac{3\pi}{4}^c$

2 $270^\circ = 3 \times 90^\circ = \dfrac{3\pi}{2}^c$

3 $100^\circ = \dfrac{100\pi}{180}^c = \dfrac{5\pi}{9}^c$

Note. We will often leave off the c, particularly when the angle is expressed in terms of π.

Students should have a deal of practice in finding the trigonometric functions of angles expressed in radians. Since students are more familiar with degrees, it is often best to convert back to degrees.

Example

Find

1 $\cos\dfrac{4\pi}{3}$ **2** $\sin\dfrac{7\pi}{6}$ **3** $\tan\dfrac{5\pi}{4}$.

Solution

1 $\cos\dfrac{4\pi}{3} = \cos 240° = -\cos 60° = -\dfrac{1}{2}$

2 $\sin\dfrac{7\pi}{6} = \sin 210° = -\sin 30° = -\dfrac{1}{2}$

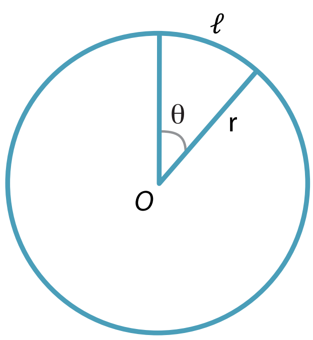
3 $\tan\dfrac{5\pi}{4} = \tan 225° = \tan 45° = 1$

Note. You can enter radians directly into your calculator to evaluate a trigonometric function at an angle in radians, but you must make sure your calculator is in **radian mode**. Students should be reminded to check what mode their calculator is in when they are doing problems involving the trigonometric functions.

Arc lengths, sectors and segments

Measuring angles in radians enables us to write down quite simple formulas for the arc length of part of a circle and the area of a sector of a circle.

In any circle of radius r, the ratio of the arc length ℓ to the circumference equals the ratio of the angle θ subtended by the arc at the centre and the angle in one revolution.



Thus, measuring the angles in radians,

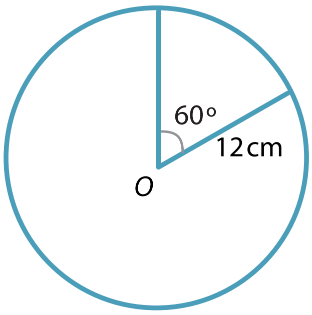
$$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi}$$

$$\implies \quad \ell = r\theta.$$

It should be stressed again that, to use this formula, we require the angle to be in *radians*.

Example

In a circle of radius 12 cm, find the length of an arc subtending an angle of 60° at the centre.



Solution

With $r = 12$ and $\theta = 60° = \frac{\pi}{3}$, we have

$$\ell = 12 \times \frac{\pi}{3} = 4\pi \approx 12.57 \text{ cm}.$$

It is often best to leave your answer in terms of π unless otherwise stated.

We use the same ratio idea to obtain the area of a sector in a circle of radius r containing an angle θ at the centre. The ratio of the area A of the sector to the total area of the circle equals the ratio of the angle in the sector to one revolution.

Thus, with angles measured in radians,

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

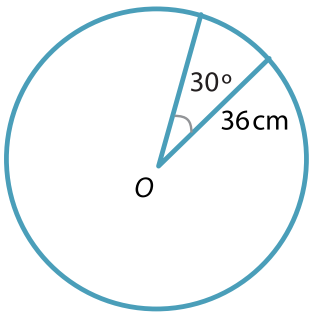
$$\implies \quad A = \frac{1}{2} r^2 \theta.$$

The arc length and sector area formulas given above should be committed to memory.

Example

In a circle of radius 36 cm, find the area of a sector subtending an angle of 30° at the centre.

Solution

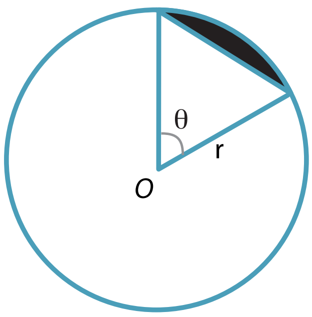


With $r = 36$ and $\theta = 30° = \dfrac{\pi}{6}$, we have

$$A = \frac{1}{2} \times 36^2 \times \frac{\pi}{6} = 108\pi \text{ cm}^2.$$

As mentioned above, in problems such as these it is best to leave your answer in terms of π unless otherwise stated.

The area A_s of a segment of a circle is easily found by taking the difference of two areas.

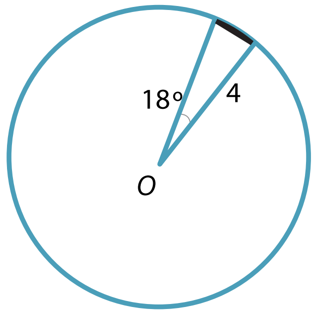


In a circle of radius r, consider a segment that subtends an angle θ at the centre. We can find the area of the segment by subtracting the area of the triangle (using $\frac{1}{2}ab\sin\theta$) from the area of the sector. Thus

$$A_s = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta).$$

Example

Find the area of the segment shown.



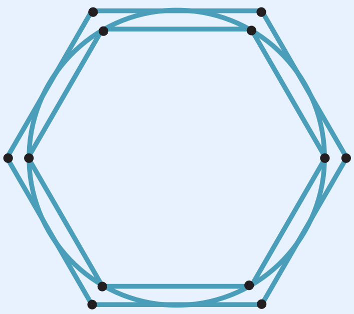
Solution

With $r = 4$ and $\theta = 18^\circ = \dfrac{18\pi}{180} = \dfrac{\pi}{10}$, we have

$$A_s = \frac{1}{2} \times 4^2 \times \left(\frac{\pi}{10} - \sin\frac{\pi}{10}\right) \approx 0.041.$$

Exercise 15

Around a circle of radius r, draw an inner and outer hexagon as shown in the diagram.



By considering the perimeters of the two hexagons, show that $3 \le \pi \le 2\sqrt{3}$.

Summary of arc, sector and segment formulas

Length of arc	$\ell = r\theta$
Area of sector	$A = \frac{1}{2}r^2\theta$
Area of segment	$A = \frac{1}{2}r^2(\theta - \sin\theta)$

Solving trigonometric equations in radians

The basic steps for solving trigonometric equations, when the solution is required in radians rather than degrees, are unchanged. Indeed, it is sometimes best to find the solution(s) in degrees and convert to radians at the end of the problem.

Example

Solve each equation for $0 \le x \le 2\pi$:

1 $\cos x = -\frac{1}{2}$ **2** $\sin 3x = 1$ **3** $4\cos^2 x + 4\sin x = 5$.

Solution

1 Since $\cos 60° = \frac{1}{2}$, the related angle is $60°$. The angle x could lie in the second or third quadrant, so $x = 180° - 60°$ or $x = 180° + 60°$. Therefore $x = 120°$ or $x = 240°$. In radians, the solutions are $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

2 Since $\sin \frac{\pi}{2} = 1$, the related angle is $\frac{\pi}{2}$. Since x lies between 0 and 2π, it follows that $3x$ lies between 0 and 6π. Thus

$$\begin{aligned} 3x &= \frac{\pi}{2},\ 2\pi + \frac{\pi}{2},\ 4\pi + \frac{\pi}{2} \\ &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}. \end{aligned}$$

Hence $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

3 In this case, we replace $\cos^2 x$ with $1 - \sin^2 x$ to obtain the quadratic

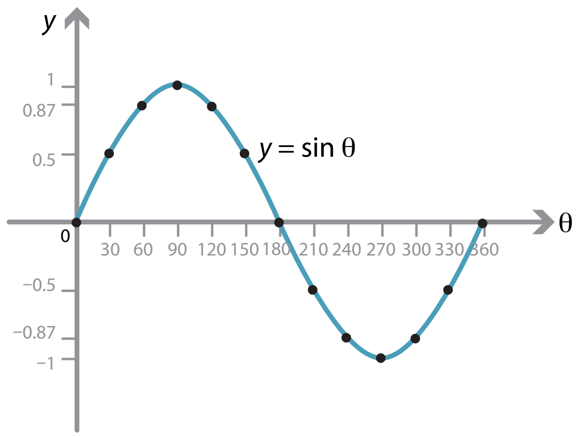
$$4(1 - \sin^2 x) + 4\sin x = 5$$

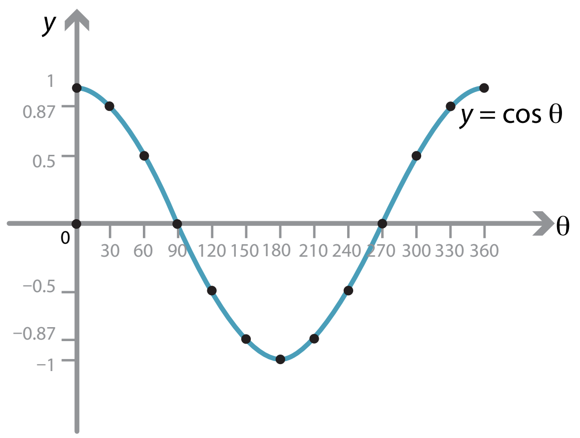
$$\implies \quad 4\sin^2 x - 4\sin x + 1 = 0.$$

This factors to $(2\sin x - 1)^2 = 0$ and so $\sin x = \frac{1}{2}$. In the given range, this has solutions $x = 30°, 150°$. So, in radians, the solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Graphing the trigonometric functions

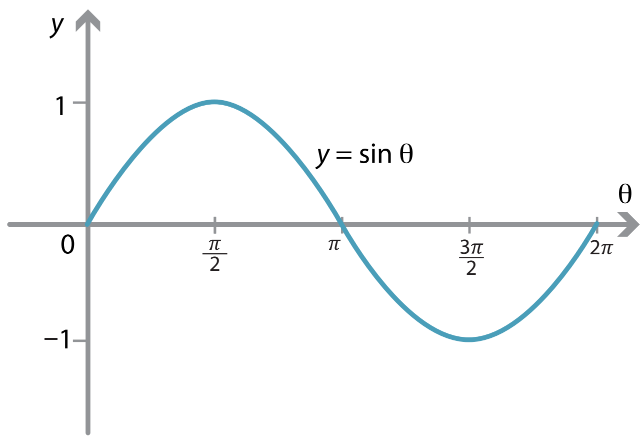
In the module *Trigonometric functions* (Year 10), we drew the graphs of the sine and cosine functions, marking the θ-axis in degrees. Using $\sin 30° = 0.5$ and $\sin 60° \approx 0.87$, we drew up a table of values and plotted them.

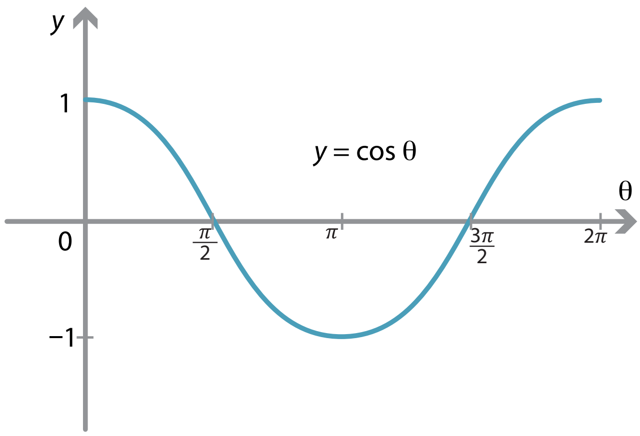




These graphs are often referred to by physicists and engineers as **sine waves**.

From now on, we will use radians as the unit on the θ-axis and so we have the following graphs for sine and cosine.

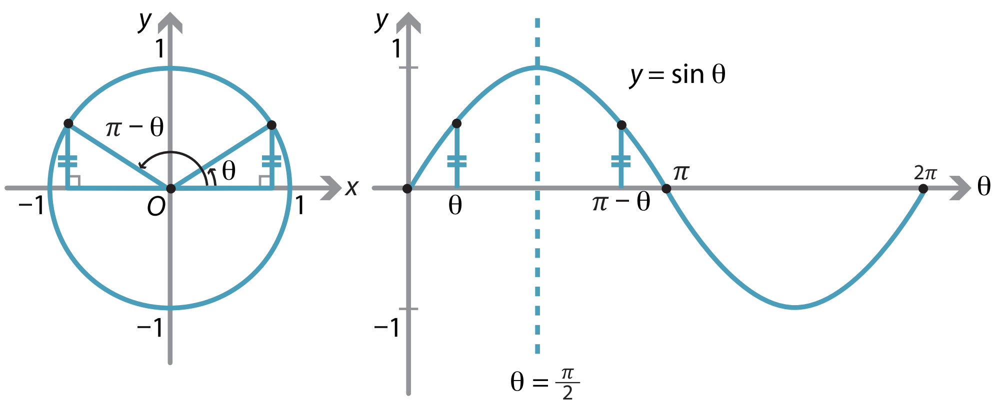




Symmetries of the sine graph

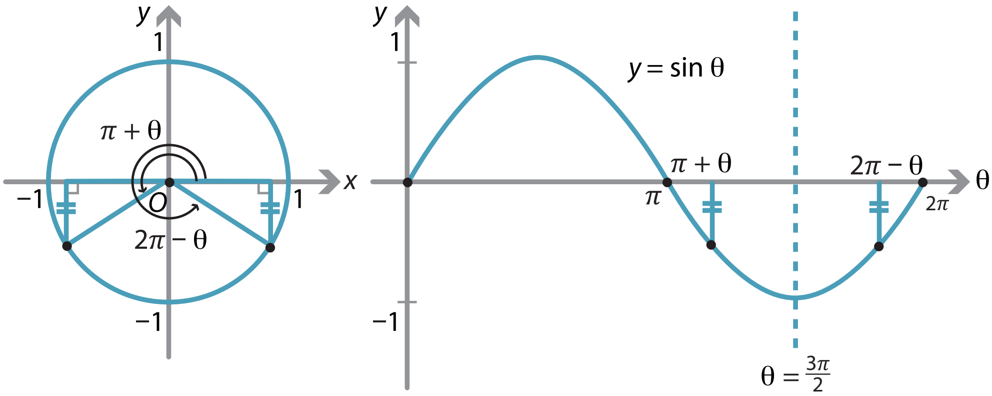
The graph of $y = \sin\theta$ has many interesting symmetries.

The model employing the unit circle helps to elucidate these. The sine of the angle θ is represented by the y-value of the point P on the unit circle. Since $\sin\theta = \sin(\pi - \theta)$, we can mark two equal intervals on the following sine graph.

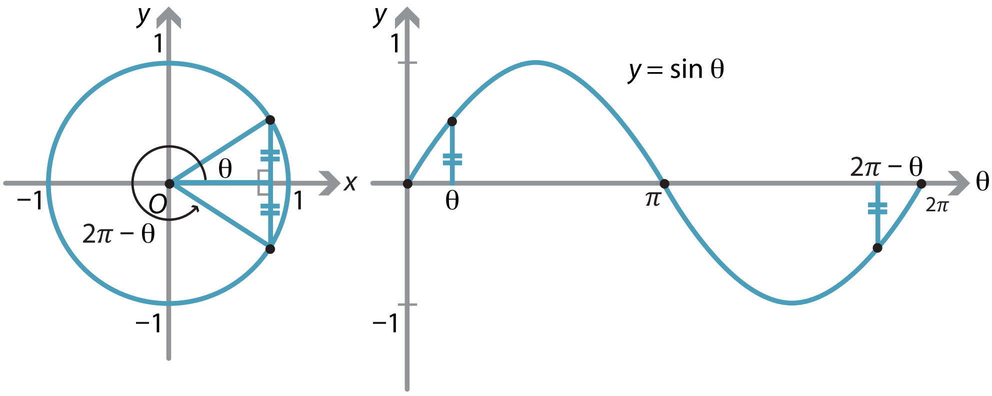


Hence, between 0 and π, the graph of $y = \sin\theta$ is symmetric about $\theta = \dfrac{\pi}{2}$.

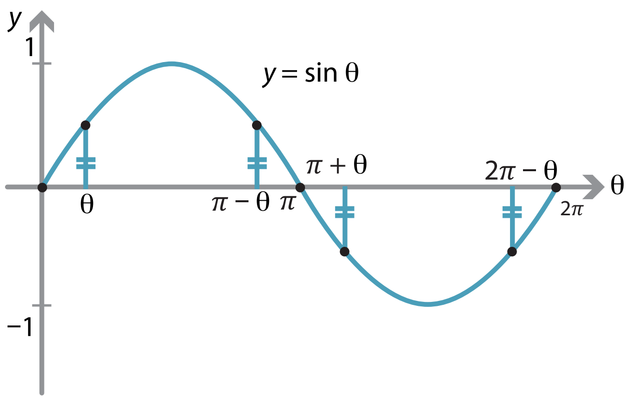
Similarly, between π and 2π, the graph is symmetric about $\theta = \dfrac{3\pi}{2}$.



Finally, the graph possesses a rotational symmetry about $\theta = \pi$ as the following diagram demonstrates.



All these observations are summarised by the following diagram. This symmetry diagram illustrates the related angle and the quadrant sign rules, as well as the symmetries discussed above.



Symmetries of the sine graph.

The extended sine graph

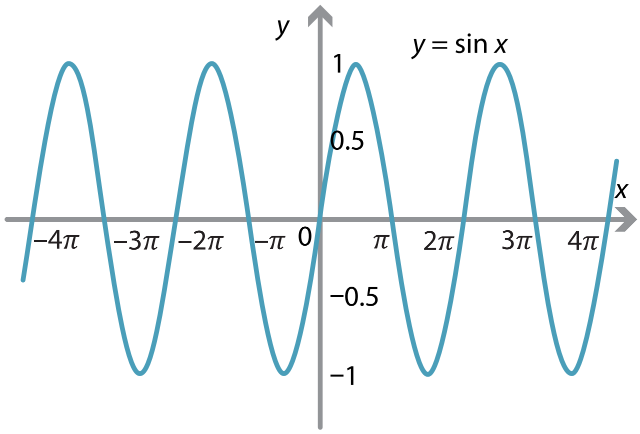
The values of sine repeat as we move through an angle of 2π, that is,

$$\sin(2\pi + x) = \sin x.$$

We say that the function $y = \sin x$ is periodic with period 2π.

Thus, the graph may be drawn for angles greater than 2π and less than 0, to produce the full (or extended) graph of $y = \sin x$.

The graph of $y = \sin x$ from 0 to 2π is often referred to as a **cycle**.



Note that the extended sine graph has even more symmetries. There is a translation (by 2π) symmetry, a reflection symmetry about any odd multiple of $\frac{\pi}{2}$ and a rotational symmetry about any even multiple of $\frac{\pi}{2}$.

Exercise 16

Sketch the graph of $y = \cos x$, for $-4\pi \le x \le 4\pi$.

Period, amplitude and phase shift

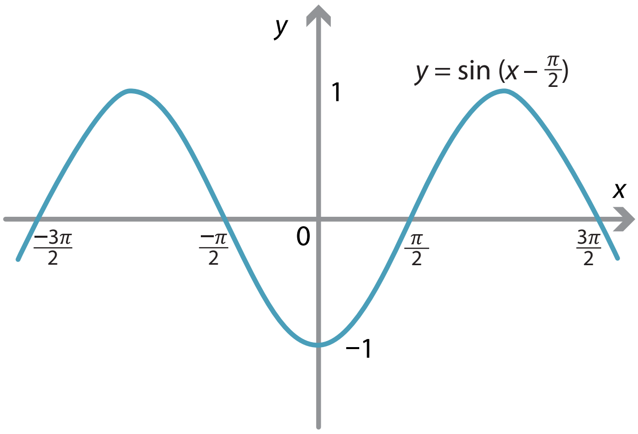
The more general form for the sine graph is

$$y = A\sin(nx + \alpha),$$

where A, n and α are constants and $A > 0$. What is the effect of varying the values of A, n and α?

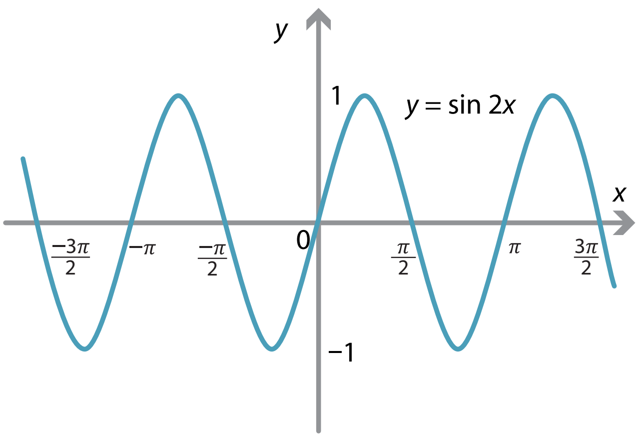
Changing α shifts the graph along the x-axis. If $\alpha > 0$, then the basic sine graph moves to the left and, if $\alpha < 0$, it moves to the right. This is sometimes referred to as a **phase shift**.

For example, we can see from the following graph that $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$.



Changing the value of A stretches the y-values from the x-axis. The sine graph $y=\sin x$ has a maximum of 1 and a minimum of -1. Hence the graph of $y=A\sin x$ (for a fixed $A>0$) has a maximum of A and a minimum of $-A$. The number A is referred to as the **amplitude**. Trigonometric graphs are used to represent the current in an AC circuit over a time period, and so the amplitude gives the maximum and minimum values of the current.

Changing the value of n stretches the graph in the x-direction. The graph of $y=\sin x$, for $0\le x\le 2\pi$, represents one cycle of the sine curve, and we say that the **period** of the graph is 2π. This means that one cycle of the graph occurs over an interval of 2π. Changing n alters the period of the graph. For example, if we draw the graph of $y=\sin 2x$, for $0\le x\le 2\pi$, we obtain two cycles of the graph and so the period becomes π, since one cycle of the graph occurs over an interval of length π.



In general, the period of the graph of $y=\sin nx$ is given by

$$\text{Period}=\frac{2\pi}{n}.$$

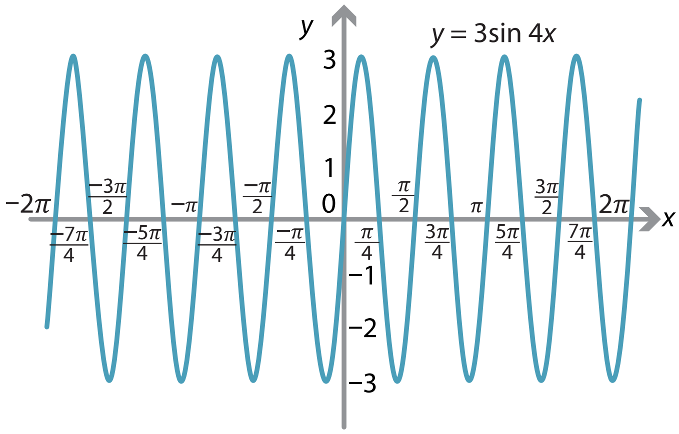
Example

Sketch the graph of $y = 3\sin 4x$, for $-2\pi \le x \le 2\pi$.

Solution

The amplitude of the graph is 3 and the period is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

The easiest way to draw the graph is to draw one cycle of a sine curve, with amplitude 3, and mark the end point as $\frac{\pi}{2}$. We can then extend the graph using periodicity to the interval $-2\pi \le x \le 2\pi$.



Cosine graphs of the form $y = A\cos(nx + \alpha)$ can be drawn by following the principles outlined above for sine graphs.

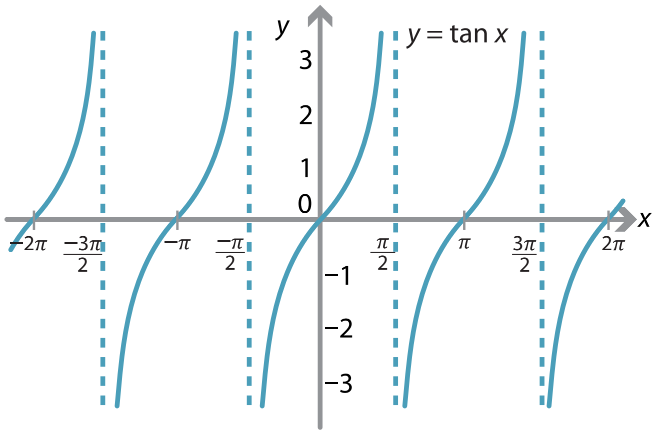
Exercise 17

Sketch the graph of $y = 2\cos 3x$, for $-3\pi \le x \le 3\pi$.

The graph of $\tan x$

The function $\tan x$ has a very different kind of graph to those for the sine and cosine functions. The period of $\tan x$ is π, rather than 2π, and the amplitude is not defined. The tangent function is not defined at $x = \pm\frac{\pi}{2}$, nor at any odd integer multiple of these values.

As x approaches $\frac{\pi}{2}$ from the left, the value of $\tan x$ increases without bound. Since tan is an odd function, as x approaches $-\frac{\pi}{2}$ from the right, $\tan x$ decreases without bound.



The period of the graph of $y = \tan nx$ is given by $\frac{\pi}{n}$.

Links forward

Multiple angles

The double angle formulas can be extended to larger multiples. For example, to find $\cos 3\theta$, we write 3θ as $2\theta + \theta$ and expand:

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta.$$

We can now apply the double angle formulas to obtain

$$\cos 3\theta = \left(\cos^2\theta - \sin^2\theta\right)\cos\theta - 2\sin\theta\cos\theta\sin\theta = \cos^3\theta - 3\sin^2\theta\cos\theta.$$

Replacing $\sin^2\theta$ with $1 - \cos^2\theta$, we have

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Exercise 18

Use the method above to find a formula for $\sin 3\theta$.

A more general approach is obtained using complex numbers. The complex number i is defined by $i^2 = -1$.

De Moivre's theorem says that, if n is a positive integer, then

$$\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta.$$

Hence, for any given n, we can expand $(\cos\theta + i\sin\theta)^n$ using the binomial theorem and equate the real and imaginary parts to find formulas for $\cos n\theta$ and $\sin n\theta$.

For example, in the case of $n = 3$,

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

and

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Equating real and imaginary parts, plus some algebraic manipulation, will produce the triple angle formulas.

Adding waves

We can add together a sine and a cosine curve. Their sum can be obtained graphically by adding the y-values of the two curves. It turns out that, if the waves have the same period, this will produce another trigonometric graph with a change in amplitude and a phase shift. Physically, this is called superimposing one wave with another.

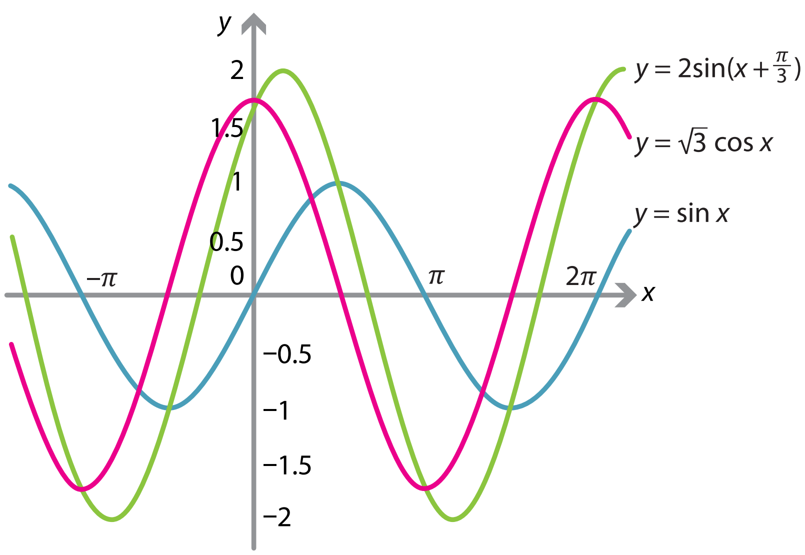
This can also be done algebraically. For example, to add $\sin x$ and $\sqrt{3}\cos x$, we proceed as follows. Divide their sum by $\sqrt{1^2 + (\sqrt{3})^2} = 2$, so

$$\sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right).$$

This looks a little like the expansion $\sin(x+\alpha) = \sin x\cos\alpha + \cos x\sin\alpha$. We thus equate $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$, which implies that $\alpha = \frac{\pi}{3}$. Hence

$$\sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right).$$

The new wave has amplitude 2 and a phase shift of $\frac{\pi}{3}$ to the left.



In general, the same method will work to add $a\sin x + b\cos x$, in which case we divide out by $\sqrt{a^2+b^2}$.

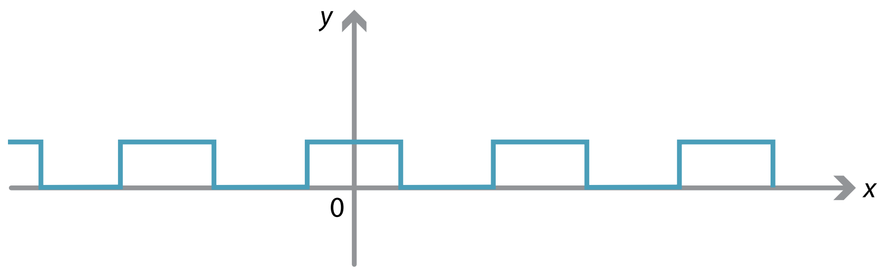
Exercise 19

Express $3\cos x - 4\sin x$ in the form $A\sin(x+\alpha)$.

Waves such as the square wave and the saw-tooth wave, which arise in physics and engineering, can be approximated using the sum of a large number of waves — this is the study of Fourier series, which is central to much of modern electrical engineering and technology.

The following square wave is the graph of the function with rule

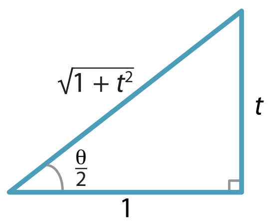
$$\frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \cdots = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n}\cos nx.$$



The t formulas

Introducing the parameter $t = \tan\dfrac{\theta}{2}$ turns out to be a very useful tool in solving certain types of trigonometric equations and also in finding certain integrals involving trigonometric functions. The basic idea is to relate $\sin\theta$, $\cos\theta$ and even $\tan\theta$ to the tangent of half the angle. This can be done using the double angle formulas.

We let $t = \tan\dfrac{\theta}{2}$ and so we can draw the following triangle with sides 1, t, $\sqrt{1+t^2}$.



Now $\sin 2\alpha = 2\sin\alpha\cos\alpha$, so replacing 2α with θ we have

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

From the triangle, we have

$$\sin\theta = 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}.$$

This is referred to as the t formula for $\sin\theta$.

Exercise 20

Use the geometric method above to derive the t formula

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

and deduce that $\tan\theta = \dfrac{2t}{1-t^2}$, for $t \neq \pm 1$.

Exercise 21

The geometric proof of the t formula for $\sin\theta$ given above assumes that the angle $\frac{\theta}{2}$ is acute. Give a general algebraic proof of the formula.

One application of the t formulas is to solving certain types of trigonometric equations.

Example

Solve $\cos\theta + \sin\theta = \frac{1}{2}$, for $0° \leq \theta \leq 360°$, correct to one decimal place.

Solution

Put $t = \tan\dfrac{\theta}{2}$. Then

$$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = \frac{1}{2}.$$

This rearranges to $3t^2 - 4t - 1 = 0$, whose solutions are $t = \dfrac{2\pm\sqrt{7}}{3} \approx 1.549, -0.215$. Taking the inverse tangents (in degrees) and doubling, we obtain the solutions $\theta \approx 114.3°, 335.7°$ in the given range.

Note. Some care is required when using the t formulas to find solutions of an equation, as there may be a solution θ for which $\tan\frac{\theta}{2}$ is undefined.

History

Some brief descriptions of the Greek approach to trigonometry via chords of circles were given in the module *Further trigonometry* (Year 10). The chord equivalents of the double angle formulas and the sine and cosine expansions, $\sin(A+B)$ and $\cos(A+B)$, were found by Hipparchos (180–125 BCE) and Ptolemy (*c.* 90–168 CE).

The Indian mathematicians Aryabhata (476–550 CE) and Bhaskara I (7th century) reworked much of the Greek chord ideas and introduced ratios that are essentially the same as our sine and cosine. They were able to tabulate the values of these ratios. In particular, Bhaskara I used a formula equivalent to

$$\sin x \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}, \qquad \text{for } 0 \le x \le \frac{\pi}{2},$$

to produce tables of values.

In the 12th century, Bhaskara II discovered the sine and cosine expansions in roughly their modern form.

Arabic mathematicians were also working in this area and, in the 9th century, Muhammad ibn Musa al-Khwarizmi produced sine and cosine tables. He also gave a table of tangents.

The first mathematician in Europe to treat trigonometry as a distinct mathematical discipline was Regiomontanus. He wrote a treatise called *De triangulis omnimodus* in 1464, which is recognisably 'modern' in its approach to the subject.

The history of the trigonometric functions really begins after the development of calculus and was largely developed by Leonard Euler (1707–1783). The works of James Gregory in the 17th century and Colin Maclaurin in the 18th century led to the development of infinite series expansions for the trigonometric functions, such as

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots.$$

It was Euler who discovered the remarkable relationship between the exponential and trigonometric functions

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which is nowadays taken as a definition of $e^{i\theta}$.

Answers to exercises

Exercise 1

Coordinates of P	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$(1,0)$	$0°$	0	1	0
$(0,1)$	$90°$	1	0	undefined
$(-1,0)$	$180°$	0	-1	0
$(0,-1)$	$270°$	-1	0	undefined
$(1,0)$	$360°$	0	1	0

Exercise 2

a $\sin 330° = -\sin 30° = -\dfrac{1}{2}$

$\cos 330° = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\tan 330° = -\tan 30° = -\dfrac{1}{\sqrt{3}}$

b If θ is in the fourth quadrant, the related angle is $360° - \theta$.

Exercise 3

a $\sin 210° = -\dfrac{1}{2}$ **b** $\cos 315° = \dfrac{1}{\sqrt{2}}$ **c** $\tan 150° = -\dfrac{1}{\sqrt{3}}$

Exercise 4

a In $\triangle BCP$, we have $h = a\sin B$. In $\triangle ACP$, we have $h = b\sin A$.

Hence $a\sin B = b\sin A \implies \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

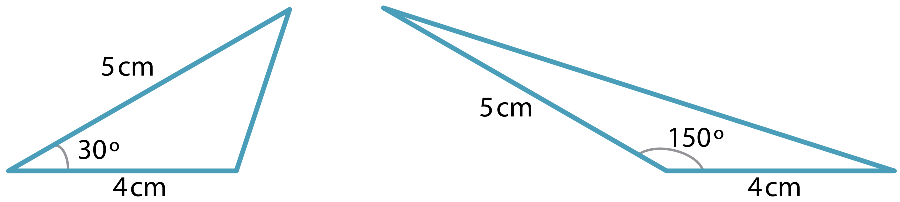
b In $\triangle BCM$, we have $h = a\sin B$. In $\triangle ACM$, we have $h = b\sin(180° - A) = b\sin A$.

Again it follows that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

Exercise 5

Using $A = \frac{1}{2}ab\sin\theta$, we have

$$\begin{aligned} 5 = \frac{1}{2} \times 5 \times 4 \times \sin\theta \implies& \sin\theta = \frac{1}{2} \\ \implies& \theta = 30°, 150°. \end{aligned}$$



Exercise 6

$$\frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A \implies a\sin B = b\sin A \implies \frac{a}{\sin A} = \frac{b}{\sin B}$$

Exercise 7

$$\text{LHS} = \frac{1}{\sec\theta + \tan\theta} = \frac{1}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{1+\sin\theta} = \text{RHS}$$

Exercise 8

The double angle formula for cosine gives $\cos 30° = 1 - 2\sin^2 15°$. So

$$\begin{aligned}\sin^2 15° &= \frac{1}{2}\big(1 - \cos 30°\big)\\ &= \frac{1}{2}\Big(1 - \frac{\sqrt{3}}{2}\Big)\\ &= \frac{1}{4}\big(2-\sqrt{3}\big).\end{aligned}$$

Hence $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$. (Take the positive square root, as $\sin 15°$ is positive.)

An alternative method is to start from $\sin 30° = 2\sin 15° \cos 15°$. Proceed by squaring both sides of the equation and using the Pythagorean identity.

Exercise 9

Using the double angle formulas, we have

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}.$$

If we assume that $\tan\theta$ is defined, then we can divide the numerator and denominator by $\cos^2\theta$ to obtain

$$\tan 2\theta = \frac{2\,\frac{\sin\theta}{\cos\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Note that $\tan 2\theta$ is defined if and only if $\cos 2\theta = \cos^2\theta - \sin^2\theta \neq 0$, in which case $\tan\theta$ cannot equal ± 1.

Exercise 10

$$\begin{aligned}\sin(A+B) &= \cos\big(90^\circ - (A+B)\big)\\ &= \cos\big((90^\circ - A) - B\big)\\ &= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B\\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

Replacing B with $-B$ gives the remaining formula.

Exercise 11

a These follow easily by considering the triangles BCD and ACD.

b Area of $\triangle ABC$ = area of $\triangle BCD$ + area of $\triangle ACD$.

So $\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ay\sin\alpha + \frac{1}{2}by\sin\beta$.

c Substitute $y = b\cos\beta$ into the first term on the right-hand side of the equation from part (b) and $y = a\cos\alpha$ into the second term. Then divide both sides by $\frac{1}{2}ab$ to obtain the sine expansion.

Exercise 12

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(By multiplying both the numerator and the denominator by $\sqrt{3}-1$, this answer can be simplified to $\tan 15^\circ = 2-\sqrt{3}$.)

Exercise 13

Put $\theta = 67\frac{1}{2}^\circ$ into the double angle formula:

$$\begin{aligned}\tan 135^\circ = \frac{2t}{1-t^2} \quad&\Longrightarrow\quad -1 = \frac{2t}{1-t^2}\\ &\Longrightarrow\quad t^2 - 2t - 1 = 0\\ &\Longrightarrow\quad t = 1 \pm \sqrt{2}.\end{aligned}$$

Hence, $\tan 67\frac{1}{2}^\circ = 1+\sqrt{2}$.

(Here we take the positive square root, since $\tan 67\frac{1}{2}^\circ$ is positive.)

Exercise 14

With $m_2 = 2$ and $\gamma = 45^\circ$, we have

$$1 = \left|\frac{m-2}{1+2m}\right|.$$

Hence $m-2=2m+1$ or $m-2=-(2m+1)$, giving $m=-3$ and $m=\frac{1}{3}$.

The two lines obtained are perpendicular.

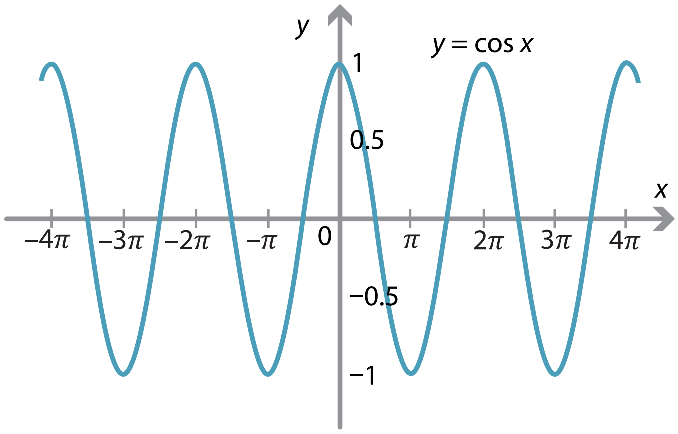
Exercise 15

Since the angles subtended at the centre of the hexagon are each 60°, we can divide the inner and outer hexagons into equilateral triangles. The side length of the inner triangles is r. After applying Pythagoras' theorem, we find that the side length of each triangle in the outer hexagon is $\frac{2}{\sqrt{3}}r$. Hence, comparing perimeters,

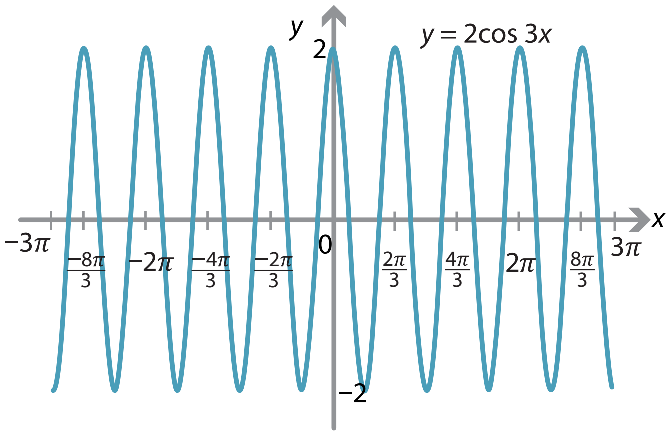
$$6r \le 2\pi r \le 6\times\frac{2}{\sqrt{3}}r.$$

Dividing by $2r$ and simplifying gives the desired result.

Exercise 16



Exercise 17



Exercise 18

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta)\\ &= \sin 2\theta\,\cos\theta + \cos 2\theta\,\sin\theta\\ &= 2\sin\theta\,\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta\\ &= 2\sin\theta\left(1-\sin^2\theta\right) + \left(1-2\sin^2\theta\right)\sin\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Exercise 19

There are a number of ways to do this problem. We could begin by factoring out 5, but we will do it as follows:

$$A\sin(x+\alpha) = A\big(\sin x\,\cos\alpha + \cos x\,\sin\alpha\big) = 3\cos x - 4\sin x.$$

Since this is an identity in x, we can equate coefficients and write

$$A\sin\alpha = 3 \qquad \text{and} \qquad A\cos\alpha = -4.$$

Squaring and adding these equations gives $A = 5$, and hence $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$. So we can take α as an angle in the second quadrant and the calculator gives $\alpha \approx 143.1°$. Hence $3\cos x - 4\sin x = 5\sin(x+\alpha)$, with $\alpha \approx 143.1°$.

Exercise 20

The double angle formula for cosine gives $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$. So, using the triangle arising from $t = \tan\frac{\theta}{2}$, we have

$$\cos\theta = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.$$

Exercise 21

Let $\alpha = \frac{\theta}{2}$ and $t = \tan\alpha$. Then

$$\frac{2t}{1+t^2} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}} = \frac{2\sin\alpha\,\cos\alpha}{\cos^2\alpha+\sin^2\alpha} = 2\sin\alpha\,\cos\alpha = \sin 2\alpha = \sin\theta,$$

using the Pythagorean identity and the double angle formula for sine.

0 1 2 3 4 5 6 7 8 9 10 11 12