Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Algebra and coordinate geometry: Module 6

The binomial theorem

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The bionomial theorem – A guide for teachers (Years 11–12)

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The binomial theorem

Assumed knowledge

- Basic skills for simplifying algebraic expressions.
- Expanding brackets.
- Factoring linear and quadratic expressions.
- Some experience in working with polynomials.

Motivation

> I am the very model of a modern Major-General,
> I've information vegetable, animal, and mineral,
> I know the kings of England, and I quote the fights historical
> From Marathon to Waterloo, in order categorical;
> I'm very well acquainted, too, with matters mathematical,
> I understand equations, both the simple and quadratical,
> About **binomial theorem** I'm teeming with a lot o' news —
> With many cheerful facts about the square of the hypotenuse.
>
> — Gilbert and Sullivan, *Pirates of Penzance*.

When you look at the following expansions you can see the symmetry and the emerging patterns. The simple first case dates back to Euclid in the third century BCE.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + 2ab + b^2\end{aligned}$$

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= (a^3+2a^2b+ab^2)+(a^2b+2ab^2+b^3)\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)^3(a+b)\\ &= (a^3+3a^2b+3ab^2+b^3)(a+b)\\ &= (a^4+3a^3b+3a^2b^2+ab^3)+(a^3b+3a^2b^2+3ab^3+b^4)\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

Notice that

- the expansion of $(a+b)^2$ has three terms and in each term the sum of the indices is 2
- the expansion of $(a+b)^3$ has four terms and in each term the sum of the indices is 3
- the expansion of $(a+b)^4$ has five terms and in each term the sum of the indices is 4.

We conjecture that the expansion of $(a+b)^n$ has $n+1$ terms and in each term the sum of the indices is n.

The coefficients of the terms follow an interesting pattern. How can we determine this pattern and how can we predict the coefficients of the expansion of $(a+b)^n$? The binomial theorem gives us the general formula for the expansion of $(a+b)^n$ for any positive integer n. It also enables us to determine the coefficient of any particular term of an expansion of $(a+b)^n$.

In this module, Pascal's triangle is centre stage. The coefficients of the expansion of $(a+b)^n$, for a particular positive integer n, are contained in sequence in the nth row of this triangle of numbers. The triangular numbers, the square numbers and the numbers of the Fibonacci sequence can be found from the triangle, and many interesting identities can be established.

For example, the triangular numbers occur in Pascal's triangle along the diagonal shown in the following diagram. The square numbers can be found by adding pairs of adjacent numbers on this diagonal.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

The triangular numbers in Pascal's triangle.

This topic combines combinatoric and algebraic results in a most productive manner.

The relationship between the expansion of $(a+b)^n$ and binomial probabilities is addressed in the module *Binomial distribution*.

Content

A look at Pascal's triangle

We begin by looking at the expansions of $(1+x)^n$ for $n = 0, 1, 2, 3, 4, 5$.

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1 + x$$

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

When the coefficients in the expansions of $(1+x)^n$ are arranged in a table, the result is known as **Pascal's triangle**.

Pascal's triangle

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's triangle is often displayed in the following way. Some of the patterns of the triangle are more apparent in this form.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

By examining Pascal's triangle, we can make the following observations, which will be proved later in this module.

1. Each number is the sum of the two numbers diagonally above it (with the exception of the 1's).
2. Each row is symmetric (i.e., the same backwards as forwards).
3. The sum of the numbers in each row is a power of 2.
4. In any row, the sum of the first, third, fifth, ... numbers is equal to the sum of the second, fourth, sixth, ... numbers. (This is not a totally obvious result.)

We can use Pascal's triangle to help us expand expressions of the form $(1+x)^n$.

Example

Expand

1 $(1+x)^6$ **2** $(1-2x)^6$.

Solution

1 The coefficients of $(1+x)^6$ are given in the sixth row of Pascal's triangle:

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$$

2 The expansion of $(1-2x)^6$ can be obtained by replacing $(-2x)$ for x in the expansion of $(1+x)^6$:

$$\begin{aligned}(1-2x)^6 &= 1 + 6(-2x) + 15(-2x)^2 + 20(-2x)^3 + 15(-2x)^4 + 6(-2x)^5 + (-2x)^6\\ &= 1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6.\end{aligned}$$

Expansions and the notation $\binom{n}{r}$

Expansions

We start by looking at the results of multiplying several binomials. With two binomials, we have

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd.$$

The expansion is obtained by multiplying each letter in the first bracket by each letter in the second and adding them. There are $2 \times 2 = 2^2 = 4$ terms. Similarly, with three binomials, we have

$$(a+b)(c+d)(e+f) = ace + acf + ade + adf + bce + bcf + bde + bdf.$$

There are $2 \times 2 \times 2 = 2^3 = 8$ terms.

Exercise 1

a How many terms are there in the expansion of $(a+b+c)(d+e)$?

b How many terms are there in the expansion of $(a+b)(c+d)(e+f)(g+h)(i+j)$?

In general, the product of any number of polynomials is equal to the sum of all the products which can be formed by choosing one term from each polynomial and multiplying these terms together.

Example

Find the coefficient of x^2 in the expansion of $(2x-1)(3x+4)(5x-6)$.

Solution

If we take the terms containing x from any two of the factors and the constant from the remaining factor and multiply these terms together, we will obtain a term containing x^2 in the expansion. If we do this in all possible ways and add, we will find the required coefficient.

The required coefficient is

$$\begin{aligned} 2\times3\times(-6)+3\times5\times(-1)+2\times5\times4 &= -36-15+40 \\ &= -11. \end{aligned}$$

Permutations and factorial notation

In how many ways can eight people line up to get into a theme-park ride?

We can draw a box diagram for this situation, with each box indicating the number of choices for each position.

$$\boxed{8}\ \boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1}$$

So the answer is $8\times7\times6\times5\times4\times3\times2\times1 = 40\,320$ ways. The notation for this is 8!. This is read as eight **factorial**.

Most calculators have a factorial key.

The number of ways to arrange n different objects in a row is $n!$. It is important to note that $n! = n(n-1)!$. For this statement to be true when $n=1$, we define $0! = 1$.

If there are eight competitors in a race, in how many ways can the first four places be filled? We can draw a box diagram for this situation, with each box indicating the number of choices for each position.

$$\boxed{8}\ \boxed{7}\ \boxed{6}\ \boxed{5}$$

So the answer is $8\times7\times6\times5 = 1680$ ways.

A **permutation** is an arrangement of elements chosen from a certain set. For example, consider the set $\{a, b, c, d, e\}$. Some of the permutations of three letters taken from this set include

$$abc, \qquad bac, \qquad cab, \qquad eda.$$

Altogether, there are $5 \times 4 \times 3 = 60$ such permutations.

The symbol ${}^nP_r$ is used to denote the number of permutations of r distinct objects chosen from n objects. We see that

$${}^nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Example

List all the permutations of the letters in the word CAT.

Solution

There are $3 \times 2 \times 1 = 6$ such permutations: CAT, CTA, TAC, TCA, ACT, ATC.

Example

Seven runners are competing in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

The gold medal can be awarded in 7 ways. The silver medal can then be awarded in 6 ways. The bronze medal can then be awarded in 5 ways. The total number of ways of awarding the medals is $7 \times 6 \times 5 = 210$ ways.

The notation $\binom{n}{r}$

Let us return to the set of five letters $\{a, b, c, d, e\}$, but this time we are only interested in choosing three of the letters with no concern for order. The three letters a, b, c can be arranged in six ways:

$$abc, \qquad acb, \qquad bac, \qquad bca, \qquad cab, \qquad cba.$$

Similarly, any group of three letters can be arranged in six ways. The number of permutations of three letters chosen from five letters is ${}^5P_3 = 5 \times 4 \times 3 = 60$. Therefore the number of ways of choosing three letters from five letters is $\frac{60}{6} = 10$. We list them here:

$$\{a,b,c\},\quad \{a,b,d\},\quad \{a,b,e\},\quad \{a,c,d\},\quad \{a,c,e\},$$
$$\{a,d,e\},\quad \{b,c,d\},\quad \{b,c,e\},\quad \{b,d,e\},\quad \{c,d,e\}.$$

Remember that each set of three elements can be arranged in six ways.

We denote the number of ways of choosing r objects from n objects by $\binom{n}{r}$, which is read as 'n **choose** r'.

In the example above, we found that

$$\binom{5}{3} = \frac{{}^5P_3}{3!} = \frac{60}{6} = 10.$$

In general, we have

$$\binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}.$$

Consider the five-element set $\{a,b,c,d,e\}$ again. There are

$\binom{5}{1} = 5$ ways of choosing 1 letter, $\binom{5}{2} = 10$ ways of choosing 2 letters,

$\binom{5}{3} = 15$ ways of choosing 3 letters, $\binom{5}{4} = 10$ ways of choosing 4 letters,

$\binom{5}{5} = 1$ way of choosing 5 letters.

Listing these as a row, including choosing no letters as the first entry:

$$\binom{5}{0} = 1,\quad \binom{5}{1} = 5,\quad \binom{5}{2} = 10,\quad \binom{5}{3} = 10,\quad \binom{5}{4} = 5,\quad \binom{5}{5} = 1.$$

This is the fifth row of Pascal's triangle.

Example

Evaluate

1 $\binom{100}{2}$ **2** $\binom{1000}{998}$.

Solution

1 $\binom{100}{2} = \frac{100 \times 99}{2} = 4950$ **2** $\binom{1000}{998} = \frac{1000 \times 999}{2} = 499\,500.$

Example

In how many ways can you choose two people from a group of seven people?

Solution

There are $\binom{7}{2} = \dfrac{7 \times 6}{2} = 21$ ways of choosing two people from seven.

Example

There are ten people in a basketball squad. Find how may ways:

1 the starting five can be chosen from the squad
2 the squad can be split into two teams of five.

Solution

1 There are $\binom{10}{5} = \dfrac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$ ways of choosing the starting five.
2 The number of ways of dividing the squad into two teams of five is $\dfrac{252}{2} = 126$.

If we expand $(a+b)^6$, we know the terms will be of the form

$$c_0 a^6, \quad c_1 a^5 b, \quad c_2 a^4 b^2, \quad c_3 a^3 b^3, \quad c_4 a^2 b^4, \quad c_5 a b^5, \quad c_6 b^6,$$

where c_i are the coefficients. With

$$(a+b)^6 = (a+b)(a+b)(a+b)(a+b)(a+b)(a+b),$$

we can use combinations to find the coefficients.

For c_5, the relevant terms when multiplying out are

$$abbbbb, \quad babbbb, \quad bbabbb, \quad bbbabb, \quad bbbbab, \quad bbbbba.$$

There are $\binom{6}{1} = 6$ ways of choosing one a from the six brackets. Equivalently, there are $\binom{6}{5} = 6$ ways of choosing five b's from the six brackets. Therefore $c_5 = 6$.

We can find the values of the other coefficients in the same way. This is done in the following example.

Example

Write out the expansion of $(a+b)^{10}$.

Solution

The terms are

$$a^{10}, \quad c_1 a^9 b, \quad c_2 a^8 b^2, \quad c_3 a^7 b^3, \quad c_4 a^6 b^4, \quad c_5 a^5 b^5,$$
$$c_6 a^4 b^6, \quad c_7 a^3 b^7, \quad c_8 a^2 b^8, \quad c_9 a b^9, \quad b^{10},$$

where c_i are the coefficients.

Using a similar argument to that given above, we have

$$c_1 = \binom{10}{1}, \qquad c_2 = \binom{10}{2}, \qquad c_3 = \binom{10}{3}, \qquad \dots.$$

Therefore

$$\begin{aligned}(a+b)^{10} = a^{10} &+ 10a^9 b + 45a^8 b^2 + 120a^7 b^3 + 210a^6 b^4 + 252a^5 b^5 \\ &+ 210a^4 b^6 + 120a^3 b^7 + 45a^2 b^8 + 10ab^9 + b^{10}.\end{aligned}$$

Exercise 2

Let c_i denote the coefficient of the term $a^{12-i} b^i$ in the expansion of $(a+b)^{12}$.

Write down the values of the coefficients c_2, c_3, c_5 and c_9 using the notation $\binom{12}{r}$, and evaluate each of these coefficients.

The binomial theorem

We are now ready to prove the binomial theorem. We will give another proof later in the module using mathematical induction.

Theorem (Binomial theorem)

For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + \binom{n}{n-1} a b^{n-1} + b^n.$$

Proof

Suppose that we have n factors each of which is $a+b$. If we choose one letter from each of the factors of

$$(a+b)(a+b)(a+b)\cdots(a+b)$$

and multiply them all together, we obtain a term of the product. If we do this in every possible way, we will obtain all of the terms.

- If we choose a from every one of the factors, we get a^n. This can only be done in one way.
- We could choose b from one of the factors and choose a from the remaining $n-1$ factors. The number of ways of choosing one b from n factors is $\binom{n}{1}$. So the term with b is $\binom{n}{1}a^{n-1}b$.
- We could choose b from two of the factors and choose a from the remaining $n-2$ factors. The number of ways of choosing two b's from n factors is $\binom{n}{2}$. So the term with b^2 is $\binom{n}{2}a^{n-2}b^2$.
- In general, we choose b from r factors and choose a from the remaining $n-r$ factors. The number of ways of choosing r b's from n factors is $\binom{n}{r}$. So the term with b^r is $\binom{n}{r}a^{n-r}b^r$.
- If we choose b from every one of the factors, we get b^n. This can be done in only one way.

Thus,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

□

The binomial theorem can also be stated using summation notation:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r.$$

Substituting with $a=1$ and $b=x$ gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

We can now display Pascal's triangle with the notation of the binomial theorem.

Pascal's triangle using the binomial theorem

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	$\binom{0}{0}$								
1	$\binom{1}{0}$	$\binom{1}{1}$							
2	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{2}{2}$						
3	$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$					
4	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$				
5	$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$			
6	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$		
7	$\binom{7}{0}$	$\binom{7}{1}$	$\binom{7}{2}$	$\binom{7}{3}$	$\binom{7}{4}$	$\binom{7}{5}$	$\binom{7}{6}$	$\binom{7}{7}$	
8	$\binom{8}{0}$	$\binom{8}{1}$	$\binom{8}{2}$	$\binom{8}{3}$	$\binom{8}{4}$	$\binom{8}{5}$	$\binom{8}{6}$	$\binom{8}{7}$	$\binom{8}{8}$

The nth row of this table gives the coefficients of $(1+x)^n$, where $\binom{n}{r}$ is the coefficient of x^r in this expansion. Numbers of the form $\binom{n}{r}$ are called **binomial coefficients**.

Pascal's triangle — the observations

We return to the observations made in the section *A look at Pascal's triangle*.

Observation 1

Each number in Pascal's triangle is the sum of the two numbers diagonally above it (with the exception of the 1*'s).*

For example, from the fifth and fourth rows of Pascal's triangle, we have $10 = 4+6$. In the notation introduced earlier in this module, this says

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

We now describe the general pattern.

Theorem (Pascal's identity)

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad \textit{for} \quad 0 < r < n.$$

Proof

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\
&= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\
&= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left(\frac{n}{(n-r)r}\right) \\
&= \frac{n!}{(n-r)!r!} \\
&= \binom{n}{r}.
\end{aligned}$$

□

We will now undertake an alternative proof of Pascal's identity in a special case, but the general proof is essentially the same.

Let us start with a six-element set

$$\{a, b, c, d, e, f\}.$$

The number of subsets with four elements is $\binom{6}{4}$. Now let us single out the letter c. The subsets with four elements can be separated into two types:

- the four-element sets that contain c
- the four-element sets that do not contain c.

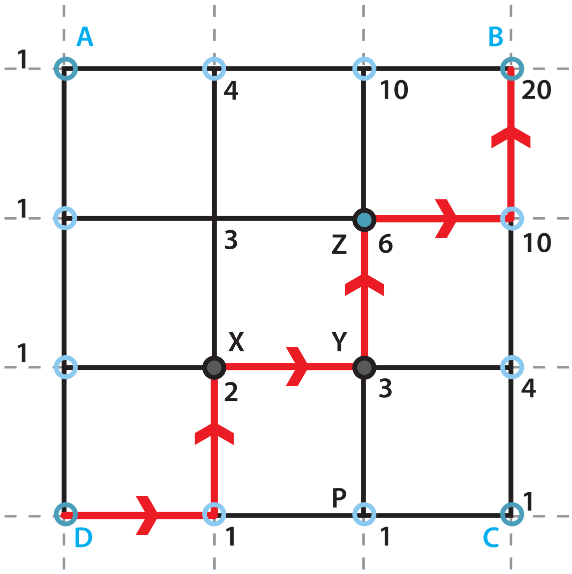
The number of four-element subsets that contain c is $\binom{5}{3}$. The number of four-element subsets that do not contain c is $\binom{5}{4}$. Hence,

$$\binom{6}{4} = \binom{5}{3} + \binom{5}{4}.$$

This gives us a combinatorial proof of Pascal's identity.

Application — Number of paths across a grid

The following diagram shows a 3×3 grid. The question arises as to how many paths there are from the bottom-left corner to the top-right corner of the grid, if you can only move to the right or up. We will call such moves 'forward' moves. One path from the bottom-left corner to the top-right corner is shown in red.



A forward path across a 3×3 grid.

The numbers on the grid indicate the number of paths to that point. These numbers form part of Pascal's triangle. You can move up and across the grid using addition to find the number of paths to any point on the grid. For example:

- To get to the point Y you must go through point P or point X. There is one path to P and two paths to X. Therefore there are three paths to Y.

In the same way you can work out all the numbers in the grid, ending with the 20 paths from D to B.

There is a way to calculate the total number of paths directly. We can think of the paths as strings of letters. We will call a move one to the right R and a move one up U. The path shown in the diagram is $RURURU$. There are six moves in each string that takes you from D to B. There must be three U's and three R's in each such string. If you choose where to place the three R's in the string, then the remaining places must be filled with U's. So there are $\binom{6}{3} = 20$ paths from D to B.

This problem can be solved for an $n \times n$ grid by thinking of the paths as strings of $2n$ letters. It takes $2n$ moves to go from the bottom-left corner of an $n \times n$ grid to the top-right corner, and n of these moves are U and n are R.

A path could be $UUURRR \dots RUR \dots RR$. There are $2n$ letters in the string and there are n U's and n R's. If you choose where to place the n R's in the string, then the remaining places must be filled with U's. There are $\binom{2n}{n}$ ways of doing this and so, for an $n \times n$ grid, there are $\binom{2n}{n}$ paths.

Exercise 3

Show that the number of paths in an $m \times n$ grid moving from the bottom-left corner to the top-right corner with only *forward* moves allowed is $\binom{m+n}{n}$.

Observation 2

Each row of Pascal's triangle is symmetric.

Clearly

$$\binom{n}{r} = \binom{n}{n-r},$$

since choosing r objects from n objects leaves $n-r$ objects, and choosing $n-r$ objects leaves r objects. This means that the coefficient of x^r in the expansion of $(1+x)^n$ is the same as the coefficient of x^{n-r}.

Observation 3

The sum of each row of Pascal's triangle is a power of 2. In fact, the sum of the entries in the nth row is 2^n.

Proof

We use the binomial theorem in the form

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

Substituting $x = 1$ gives

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{r} + \cdots + \binom{n}{n-1} + \binom{n}{n}. \qquad \square$$

Application — Number of subsets of a set

There is another way of proving the observation above. Here is an example to illustrate the idea. There are three different chocolates A, B and C, and Alison can choose to eat whichever ones she wants. How many ways can this choice be made?

Let $S = \{A, B, C\}$. The subsets are

$$\emptyset, \quad \{A\}, \quad \{B\}, \quad \{C\}, \quad \{A, B\}, \quad \{A, C\}, \quad \{B, C\}, \quad \{A, B, C\}.$$

There are

$$\binom{3}{0} = 1 \text{ set with no elements,} \qquad \binom{3}{1} = 3 \text{ sets with one element,}$$

$$\binom{3}{2} = 3 \text{ sets with two elements,} \qquad \binom{3}{3} = 1 \text{ set with three elements.}$$

Hence the total number of choices is

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}.$$

Now think of each chocolate one at a time:

- chocolate A can be chosen or not (2 ways)
- chocolate B can be chosen or not (2 ways)
- chocolate C can be chosen or not (2 ways).

So there are a total number of $2^3 = 8$ ways of choosing chocolates (including not choosing any). This gives

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3.$$

The argument can easily be extended to prove the result for a set of n elements.

Observation 4

In any row of Pascal's triangle, the sum of the first, third, fifth, ... numbers is equal to the sum of the second, fourth, sixth, ... numbers.

To prove this observation, we use the binomial theorem in the form

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

Let $x = -1$. Then

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^r\binom{n}{r} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}.$$

Therefore

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

which is the observation.

This shows that, for the expansion of $(1+x)^n$, the sum of the coefficients of the even powers is equal to the sum of the coefficients of the odd powers. Using Observation 3, it follows that each of these sums is equal to 2^{n-1}.

For example, for $n = 6$ we have

$$\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6} = 2^5 = \binom{6}{1}+\binom{6}{3}+\binom{6}{5},$$

and for $n = 7$ we have

$$\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6} = 2^6 = \binom{7}{1}+\binom{7}{3}+\binom{7}{5}+\binom{7}{7}.$$

Exercise 4

a Find an identity by substituting $x = 2$ into the binomial expansion for $(1+x)^n$.

b Find an identity by substituting $x = -2$ into the binomial expansion for $(1+x)^n$.

Applying the binomial theorem

In this section, we give some examples of applying the binomial theorem.

Example

Expand $(2x+3)^4$.

Solution

Using the binomial theorem:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Let $a = 2x$ and $b = 3$. Then

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + 4(2x)^3 \times 3 + 6(2x)^2 \times 3^2 + 4(2x) \times 3^3 + 3^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81.\end{aligned}$$

Example

Expand $(1-x)^{10}$.

Solution

Using the binomial theorem:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r.$$

Let $a = 1$, $b = -x$ and $n = 10$. Then

$$(1-x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-x)^r$$

$$= 1 - 10x + 45x^2 - 120x^3 + 210x^4 - 252x^5 + 210x^6 - 120x^7 + 45x^8 - 10x^9 + x^{10}.$$

Exercise 5

Expand

a $(2x-3y)^4$ **b** $\left(x - \dfrac{2}{x}\right)^4$.

The general term in the binomial expansion

The general term in the expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r} b^r, \qquad \text{where} \quad 0 \le r \le n.$$

Example

For the expansion of $(x-3z^2)^5$, find the general term and find the coefficient of $x^3 z^4$.

Solution

The general term is

$$\binom{5}{r} x^{5-r} (-3z^2)^r, \qquad \text{where} \quad 0 \le r \le 5.$$

To find the coefficient of $x^3 z^4$, we want $5 - r = 3$ and $r = 2$.

Thus the coefficient of $x^3 z^4$ is $\binom{5}{2}(-3)^2 = 90$.

Exercise 6

Consider the expression $(a-2b^3)^7$. How many terms are there in the expansion? Find a formula for the general term.

Example

Find the constant term in the expansion of $\left(x^2+\dfrac{1}{x^2}\right)^6$.

Solution

The general term is

$$\begin{aligned}\binom{6}{r}(x^2)^{6-r}\left(\frac{1}{x^2}\right)^r &= \binom{6}{r}x^{12-2r}\left(\frac{1}{x^{2r}}\right)\\ &= \binom{6}{r}x^{12-4r}.\end{aligned}$$

This term will be a constant when $12-4r=0$, that is, when $r=3$.

Hence the constant term in the expansion of $\left(x^2+\dfrac{1}{x^2}\right)^6$ is $\dbinom{6}{3}=20$.

Exercise 7

Find the constant term in the expansion of $\left(x+\dfrac{1}{x^2}\right)^6$.

Example

Find the coefficient of x^5 in the expansion of $(1-2x+3x^2)^5$.

Solution

Bracket $1-2x$ and expand:

$$(1-2x+3x^2)^5 = (1-2x)^5 + 5(1-2x)^4(3x^2) + 10(1-2x)^3(3x^2)^2 + \cdots.$$

- The coefficient of x^5 in $(1-2x)^5$ is $(-2)^5=-32$.
- The coefficient of x^5 in $5(1-2x)^4(3x^2)$ is $15\binom{4}{3}(-2)^3=-480$.
- The coefficient of x^5 in $10(1-2x)^3(3x^2)^2$ is $90\binom{3}{1}(-2)=-540$.
- The remaining terms do not contain x^5.

Therefore the required coefficient is $-32-480-540=-1052$.

The middle term

When n is even, there will be an odd number of terms in the expansion of $(a+b)^n$, and hence there will be a middle term. Let $n = 2m$, for some positive integer m. Then, when the expansion of $(a+b)^n$ is arranged with terms in descending or ascending order, the middle term is

$$\binom{2m}{m}a^m b^m.$$

For example, the middle term of $(a+b)^8$ is

$$\binom{8}{4}a^4 b^4 = 70a^4 b^4.$$

When n is odd, there will be two middle terms of $(a+b)^n$. Let $n = 2m+1$, for some positive integer m. Then the middle terms are

$$\binom{2m+1}{m}a^{m+1}b^m \qquad \text{and} \qquad \binom{2m+1}{m+1}a^m b^{m+1}.$$

Exercise 8

Find the middle term of $(2x-3y)^6$.

Greatest coefficients

When n is even, we can show that the greatest coefficient of $(1+x)^n$ is the coefficient of the middle term, which is

$$\binom{n}{\frac{n}{2}}.$$

When n is odd, there are two greatest coefficients, which are the coefficients of the two middle terms

$$\binom{n}{\frac{1}{2}(n-1)} \qquad \text{and} \qquad \binom{n}{\frac{1}{2}(n+1)}.$$

In general, finding the greatest coefficients for an expansion can be undertaken in a systematic fashion, as shown in the following example.

Example

Find the greatest coefficient in the expansion of $(1+3x)^{21}$.

Solution

The general term of this expansion is $\binom{21}{k}(3x)^k$, and its coefficient is $c_k = \binom{21}{k}3^k$. The next coefficient is $c_{k+1} = \binom{21}{k+1}3^{k+1}$.

We have

$$\begin{aligned}\frac{c_{k+1}}{c_k} &= \frac{\binom{21}{k+1}3^{k+1}}{\binom{21}{k}3^k}\\ &= \frac{21!}{(20-k)!(k+1)!} \times \frac{(21-k)!k!}{21!} \times 3\\ &= \frac{63-3k}{k+1}.\end{aligned}$$

To find where the coefficients are increasing, we solve $c_{k+1} > c_k$, that is, $\frac{c_{k+1}}{c_k} > 1$.

From above, $\frac{63-3k}{k+1} > 1$, which is equivalent to $k < 15\frac{1}{2}$.

Now k is an integer, and hence $c_{k+1} > c_k$ for $k = 0, 1, 2, \dots, 14, 15$.

The sequence of coefficients is increasing from c_0 to c_{16} and decreasing from c_{16} to c_{21}. Hence c_{16} is the largest coefficient:

$$c_{16} = \binom{21}{16}3^{16} = 875\,957\,725\,629.$$

Exercise 9

Find the greatest coefficient of $(2x+3y)^{15}$.

Proof of the binomial theorem by mathematical induction

In this section, we give an alternative proof of the binomial theorem using mathematical induction. We will need to use Pascal's identity in the form

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}, \qquad \text{for} \quad 0 < r \le n.$$

We aim to prove that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

We first note that the result is true for $n = 1$ and $n = 2$.

Let k be a positive integer with $k \geq 2$ for which the statement is true. So

$$(a+b)^k = a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k.$$

Now consider the expansion

$$\begin{aligned}
&(a+b)^{k+1}\\
&= (a+b)(a+b)^k\\
&= (a+b)\left(a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k\right)\\
&= a^{k+1} + \left[1 + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \cdots\\
&\qquad \cdots + \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k-r+1}b^r + \cdots + \left[\binom{k}{k-1} + 1\right]ab^k + b^{k+1}.
\end{aligned}$$

From Pascal's identity, it follows that

$$(a+b)^{k+1} = a^{k+1} + \binom{k+1}{1}a^k b + \cdots + \binom{k+1}{r}a^{k-r+1}b^r + \cdots + \binom{k+1}{k}ab^k + b^{k+1}.$$

Hence the result is true for $k+1$. By induction, the result is true for all positive integers n.

Further identities and results

Sum of the squares of the binomial coefficients

In the following diagram of Pascal's triangle, we see that by summing the squares of the numbers in the fourth row,

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70,$$

we get the middle number of the eighth row.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

Sum of the squares for a row of Pascal's triangle.

We now explain the general pattern.

Using the fact that

$$\binom{n}{r} = \binom{n}{n-r},$$

we can write the binomial expansion in two different ways:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n,$$

$$(1+x)^n = \binom{n}{n} + \binom{n}{n-1}x + \binom{n}{n-2}x^2 + \cdots + \binom{n}{r}x^{n-r} + \cdots + \binom{n}{1}x^{n-1} + \binom{n}{0}x^n.$$

Multiplying these two expansions together, we see that in $(1+x)^n(1+x)^n$ the coefficient of x^n is

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{r}^2 + \cdots + \binom{n}{n}^2.$$

Also, since

$$(1+x)^n(1+x)^n = (1+x)^{2n},$$

the coefficient of x^n in this expansion is $\binom{2n}{n}$. Equating the two expressions for the coefficient of x^n we have

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{r}^2 + \cdots + \binom{n}{n}^2.$$

Using summation notation, this identity is

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2.$$

The case for $n = 4$ was given at the start of this section:

$$\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2.$$

Exercise 10

The number of forward paths from the bottom left to the top right of an $n \times n$ grid is $\binom{2n}{n}$. Use an argument involving paths in the grid to prove

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{r}^2 + \cdots + \binom{n}{n}^2.$$

(*Hint.* Draw a diagonal line across the grid from the top-left corner to the bottom-right corner. Choose a point of the grid on the diagonal line. Consider the number of forward paths from the bottom-left corner to the point and then the number of paths from the point to the top-right corner. Repeat for each point on the diagonal line.)

Yet more results and identities

We give some further results that come from applying the binomial theorem to $(1+x)^n$.

Exercise 11

Each of the following expansions has three coefficients in arithmetic progression:

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7,$$

$$(1+x)^{14} = 1 + 14x + 91x^2 + 364x^3 + 1001x^4 + 2002x^5 + 3003x^6 + 3432x^7 + \cdots,$$

$$(1+x)^{23} = \cdots + 490\,314x^8 + 817\,190x^9 + 1\,144\,066x^{10} + \cdots.$$

In the first expansion, where $n = 7$, the arithmetic progression is 7, 21, 35. In the second expansion, where $n = 14$, the progression is 1001, 2002, 3003.

For the expansion of $(1+x)^n$, where $n > 2$, prove that if the coefficients of three consecutive powers are in arithmetic progression, then $n+2$ is a perfect square.

(*Hint.* Consider three coefficients $\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$.)

Example

Prove the identity

$$n \times 2^{n-1} = \sum_{k=0}^{n} k \times \binom{n}{k}$$

and check in the case when $n = 4$.

Solution

We begin with the expansion of $(1+x)^n$ from the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiating both sides of this identity with respect to x gives

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}.$$

Now substitute $x = 1$ to obtain the result:

$$n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}.$$

When $n = 4$, the left-hand side is $4 \times 2^3 = 32$ and the right-hand side is

$$\begin{aligned} 4 + 2 \times \binom{4}{2} + 3 \times \binom{4}{3} + 4 &= 4 + 2 \times 6 + 3 \times 4 + 4 \\ &= 32. \end{aligned}$$

If we substitute $x = -1$ into the identity from the previous example,

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1},$$

then we obtain

$$\sum_{k=0}^{n} k \binom{n}{k} (-1)^{k-1} = 0.$$

For example, when $n = 6$ this gives

$$\binom{6}{1} + 3\binom{6}{3} + 5\binom{6}{5} = 2\binom{6}{2} + 4\binom{6}{4} + 6\binom{6}{6}.$$

A link to the Fibonacci numbers

Let $F(n)$ denote the sum

$$F(n) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

where we keep summing until the lower number exceeds the top one. Then

$$F(n+1) = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots.$$

Adding the first term in line 1 with the second in line 2 and so on, we get

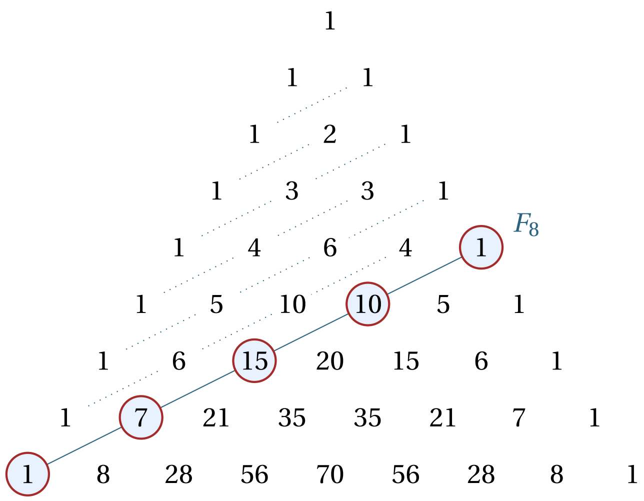
$$F(n+1) + F(n) = \binom{n+1}{0} + \left[\binom{n}{0} + \binom{n}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] + \cdots.$$

Using Pascal's identity, we obtain

$$F(n+1) + F(n) = \binom{n+2}{0} + \binom{n+1}{1} + \binom{n}{2} + \cdots = F(n+2).$$

Since $F(0) = F(1) = 1$, we have shown that $F(0)$, $F(1)$, $F(2)$, $F(3)$, ... is the sequence of **Fibonacci numbers**.

This result can be demonstrated in terms of Pascal's triangle.



$$1 + 7 + 15 + 10 + 1 = 34 = F_8$$

The Fibonacci numbers in Pascal's triangle.

Links forward

A result using complex numbers

Suppose we expand out the expressions $(1+1)^n$, $(1+i)^n$, $(1-1)^n$, $(1-i)^n$ using the binomial theorem. This gives (for $n > 0$),

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n$$

$$\binom{n}{0}+i\binom{n}{1}-\binom{n}{2}-i\binom{n}{3}+\cdots+i^n\binom{n}{n}=(1+i)^n$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n}=0^n$$

$$\binom{n}{0}-i\binom{n}{1}-\binom{n}{2}+i\binom{n}{3}+\cdots+(-i)^n\binom{n}{n}=(1-i)^n.$$

Now add these equations and divide by 4 to get

$$\binom{n}{0}+\binom{n}{4}+\binom{n}{8}+\cdots=\frac{1}{4}\Big(2^n+(1-i)^n+(1+i)^n\Big).$$

It is easy to show, using the polar form, that

$$(1-i)^n+(1+i)^n=2(\sqrt{2})^n\cos\big(\frac{n\pi}{4}\big).$$

Thus, we have

$$\binom{n}{0}+\binom{n}{4}+\binom{n}{8}+\cdots=\frac{1}{4}\Big(2^n+2(\sqrt{2})^n\cos\big(\frac{n\pi}{4}\big)\Big).$$

This is a rather amazing and beautiful result.

Series for e

One way of defining the number e is

$$\lim_{n\to\infty}\Big(1+\frac{1}{n}\Big)^n.$$

The binomial theorem gives

$$\Big(1+\frac{1}{n}\Big)^n=\binom{n}{0}+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\Big(\frac{1}{n}\Big)^2+\cdots+\binom{n}{r}\Big(\frac{1}{n}\Big)^r+\cdots+\binom{n}{n-1}\Big(\frac{1}{n}\Big)^{n-1}+\binom{n}{n}\Big(\frac{1}{n}\Big)^n.$$

Using the result that

$$\lim_{n\to\infty}\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{1}{k!},$$

we obtain the result

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots.$$

Formally, this argument involves some difficult limiting processes.

The generalised binomial theorem

The generalised binomial theorem was discovered by Isaac Newton (1642–1727). It states that, for $|x| < 1$,

$$(1+x)^r = \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k$$

where r is any real number.

For $r = -1$, this is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

The expansion for $(1-x)^{-1}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

These are examples of infinite geometric series. We know that they are only convergent for $|x| < 1$.

History and applications

Applications

The derivative of x^n

The binomial theorem can be used to give a proof of the derivative of x^n.

The derivative of x^n with respect to x is given by

$$\frac{d(x^n)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Using the binomial theorem,

$$\begin{aligned}\frac{(x+h)^n - x^n}{h} &= \frac{1}{h}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k - x^n\right)\\ &= \frac{1}{h}\left(\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k\right)\\ &= \sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}.\end{aligned}$$

Thus

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Approximations

The binomial theorem can be used to find approximations. For example for 1.01^5 we have

$$(1+0.01)^5 \approx 1 + 5\times 0.01 + 10\times 0.01^2 = 1 + 0.05 + 10\times 0.0001 = 1.051.$$

Of course, with calculators, it is not necessary to go through such a process. However we can see that $(1+x)^n > 1+nx$ for $x>0$, which is a very useful inequality.

History

The expansion $(a+b)^2 = a^2 + 2ab + b^2$ appears in Book 2 of Euclid. There it is stated in geometric terms:

> *If a line segment is cut at random, the square on the whole is equal to the square on each of the line segments which have been formed and twice the rectangle which has these line segments as length and width.*

The triangle of numbers that we refer to as Pascal's triangle was known before Pascal. Pascal developed many uses of it and was the first one to organise all the information together in his 1653 treatise.

The triangle had been discovered centuries earlier in India and China. In the 13th century, Yang Hui (1238–1298) knew of this triangle of numbers. In China, Pascal's triangle is called Yang Hui's triangle. The triangle was known in China in the early 11th century by the mathematician Jia Xian (1010–1070). It was also discussed by the Persian poet-astronomer-mathematician Omar Khayyam (1048–1131). In Iran, the triangle is referred to as the Khayyam–Pascal triangle or simply the Khayyam triangle. Several theorems related to the triangle were known, including the binomial theorem for non-negative integer exponents. In Europe, it first appeared as the frontispiece of a book by Petrus Apianus (1495–1552) and this is the first printed record of the triangle in Europe. In Italy, it is still referred to as Tartaglia's triangle, named for the Italian algebraist Niccolo Tartaglia (1500–1577).

Although Pascal was not the first to study this triangle, his work on the topic was the most important. Pascal's work on the binomial coefficients led to Newton's discovery of the general binomial theorem for fractional and negative powers.

Answers to exercises

Exercise 1

a There are $3 \times 2 = 6$ terms in the expansion.

b There are $2^5 = 32$ terms in the expansion.

Exercise 2

$c_2 = \binom{12}{2} = 66, \quad c_3 = \binom{12}{3} = 220, \quad c_5 = \binom{12}{5} = 792, \quad c_9 = \binom{12}{9} = 220.$

Exercise 3

The possible strings of R's and U's have length $m+n$. If the grid has height m and width n, then there must be m U's and n R's. So the number of ways of choosing the paths is $\binom{m+n}{n} = \binom{m+n}{m}$.

Exercise 4

The binomial theorem gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

When $x = 2$, we have

$$\begin{aligned} 3^n &= (1+2)^n \\ &= \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \cdots + \binom{n}{r}2^r + \cdots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n \\ &= 1 + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^r\binom{n}{r} + \cdots + 2^{n-1}\binom{n}{n-1} + 2^n. \end{aligned}$$

When $x = -2$, we have

$$\begin{aligned}
(-1)^n &= (1-2)^n \\
&= \binom{n}{0} + \binom{n}{1}(-2) + \binom{n}{2}(-2)^2 + \cdots + \binom{n}{r}(-2)^r + \cdots + \binom{n}{n-1}(-2)^{n-1} + \binom{n}{n}(-2)^n \\
&= 1 - 2\binom{n}{1} + 4\binom{n}{2} + \cdots + (-2)^r\binom{n}{r} + \cdots + (-2)^{n-1}\binom{n}{n-1} + (-2)^n.
\end{aligned}$$

Exercise 5

a $(2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

b $\left(x - \dfrac{2}{x}\right)^4 = 24 + \dfrac{16}{x^4} - \dfrac{32}{x^2} - 8x^2 + x^4$

Exercise 6

There are eight terms in the expansion.

The general term is $(-2)^r\binom{7}{r}a^{7-r}b^{3r}$, for $0 \le r \le 7$.

Exercise 7

The constant term is 15.

Exercise 8

The middle term is $-4320x^3y^3$.

Exercise 9

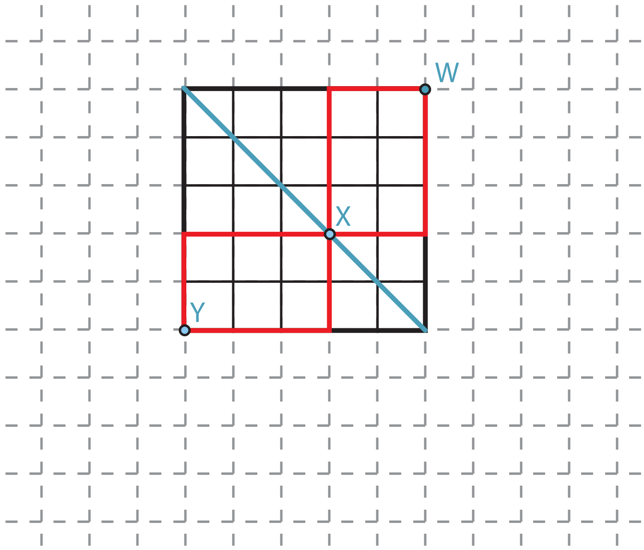
$c_9 = 2^6 \times 3^9\binom{15}{9} = 6\,304\,858\,560$

Exercise 10

In the following diagram, there are $\binom{3+2}{3}$ paths from Y to X, and $\binom{3+2}{2}$ paths from X to W. So the number of paths from Y to W that pass through X is

$$\binom{3+2}{3}\binom{3+2}{2} = \binom{5}{3}\binom{5}{2} = \binom{5}{2}^2.$$

Every path from Y to W meets the diagonal, and so the total number of paths from Y to W can be found by summing the number of such paths through each point on the diagonal. The general result follows from the same argument.



Exercise 11

Assume that the coefficients of three consecutive powers of $(1+x)^n$ are in arithmetic progression. Let the three coefficients be

$$\binom{n}{r-1}, \qquad \binom{n}{r}, \qquad \binom{n}{r+1}.$$

Then

$$\binom{n}{r}-\binom{n}{r-1}=\binom{n}{r+1}-\binom{n}{r}$$

$$\frac{n!}{(n-r)!r!}-\frac{n!}{(n-r+1)!(r-1)!}=\frac{n!}{(n-r-1)!(r+1)!}-\frac{n!}{(n-r)!r!}$$

$$\frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{r}-\frac{1}{n-r+1}\right)=\frac{n!}{(n-r-1)!r!}\left(\frac{1}{r+1}-\frac{1}{n-r}\right)$$

$$\frac{n-2r+1}{n-r+1}=\frac{n-2r-1}{r+1}$$

$$(n-2r)^2=n+2.$$

For this to be possible, $n+2$ is a perfect square.