VCAA-AMSI maths modules

A guide for teachers – Years 11 and 12

Geometry

Transformations

DRAFT

Years 11 & 12

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Transformations – A guide for teachers (Years 11–12)

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Transformations

Assumed knowledge

- Familiarity with functions, including injective, surjective and inverse functions;
- Familiarity with linear equations, including solving simultaneous equations.
- Familiarity with vectors and 2×2 matrices, including position vectors and matrix multiplication and inverses;
- Familiarity with plane Euclidean geometry.

Motivation

> Give me a place to stand and I will move the earth.
>
> – Archimedes

As in many aspects of life, it's often useful in mathematics to see things from different perspectives. Changing your point of view involves some kind of *transformation*.

You see geometric transformations every minute of every day: whenever you walk around an object, you see a rotation. When you see an image of an object, it is a projection. When you move an object, you see a translation. When you look in a mirror, you see a reflection.

In this module we focus on transformations of the *plane*. Plane transformations encompass a great deal of geometry, including rotations, reflections, projections, translations, and many other operations.

We mostly focus on transformations of a very particular type: *linear transformations*. Linear transformations form a narrow, but useful, class of plane transformations. We will see how such transformations can be described by some relatively simple algebra.

Studying plane transformations brings together algebra and geometry in a coherent and elegant way. We will see that the algebra of *linear functions* and *matrices* efficiently describes a wide array of geometric transformations.

Content

Transformations of the plane

> Not everything that is faced can be changed. But nothing can be changed until it is faced. – James Baldwin

A *plane transformation* is just a function from the plane to itself, i.e. a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. Such an F takes a point (x, y) to another point, which we can denote (x', y') or $F(x, y)$.[1] You can think of x, y as inputs to f, and x', y' as outputs of f.

By our definition, any function $\mathbb{R}^2 \to \mathbb{R}^2$ whatsoever is a plane transformation. E.g.:

$$F(x, y) = (0, 5), \quad F(x, y) = (2x + y, 5x - 3y), \quad F(x, y) = (x - 2y + 3, -4x + 5y - 6),$$

$$F(x, y) = (x^2 + y^3, e^y \sqrt[3]{x}), \quad F(x, y) = (\lfloor xy \rfloor, \sin(x + y)).$$

(However, in this module we will not deal with functions as complicated as the last two.)

It's not possible to draw a graph of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ in the same way we draw the graph $y = f(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$. For a function $f : \mathbb{R} \to \mathbb{R}$, its graph consists of a plot of all points (x, y) such that $y = f(x)$. To draw a similar graph of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, we would need to plot all (x, y, x', y') such that $(x', y') = F(x, y)$, which would require *four* coordinates — it would have to be drawn in four-dimensional space! As we will see, however, there are other ways to visualise a plane transformation.

In this module, we will do the following.

- We will briefly examine some *geometry*, introducing a few examples of plane transformations arising from geometric operations.
- Then, we will examine some *algebra* of vectors and *matrices*, and discuss *linear* transformations, which arise from this algebra.
- Next, we will examine some geometric transformations in depth: rotations, translations, projections, dilations, and more. We will investigate the relationship between the geometry and algebra of these operations.
- Finally, we will discuss transformations of *graphs*, discussing what happens to a graph $y = f(x)$ when it is subject to a transformation.
- In the *Links Forward* section, we will discuss some further concepts such as *determinants*, *isometries*, and how the ideas of this module can be generalised from the 2-dimensional plane to 3-dimensional space, and even to higher dimensions.

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[1] Note the primes on x', y' have nothing to do with derivatives!

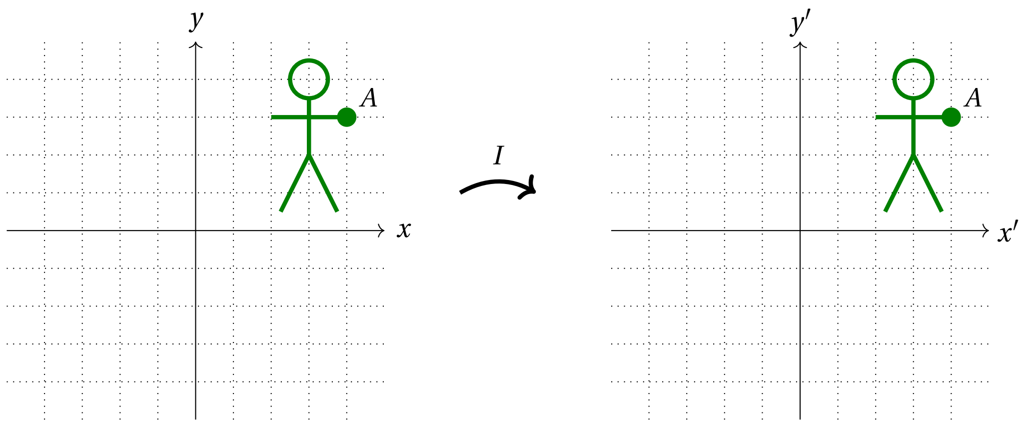
Warning regarding notation. Throughout this module, we often blur the distinction between a point (x, y) in the plane, and its position vector $\begin{bmatrix} x \\ y \end{bmatrix}$. This is a slight abuse of notation, but it may be useful, and we hope it will not be too confusing.

Geometric examples of transformations

Some transformations arise from very natural and intuitive geometric operations.

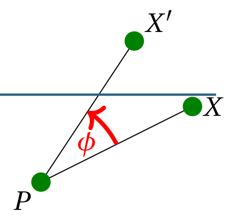
Identity

The laziest plane transformation of all is the one that *does nothing.* This transformation is known as the *identity* function $I : \mathbb{R}^2 \to \mathbb{R}^2$. It sends every point to itself, $I(x, y) = (x, y)$.

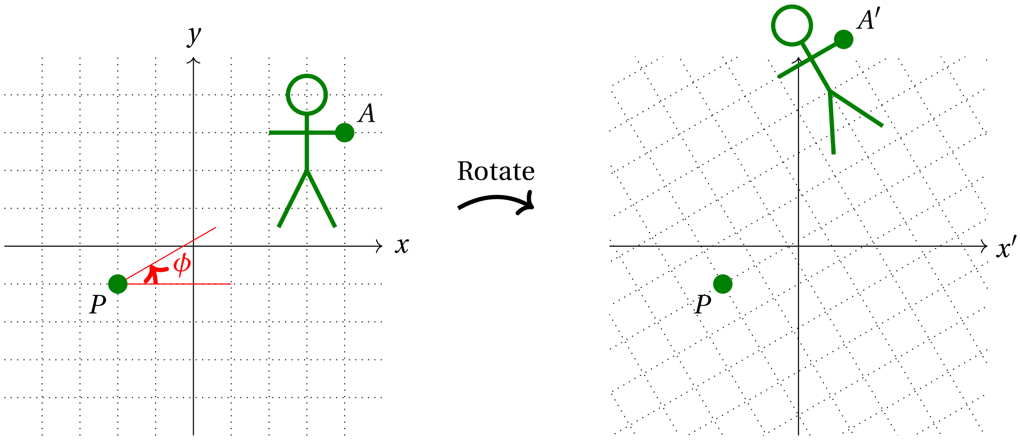


Rotations

Given a point P in the plane, and an angle ϕ,[2] we can consider rotating points around P by the angle ϕ. Any point X can be rotated in this way, giving rise to a plane transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

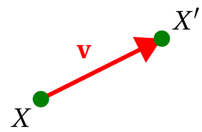


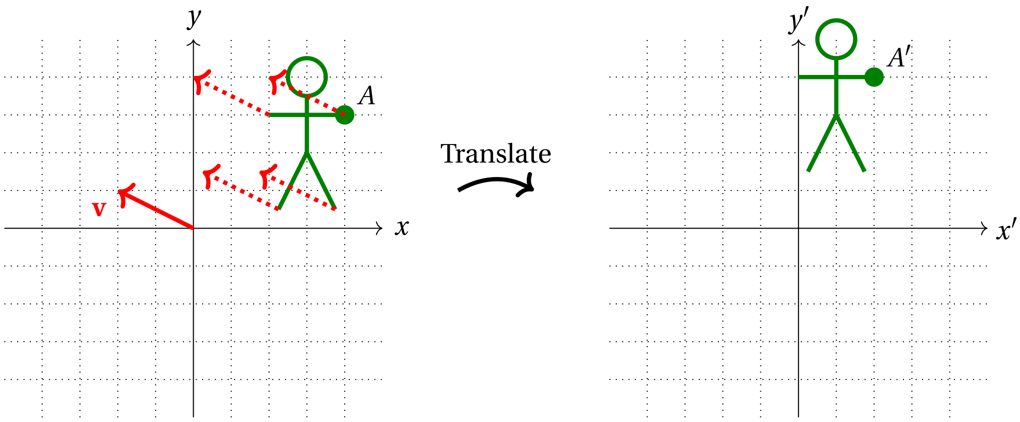
[2] This is the Greek letter phi.



Translations

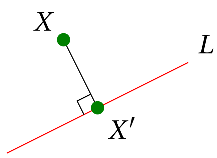
Given a vector $\mathbf{v}$, we can consider *translating* points by $\mathbf{v}$. Any point X can be moved along by the vector $\mathbf{v}$ to obtain a point X', and in this way we obtain a plane transformation.

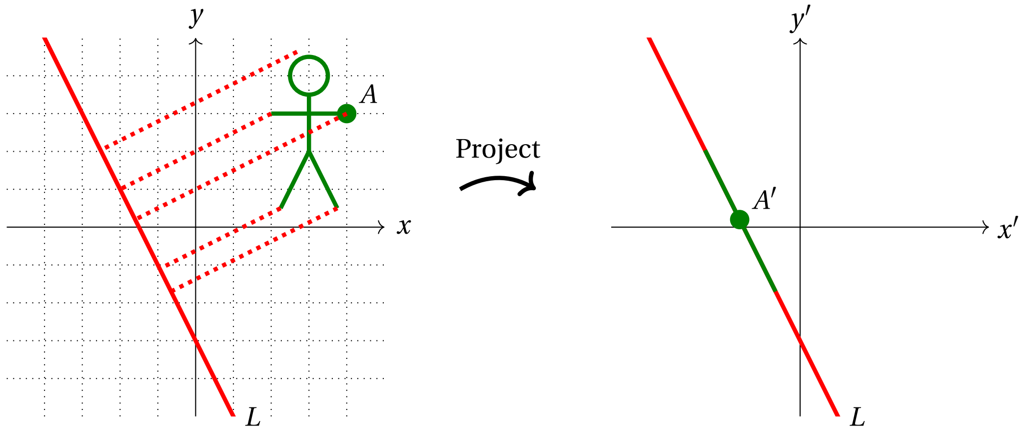




Projections

Given a line L in the plane, we can consider *projecting* points onto L. From a point X, draw the line through X perpendicular to L; the intersection of this line with L is the *projection* X' of X onto L. Projecting all points of the plane onto L, we obtain a plane transformation.



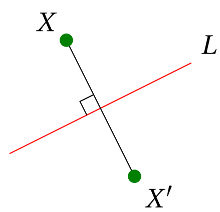


Exercise 1

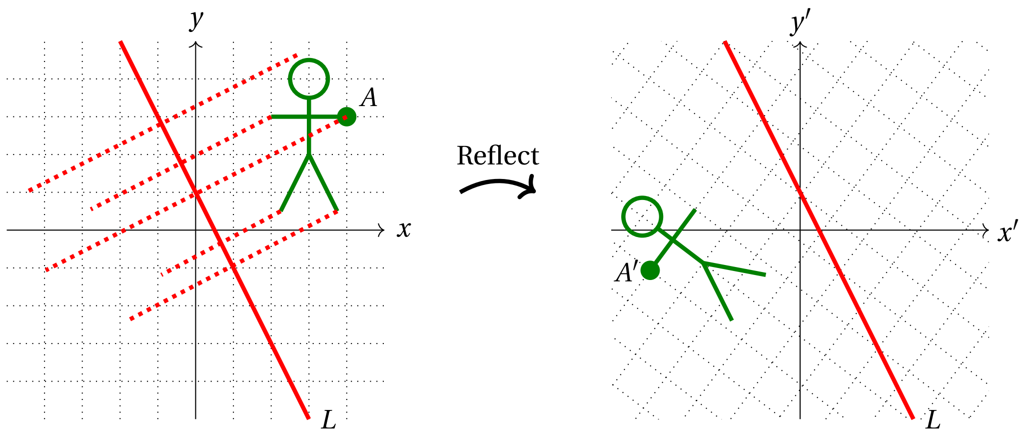
Show that the projection X' of X onto L is the closest point on L to X.

Reflections

Take a line L and think of it as a mirror; we can then reflect points in L. Given a point X not on L, suppose you are standing at X; your reflection X' in L is where you appear to be when you look in the mirror. Reflecting each point in this way we obtain a plane transformation.

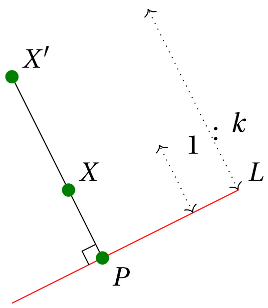


To find X', we can draw the line perpendicular to L through X; X' is the point on this perpendicular, on the other side of L, at the same distance from L as X.

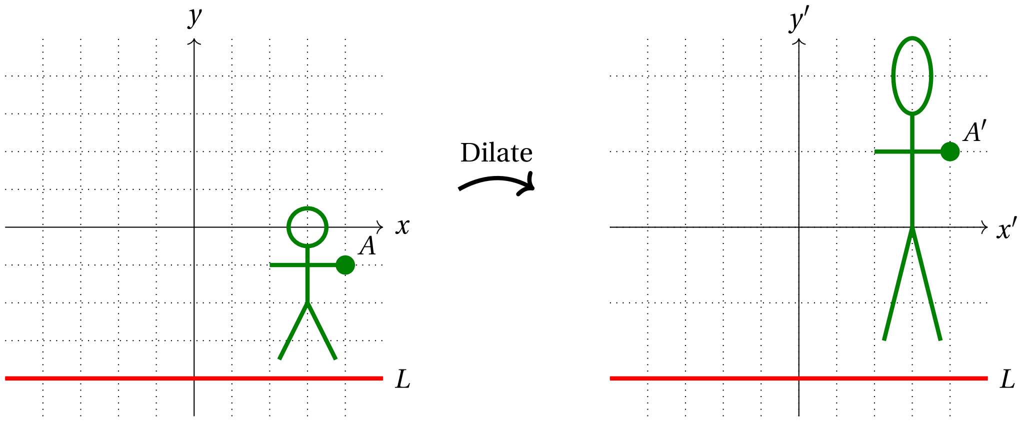


Dilations

Given a line L and a real number k, we can *dilate* points from L by a factor k. From a point X not on L, once more we draw the perpendicular from X to L. The dilation X' of X from L with factor k is the point X' on the perpendicular which is k times as far away from L as X is. In other words, if P is the foot of the perpendicular from X to L, then $\overrightarrow{PX'} = k\,\overrightarrow{PX}$.



Again we have a plane transformation, which stretches the plane out from L by a factor of k. Note that if k is positive, then X' lies on the same side of L as X; if k is negative, then X' lies on the opposite side of L from X.



Linear transformations and matrices

We will now take a more algebraic approach to transformations of the plane.

As it turns out, *matrices* are very useful for describing transformations. Whenever we have a 2×2 *matrix* of real numbers

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we can naturally define a plane transformation $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_M(\mathbf{v}) = M\mathbf{v}.$$

That is, T_M takes a vector $\mathbf{v}$ and multiplies it on the left by the matrix M. If $\mathbf{v}$ is the position vector of the point (x, y), then

$$T_M(\mathbf{v}) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

or equivalently, $T_M(x,y) = (ax+by, cx+dy)$.

(Note that here we used the notation (x,y) and $\begin{bmatrix} x \\ y \end{bmatrix}$ interchangeably; we will be doing this throughout this module.)

As it turns out, matrices give us a powerful systematic way to describe a wide variety of transformations: they can describe rotations, reflections, dilations, and much more.

Example

Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

1. Write an expression for T_M.
2. Find $T_M(1,0)$ and $T_M(0,1)$.
3. Find all points (x,y) such that $T_M(x,y) = (1,0)$.

Solution

1. $T_M(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+7y \end{bmatrix} = (x+2y, 3x+7y)$.
2. Using the formula from the previous part, $T_M(1,0) = (1,3)$ and $T_M(0,1) = (2,7)$
3. We have $T_M(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

   $$x+2y = 1, \quad 3x+7y = 0.$$

   Solving these equations yields $x = 7, y = -3$; and this is the only solution. So the only point (x,y) such that $T_M(x,y) = (1,0)$ is $(x,y) = (7,-3)$.

Note that $T_M(1,0)$ and $T_M(0,1)$ are precisely the *columns* of the matrix M. This is an important fact, which we will discuss later.

While every matrix describes a plane transformation, not every plane transformation can be described by a matrix. Matrices correspond to a specific type of plane transformation which sends (x,y) to $(ax+by, cx+dy)$, for some real numbers a,b,c,d.

A transformation T_M arising from a matrix M obeys some "distributive laws". For any 2×2 matrix M and vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$, it is true that

$$M(\mathbf{v}+\mathbf{w}) = M\mathbf{v} + M\mathbf{w} \quad \text{and hence} \quad T_M(\mathbf{v}+\mathbf{w}) = T_M(\mathbf{v}) + T_M(\mathbf{w}).$$

Moreover, for any real number (scalar) c,

$$M(c\mathbf{v}) = cM\mathbf{v} \quad \text{and hence} \quad T_M(c\mathbf{v}) = cT_M(\mathbf{v}).$$

In fact, if a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies these two distributive laws, then it must arise from a matrix. Why? If $F(\mathbf{v}+\mathbf{w}) = F(\mathbf{v}) + F(\mathbf{w})$ and $F(c\mathbf{v}) = cF(\mathbf{v})$, then for any point (x, y),

$$\begin{aligned} F\begin{bmatrix} x \\ y \end{bmatrix} &= F\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = F\begin{bmatrix} x \\ 0 \end{bmatrix} + F\begin{bmatrix} 0 \\ y \end{bmatrix} \qquad \text{(using } F(\mathbf{v}+\mathbf{w}) = F(\mathbf{v}) + F(\mathbf{w})\text{)} \\ &= xF\begin{bmatrix} 1 \\ 0 \end{bmatrix} + yF\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(using } F(c\mathbf{v}) = cF(\mathbf{v})\text{)} \end{aligned}$$

Letting $F(1,0) = (a,c)$ and $F(0,1) = (b,d)$ then, we have

$$F\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

so F corresponds to a matrix.

Therefore, we have two equivalent ways to define linear transformations. A plane transformation F is *linear* if either of the following equivalent conditions holds:

- $F(x, y) = (ax+by, cx+dy)$ for some real a, b, c, d. That is, F arises from a matrix.
- For any scalar c and vectors $\mathbf{v}, \mathbf{w}$, $F(c\mathbf{v}) = cF(\mathbf{v})$ and $F(\mathbf{v}+\mathbf{w}) = F(\mathbf{v}) + F(\mathbf{w})$.

Exercise 2

Show that any linear transformation sends the origin to the origin.

The effect of a linear transformation

Suppose you are given a matrix M. You now know that M determines a linear transformation T_M. But what does T_M do, geometrically? We will now see a method to analyse this question.

Let's begin by restricting our attention to two vectors: $(1,0)$ and $(0,1)$. These two vectors are sometimes called the *standard basis* for $\mathbb{R}^2$.

Multiplying any matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by the standard basis vectors gives its *columns*:

$$M\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

So, just by looking at a matrix M, reading down its columns, you can see where T_M sends $(1,0)$ and $(0,1)$. Once you know where T_M sends $(1,0)$ and $(0,1)$, you can use the "distributive" property of linear transformations to see where any other point goes.

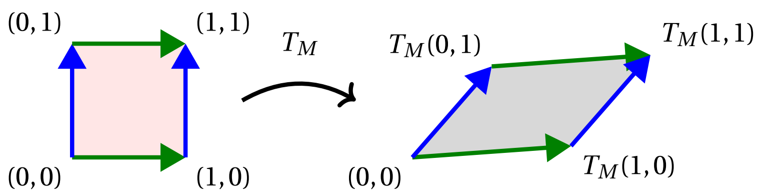
For instance, if you want to know where T_M sends $(2,3)$, you could note that $(2,3) = 2(1,0)+3(0,1)$, so that

$$T_M(2,3) = 2T_M(1,0)+3T_M(0,1).$$

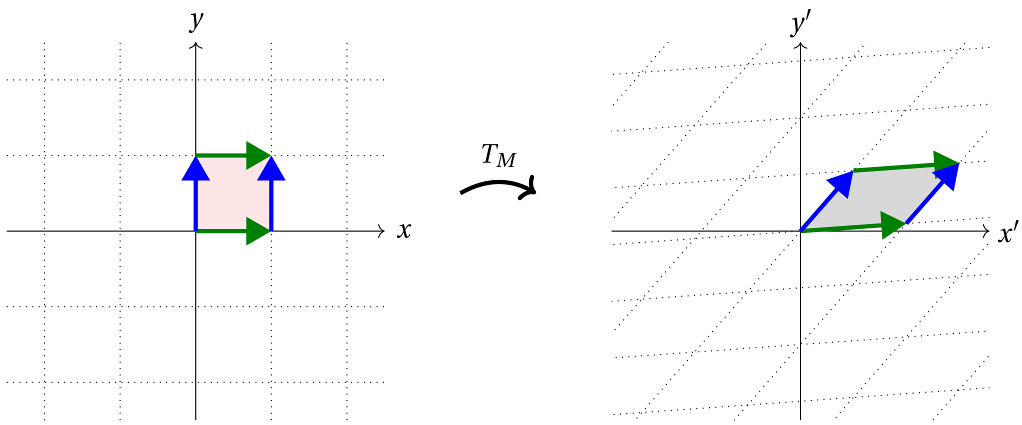
Hence $T_M(2,3)$ is given by 2 times the first column of M, plus 3 times the second column.

In the xy-plane we can consider the unit square, with vertices $(0,0),(1,0),(0,1),(1,1)$. Unit squares tiles the plane, giving a tessellation whose vertices are precisely the points with integer coordinates. Where does T_M send these points?

- T_M sends the origin to the origin (exercise **??**).
- T_M sends $(1,0)$ and $(0,1)$ to the vectors given by the first and second columns of M.
- T_M sends $(1,1) = (1,0)+(0,1)$, to $T_M(1,0)+T_M(0,1)$, the sum of the two columns of M.
- In general, T_M sends (a,b) to $aT_M(1,0)+bT_M(0,1)$.



So the unit square is sent by T_M to a *parallelogram*. The distributive property implies that the whole tessellation of the plane by unit squares is sent by T_M to a tessellation of the plane by parallelograms. We can then draw a picture of the entire transformation T_M.



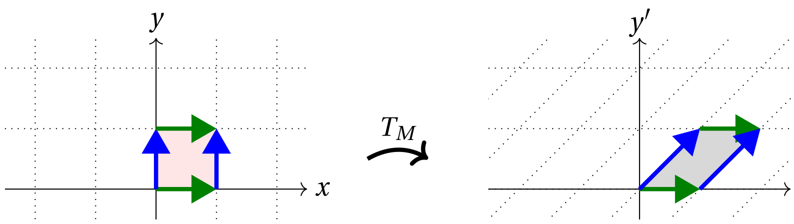
Let us now use these ideas to describe some specific linear transformations.

Example

Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Describe the linear transformation T_M geometrically.

Solution

Reading the columns of T_M tells us that $T_M(1,0) = (1,0)$ and $T_M(0,1) = (1,1)$; the transformation T_M thus turns the unit square into a parallelogram with base 1 and height 1 as shown. This transformation is called a *shear*.



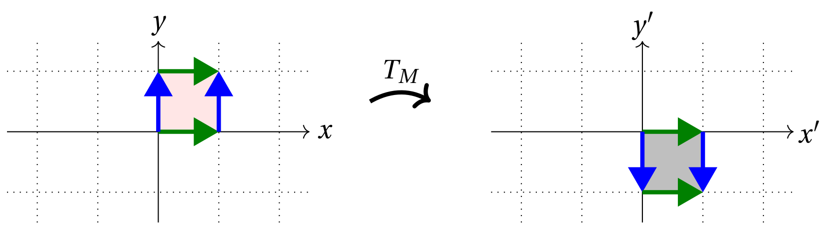
In the above figure, we draw a parallelogram (shaded grey) with two vectors $\mathbf{v} = T_M(1,0)$ (green) and $\mathbf{w} = T_M(0,1)$ (blue) tail-to-tail at the origin. We call this the parallelogram *spanned* by $\mathbf{v}$ and $\mathbf{w}$.

Example

Let $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Describe the linear transformation T_M geometrically.

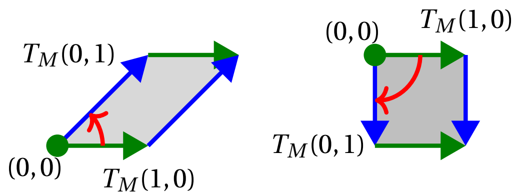
Solution

Reading the columns of M, we have $T_M(1,0) = (1,0)$ and $T_M(0,1) = (0,-1)$. So T_M fixes $(1,0)$ and "flips" $(0,1)$ to its negative $(0,-1)$. Thus vertical directions are "flipped", so that (x,y) is sent to $(x,-y)$, and T_M is reflection in the x-axis.



Note that in the two examples above, the parallelograms spanned by $T_M(1,0)$ and $T_M(0,1)$ have *different orientations*. In the first example, if we consider sweeping around the origin, through the parallelogram, from $T_M(1,0)$ to $T_M(0,1)$, we go *anticlockwise*; in the second example, we go *clockwise*. Accordingly, we say the first parallelogram is *positively* oriented, and the second is *negatively* oriented.

In the standard unit square, sweeping around the origin from $(1,0)$ to $(0,1)$ goes anticlockwise, so it is positively oriented. So the first example *preserved* this orientation, and the second example *reversed* it. In general, we say a linear transformation *preserves* or *reverse* orientation accordingly as the parallelogram spanned by $T_M(1,0)$ and $T_M(0,1)$ is positively or negatively oriented.



Exercise 3

For each matrix M below, describe the linear transformation T_M geometrically.

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Composing linear transformations and matrix multiplication

Let's now suppose we have two matrices M, N representing two linear transformations T_M, T_N. What happens if we apply one transformation then the other? What is the composition $T_M \circ T_N$?

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. So

$$T_M(x,y) = (ax+by, cx+dy) \quad \text{and} \quad T_N(x,y) = (ex+fy, gx+hy).$$

Composing these two functions gives

$$\begin{aligned} T_M \circ T_N(x,y) = T_M(T_N(x,y)) &= T_M(ex+fy, gx+hy) \\ &= \big(a(ex+fy)+b(gx+hy),\ c(ex+fy)+d(gx+hy)\big) \\ &= \big((ae+bg)x+(af+bh)y,\ (ce+dg)x+(cf+dh)y\big). \end{aligned}$$

Remembering that a,b,c,d,e,f,g,h are just real constants, this is another linear transformation, and the corresponding matrix is

$$\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$

This matrix is precisely the *product* of the two matrices M and N:

$$MN = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$

We conclude that the composition of the linear transformations of M and N is the linear transformation of MN, proving the following theorem.

Theorem

For any matrices M and N, $T_M \circ T_N = T_{MN}$. □

This theorem gives us a very quick way to compose linear transformations: just multiply the corresponding matrices!

Example

Find the matrix for the composition $g \circ f$ of the two linear transformations $f(x,y) = (x+y, y)$ and $g(x,y) = (y, x+y)$.

Solution

We have $f = T_M$ and $g = T_N$ where $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So the matrix of the composition $g \circ f = T_N \circ T_M = T_{NM}$ is the product NM:

$$NM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Exercise 4

Show that if the linear transformation $f(x,y) = (y, -x+y)$ is composed with itself six times, the result is the identity transformation.

The identity transformation and identity matrix

Recall the identity transformation $I : \mathbb{R}^2 \to \mathbb{R}^2$ sends every point to itself, $I(x,y) = (x,y)$. We can now note I is a linear transformation, corresponding to the *identity* matrix

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \text{In other words,} \quad T_{\mathrm{Id}} = I.$$

The inverse of a linear transformation and inverse matrices

Recall that a function has an inverse if and only if it is *bijective*, i.e. injective and surjective. (A function $f : X \to Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$; and f is surjective if its image is all of Y, i.e. for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.)

Some transformations of the plane are bijective, and some are not; so some have inverses, and others do not. As we proceed, we will see many examples of transformations that have inverses, and many that don't.

A bijective transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$, has an inverse $F^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. These functions F and F^{-1} "undo" each other. So if $F(\mathbf{x}) = \mathbf{y}$ then $F^{-1}(\mathbf{y}) = \mathbf{x}$, and vice versa.

Recall that matrices can have inverses too. If

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(provided } ad-bc \neq 0\text{).}$$

We say M^{-1} is the *inverse* of M because M and M^{-1} multiply to give the identity.

Exercise 5

Verify that $MM^{-1} = M^{-1}M = \mathrm{Id}$.

If we compose the linear transformations T_M and $T_{M^{-1}}$, we obtain the identity:

$$T_M \circ T_{M^{-1}} = T_{MM^{-1}} = T_{\mathrm{Id}} = I, \quad T_{M^{-1}} \circ T_M = T_{M^{-1}M} = T_{\mathrm{Id}} = I.$$

Hence T_M and $T_{M^{-1}}$ undo each other; they are *inverse* transformations.

Example

What is the inverse of the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (x+3y, x+5y)$?

Solution

The transformation F is linear and corresponds to the matrix

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}, \quad \text{which has inverse} \quad M^{-1} = \frac{1}{1\cdot 5 - 3\cdot 1}\begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

The inverse of $F = T_M$ is then $F^{-1} = T_{M^{-1}}$,

$$F^{-1}(x,y) = \left(\frac{5}{2}x - \frac{3}{2}y,\ \frac{-1}{2}x + \frac{1}{2}y\right).$$

To show that a function does *not* have an inverse, one can show that it is not injective, or that it is not surjective. In the following example we do both, giving two solutions.

Example

Show that the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x, 0)$ has no inverse.

Solution

Solution 1. Since (for instance) $F(0,0) = F(0,1) = (0,0)$, F is not injective. Hence F has no inverse.

Solution 2. Any point in the image of F has second coordinate zero. So there is no (x, y) such that (for instance) $F(x, y) = (0,1)$. Hence F is not surjective, and has no inverse.

Exercise 6

Show that $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + 2y, 2x + 4y)$ has no inverse.

The following example illustrates a useful technique for finding a linear transformation, from its value at two points.

Example

Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $(1,1)$ to $(-1,4)$ and $(-1,3)$ to $(-7,0)$

Solution

Let M be the matrix of the desired linear transformation. We have

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \text{and} \quad M \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}.$$

In fact, we can put these two equations together into a single matrix equation

$$M \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ 4 & 0 \end{bmatrix}$$

which we can then solve for M:

$$M = \begin{bmatrix} -1 & -7 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} -1 & -7 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & -8 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

Hence the only such transformation is $T_M(x, y) = (x - 2y, 3x + y)$.

Exercise 7

Find the linear transformation that sends $(3,1)$ to $(1,2)$ and $(-1,2)$ to $(2,-3)$

Describing geometric transformations algebraically

We have now seen several geometric plane transformations. We have also seen how *matrices* can describe some transformations (linear ones) in a simple algebraic way.

We now investigate these geometric operations from an algebraic point of view. We ask whether, and how, they can be represented by matrices. This gives us a very interesting way to relate everyday geometric operations to algebra.

Rotations

> To everything — turn, turn, turn
>
> – Pete Seeger

As discussed previously, given a point P in the plane, and an angle ϕ, we can consider the rotation around P by angle ϕ. By convention, we measure angles anticlockwise and in radians. We denote[3] this rotation by

$$\mathrm{Rot}_{P,\phi} : \mathbb{R}^2 \to \mathbb{R}^2.$$

Note the following preliminary facts about rotations:

1. A rotation by angle 0 is just the identity transformation, as is rotation by 2π.
2. Rotation by $\phi + 2\pi$ is the same transformation as rotation by ϕ.
3. Rotations of angles θ and ϕ about the same point P give a rotation of $\theta + \phi$ about P.
4. Any rotation is *bijective*. Indeed, a rotation by ϕ about P can be undone by rotating by $-\phi$, so the transformation $\mathrm{Rot}_{P,\phi}$ has inverse $\mathrm{Rot}_{P,-\phi}$.

That is,

$$\mathrm{Rot}_{P,0} = \mathrm{Rot}_{P,2\pi} = I, \quad \mathrm{Rot}_{P,\phi+2\pi} = \mathrm{Rot}_{P,\phi},$$

$$\mathrm{Rot}_{P,\phi} \circ \mathrm{Rot}_{P,\theta} = \mathrm{Rot}_{P,\theta+\phi}, \quad \left(\mathrm{Rot}_{P,\phi}\right)^{-1} = \mathrm{Rot}_{P,-\phi}.$$

[3] Please note: This is not a standard notation. We find it useful, but it may not be used by others!

Exercise 8

Consider $\mathrm{Rot}_{\mathbf{0},\,\pi/2}$, rotation about the origin by $\pi/2$. Show that $\mathrm{Rot}_{\mathbf{0},\,\pi/2}(x,y) = (-y,x)$.

Now, when we rotate around the *origin* $\mathbf{0}$, it turns out that $\mathrm{Rot}_{\mathbf{0},\,\phi}$ is a *linear* transformation. We will show why this is true, and find the corresponding matrix.

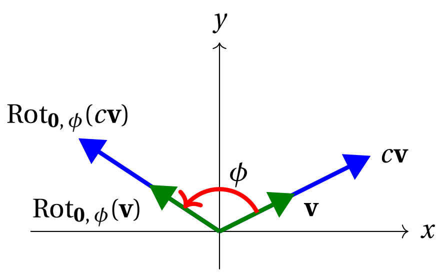
To show that $\mathrm{Rot}_{\mathbf{0},\,\phi}$ is linear, we will show that it satisfies the "distributive" laws

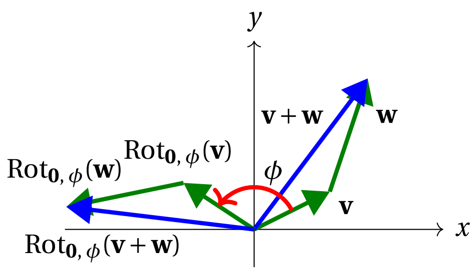
$$\mathrm{Rot}_{\mathbf{0},\phi}(c\mathbf{v}) = c\mathrm{Rot}_{\mathbf{0},\phi}(\mathbf{v}) \quad \text{and} \quad \mathrm{Rot}_{\mathbf{0},\phi}(\mathbf{v}+\mathbf{w}) = \mathrm{Rot}_{\mathbf{0},\phi}(\mathbf{v}) + \mathrm{Rot}_{\mathbf{0},\phi}(\mathbf{w}),$$

for any vectors $\mathbf{v},\mathbf{w}$ and real number c.

For the first law, note that the two vectors $c\mathbf{v}$ and $\mathbf{v}$ point in the same direction; $c\mathbf{v}$ is c times longer than $\mathbf{v}$. After rotation by ϕ about $\mathbf{0}$, the two resulting vectors still point in the same direction and one is still c times longer than the other, so

$$\mathrm{Rot}_{\mathbf{0},\phi}(c\mathbf{v}) = c\mathrm{Rot}_{\mathbf{0},\phi}(\mathbf{v}) \quad \text{as desired.}$$





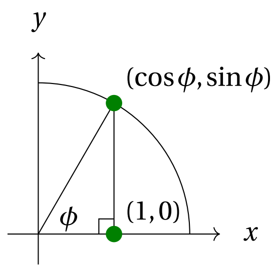
For the second law, place $\mathbf{v}$ at the origin and place $\mathbf{w}$ head-to-tail with it, as shown, to obtain the vector $\mathbf{v}+\mathbf{w}$. After rotating these vectors all by ϕ, we have head-to-tail vectors $\mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{v})$ and $\mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{w})$, summing to $\mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{v}+\mathbf{w})$. Thus

$$\mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{v}) + \mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{w}) = \mathrm{Rot}_{\mathbf{0},\,\phi}(\mathbf{v}+\mathbf{w}) \quad \text{as desired.}$$

Now that we know $\mathrm{Rot}_{\mathbf{0},\,\phi}$ is a linear transformation, we know that it corresponds to a matrix: the next question is to find the matrix.

To find the matrix, consider where the standard basis vectors $(1,0)$ and $(0,1)$ go under the rotation: this will give us the two columns of the matrix, as discussed earlier.

Rotating $(1,0)$ by an angle of ϕ about the origin, we obtain the point $(\cos\phi,\sin\phi)$, by elementary trigonometry in the right-angle triangle shown.



Exercise 9

Show that rotating $(0,1)$ by ϕ about the origin yields the point $(-\sin\phi, \cos\phi)$.

Thus we have found the columns of the matrix of $\mathrm{Rot}_{\mathbf{0},\phi}$. The matrix is

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Such a matrix is often called a *rotation matrix* with angle ϕ.

Example

Check directly that the inverse of the rotation matrix with angle ϕ is the rotation matrix with angle $-\phi$. Hence verify algebraically that $\left(\mathrm{Rot}_{\mathbf{0},\phi}\right)^{-1} = \mathrm{Rot}_{\mathbf{0},-\phi}$.

Solution

The inverse or the rotation matrix with angle ϕ is

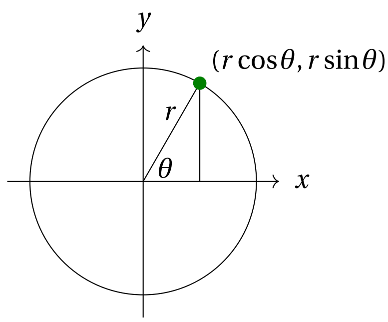
$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{-1} = \frac{1}{\cos^2\phi + \sin^2\phi}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix},$$

which is the rotation matrix with angle $-\phi$. So the inverse of $\mathrm{Rot}_{\mathbf{0},\phi}$ is $\mathrm{Rot}_{\mathbf{0},-\phi}$.

Exercise 10

Show that the product of the rotation matrices with angles θ and ϕ is the rotation matrix with angle $\theta+\phi$. Hence verify algebraically that $\mathrm{Rot}_{\mathbf{0},\theta} \circ \mathrm{Rot}_{\mathbf{0},\phi} = \mathrm{Rot}_{\mathbf{0},\theta+\phi}$.

The formula for a rotation about the origin can also be derived directly. We can do this using *polar coordinates*. Instead of describing a point by its Cartesian coordinates (x, y), we can use its *distance from the origin* r, and its *direction* θ from the origin. The angle θ is measured anticlockwise from the positive x-axis.



Basic trigonometry in the triangle shown gives $x = r\cos\theta$ and $y = r\sin\theta$. When we apply a rotation about the origin of angle ϕ, the point (x, y) moves an angle of ϕ around the origin, but its distance from the origin r is unchanged. Hence the point $(x, y) = (r\cos\theta, r\sin\theta)$ is mapped to $(x', y') = (r\cos(\theta+\phi), r\sin(\theta+\phi))$. We can now use the sine and cosine addition formulae to rewrite this expression:

$$\begin{aligned}\mathrm{Rot}_{\mathbf{0},\phi}\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}r\cos(\theta+\phi)\\r\sin(\theta+\phi)\end{bmatrix} = \begin{bmatrix}r\cos\theta\cos\phi - r\sin\theta\sin\phi\\r\sin\theta\cos\phi + r\cos\theta\sin\phi\end{bmatrix}\\ &= \begin{bmatrix}(\cos\phi)x-(\sin\phi)y\\(\sin\phi)x+(\cos\phi)y\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.\end{aligned}$$

This gives us a formula for $\mathrm{Rot}_{\mathbf{0},\phi}$ and the rotation matrix directly.

While a rotation about the *origin* gives a linear transformation, a rotation about any *other* point is *not* linear, and hence is *not* given by a matrix. It is a plane transformation, but not a linear one. You can prove this in the next exercise.

Exercise 11

If ϕ is not a multiple of 2π, and P is not the origin, show that $\mathrm{Rot}_{P,\phi}(\mathbf{0}) \neq \mathbf{0}$. Using exercise **??**, conclude that $\mathrm{Rot}_{P,\phi}(\mathbf{0})$ is not linear.

In the section on *Translations*, you can find a formula for a rotation about any point.

Translations

> I like to move it, move it
>
> – Reel 2 Real[4]

Picking up an object and moving it is a very familiar concept. As discussed previously, given a vector $\mathbf{v}$, we can consider the translation of the plane by the vector $\mathbf{v}$. We denote this transformation by[5]

$$\mathrm{Trans}_{\mathbf{v}} : \mathbb{R}^2 \to \mathbb{R}^2.$$

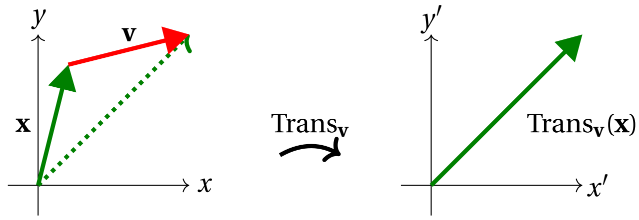
Let $\mathbf{v} = (a, b)$. When we translate a point, with position vector $\mathbf{x} = (x, y)$, by $\mathbf{v}$, we simply *add* the two vectors, as shown below. Thus

$$\mathrm{Trans}_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}+\mathbf{v}, \quad \text{or equivalently,} \quad \mathrm{Trans}_{\mathbf{v}}(x, y) = (x+a, y+b).$$

---

[4] Not to be confused with $\mathbb{R} \to \mathbb{R}$!

[5] Again, this is not a standard notation.



Translation by $\mathbf{0}$ does not move points very far! It is the identity: $\text{Trans}_\mathbf{0} = I$.

When $\mathbf{v} \neq \mathbf{0}$, translation by $\mathbf{v}$ is *not* a linear transformation. The constant terms a, b in the formula $(x + a, y + b)$ are not allowed under our definition of linear transformation. Another way to see that $\text{Trans}_\mathbf{v}$ is not linear is to note that $\text{Trans}_\mathbf{v}(\mathbf{0}) = \mathbf{v}$, but a linear transformation must take the origin $\mathbf{0}$ to $\mathbf{0}$ (exercise **??**). (Transformations which allow both linear and constant terms are called *affine transformations*.)

Exercise 12

Show that for any vector $\mathbf{v}$, the translation $\text{Trans}_\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective. Find the inverse transformation, and give it an geometric interpretation.

Exercise 13

(This exercise may be easier after reading the rest of this module.) Let $P = (a, b)$ be a point in the plane with position vector $\mathbf{v}$. Show that translation by $-\mathbf{v}$, followed by rotation by angle ϕ about the origin $\mathbf{0}$, followed by translation by $\mathbf{v}$, gives the plane transformation which rotates by angle ϕ about P. That is, show that

$$\text{Trans}_\mathbf{v} \circ \text{Rot}_{\mathbf{0},\phi} \circ \text{Trans}_{-\mathbf{v}} = \text{Rot}_{P,\phi}.$$

Hence give a formula for $\text{Rot}_{P,\theta}(x, y)$, the rotation of (x, y) by angle θ around $P = (a, b)$.

Projections

As discussed earlier, given a line L on the plane, we can consider the plane transformation which projects points onto L. We can denote[6] this transformation by

$$\text{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2.$$

Such projections are very familiar from everyday life. If the plane is a map, L is a road, and P is your position, then $\text{Proj}_L(P)$ is the point on the road closest to you. If the line L is horizontal, representing the ground, P is above ground, and the sun is directly above, then $\text{Proj}_L(P)$ is the shadow of P on the ground.

[6] Again, beware that this notation is not standard.

We have some preliminary facts about projections:

1. A projection Proj_L is not surjective. Its image consists only of points on L.
2. A projection Proj_L is not injective. Many points project to any given point of L.
3. If P is a point on L, then $\mathrm{Proj}_L(P) = P$.
4. After projecting a point onto L, projecting onto L again has no further effect. That is, $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$.

A function f such that $f \circ f = f$ is called an *idempotent*. The last property above says a projection is an idempotent.

Exercise 14

Give proofs of the above facts.

As it turns out, when the line L passes through the origin, Proj_L is a linear transformation. We will show why, and find the corresponding matrix.

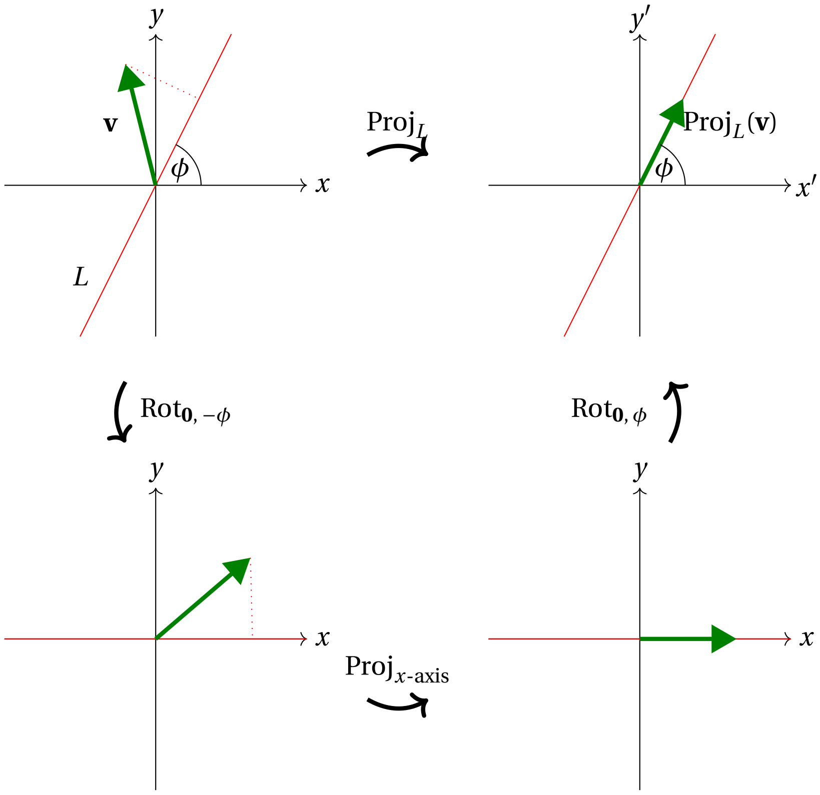
First of all, suppose L is the x-axis. Projection onto L in this case just sets the y-coordinate of a point to 0, and leaves the x-coordinate unchanged: $\mathrm{Proj}_{x\text{-axis}}(x, y) = (x, 0)$. Similarly $\mathrm{Proj}_{y\text{-axis}}(x, y) = (0, y)$. These are linear transformations:

$$\mathrm{Proj}_{x\text{-axis}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathrm{Proj}_{y\text{-axis}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now suppose that L is any line through the origin, making an angle of ϕ with the positive x-axis (as always, measured anticlockwise in radians), and let $\mathbf{v} = (x, y)$.

The key idea to calculate $\mathrm{Proj}_L(\mathbf{v})$ is to perform the projection in several steps. We first *rotate* the plane by $-\phi$ about the origin. This rotates L to the x-axis, and rotates $\mathbf{v}$ too. Then, we project onto the x-axis. Finally, we rotate back by ϕ around the origin. The result is that $\mathbf{v}$ has been projected onto L. In symbols,

$$\mathrm{Proj}_L = \mathrm{Rot}_{\mathbf{0},\phi} \circ \mathrm{Proj}_{x\text{-axis}} \circ \mathrm{Rot}_{\mathbf{0},-\phi}.$$



Why did we do this? Because we know all the matrices for the transformations on the right hand side! The matrix for $\text{Proj}_{x\text{-axis}}$ is given above, and

$$\text{Rot}_{\mathbf{0},\phi}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},\quad \text{Rot}_{\mathbf{0},-\phi}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Hence Proj_L is linear, and its matrix is given by multiplying these matrices together:

$$\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}=\begin{bmatrix}\cos^2\phi & \sin\phi\cos\phi\\ \sin\phi\cos\phi & \sin^2\phi\end{bmatrix}.$$

So projection onto L is a linear transformation corresponding to this matrix:

$$\text{Proj}_L\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos^2\phi & \sin\phi\cos\phi\\ \sin\phi\cos\phi & \sin^2\phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

When $\phi=0$, this matrix agrees with our calculation for projection onto the x-axis.

Example

Let L be the line $y=x$. Find a formula for projection onto L.

Solution

The line L passes through the origin and makes an angle $\phi = \pi/4$ with the positive x-axis. Hence Proj_L is a linear transformation with matrix

$$\begin{bmatrix} \cos^2\frac{\pi}{4} & \sin\frac{\pi}{4}\cos\frac{\pi}{4} \\ \sin\frac{\pi}{4}\cos\frac{\pi}{4} & \sin^2\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Therefore,

$$\mathrm{Proj}_L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}.$$

Exercise 15

Verify the above formula for projection onto $y = x$ directly as follows. Given a point (x_0, y_0), find the intersection of the line through this point with gradient -1, with the line $y = x$. Conclude that $\mathrm{Proj}_L(x_0, y_0) = (\frac{x_0+y_0}{2}, \frac{x_0+y_0}{2})$ as above.

Exercise 16

What is the projection of the point $(8,3)$ onto the line $y = 2x$?

An alternative derivation of the projection matrix is given in the following exercise.

Exercise 17

Let L be a line through the origin, making an angle ϕ with the positive real axis.

a Show that $\mathrm{Proj}_L(c\mathbf{v}) = c\mathrm{Proj}_L(\mathbf{v})$ and $\mathrm{Proj}_L(\mathbf{v}+\mathbf{w}) = \mathrm{Proj}_L(\mathbf{v}) + \mathrm{Proj}_L(\mathbf{w})$, for any vectors $\mathbf{v}, \mathbf{w}$ and real number c.

b Show that the projection of the point $(1,0)$ onto L is $(\cos^2\phi, \sin\phi\cos\phi)$, and the projection of $(0,1)$ onto L is $(\sin\phi\cos\phi, \sin^2\phi)$.

c Conclude that Proj_L is as given above.

It's now possible to see some facts about projections from the algebraic properties of a projection matrix.

Exercise 18

Let L be the line through the origin making an angle of ϕ with the positive x-axis.

a Let $P = (x, y)$ be a point on L. Show that

$$\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and hence} \quad \mathrm{Proj}_L(P) = P.$$

b Show that $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$ by demonstrating that

$$\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}^2 = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}.$$

While projection onto a line *through the origin* is linear, projection onto a line *not through the origin* is *not* linear, and hence not given by a matrix, as you can prove now.

Exercise 19

If L does not pass through the origin, show Proj_L is *not* linear. (Hint: exercise **??**.)

Reflections

As discussed earlier, given a line L in the plane, we can consider the plane transformation which reflects points in L. We denote[7] this transformation by

$$\mathrm{Ref}_L : \mathbb{R}^2 \to \mathbb{R}^2.$$

Here are some elementary facts about reflections:

1. Reflecting a point P in L, and reflecting it in L again, returns you to P. Hence,

$$\mathrm{Ref}_L \circ \mathrm{Ref}_L = I.$$

2. A reflection is bijective; in fact, the inverse of a reflection is *itself*:

$$(\mathrm{Ref}_L)^{-1} = \mathrm{Ref}_L.$$

3. In P is a point on L, then its reflection in L is just P again; $\mathrm{Ref}_L(P) = P$.

A function f such that $f \circ f$ is the identity — equivalently, a function which is its own inverse — is called an *involution*. So reflections are involutions.

When L is the x-axis, reflection in L fixes the x-coordinate and "flips" the y-coordinate. Similarly, reflection in the y-axis "flips" the x-coordinate:

$$\mathrm{Ref}_{x\text{-axis}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathrm{Ref}_{y\text{-axis}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

---

[7] Again, notation not standard.

So reflection in either axis is a linear transformation.

In fact, for any line L passing through the origin, Ref_L is a linear transformation. Letting ϕ be the angle between L and the positive x-axis, the corresponding matrix is given by

$$\mathrm{Ref}_L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In the following two exercises, we outline two methods to derive this result, similar to those seen previously for projections.

Exercise 20

Show that rotation of angle $-\phi$ about the origin $\mathbf{0}$, followed by reflection in the x-axis, followed by rotation of ϕ about $\mathbf{0}$, is the same as reflection in L, i.e.

$$\mathrm{Ref}_L = \mathrm{Rot}_{\mathbf{0},\phi} \circ \mathrm{Ref}_{x\text{-axis}} \circ \mathrm{Rot}_{\mathbf{0},-\phi}.$$

Hence show that Ref_L is a linear transformation and find its matrix.

Exercise 21

a Show that $\mathrm{Ref}_L(c\mathbf{v}) = c\mathrm{Ref}_L(\mathbf{v})$ and $\mathrm{Ref}_L(\mathbf{v}+\mathbf{w}) = \mathrm{Ref}_L(\mathbf{v}) + \mathrm{Ref}_L(\mathbf{w})$, for any vectors $\mathbf{v}$, $\mathbf{w}$ and real number c. Hence show Ref_L is a linear transformation.

b Show that the reflection of the point $(1,0)$ in L is $(\cos(2\phi), \sin(2\phi))$, and that the reflection of $(0,1)$ in L is $(\sin(2\phi), -\cos(2\phi))$.

c Hence find the matrix for Ref_L.

Reflection in one particular line $y = x$ has a useful effect: it *swaps coordinates*.

Example

Let L be the line $y = x$. Show that reflection in L takes a point (x, y) to (y, x).

Solution

Since L passes through the origin with angle $\phi = \pi/4$, Ref_L is linear, with matrix

$$\begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$\mathrm{Ref}_L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{as desired.}$$

Exercise 22

Give a formula for reflection in the line $y = 2x$.

The following exercise gives an interesting relationship between reflections and projections, and leads to a *third* way to derive the reflection matrix.

Exercise 23

Let L be a line in the plane, and $\mathbf{x}$ a vector. Show that

$$\mathrm{Ref}_L(\mathbf{x}) = 2\mathrm{Proj}_L(\mathbf{x}) - \mathbf{x}, \quad \text{or equivalently,} \quad \mathrm{Ref}_L = 2\mathrm{Proj}_L - I.$$

(Hint: Show $\mathbf{x} - \mathrm{Proj}_L(\mathbf{x}) = -\big(\mathrm{Ref}_L(\mathbf{x}) - \mathrm{Proj}_L(\mathbf{x})\big)$.)

(The above exercise holds for *any* line L — it need not pass through the origin!)

Exercise 24

Let L be a line through the origin, making an angle of ϕ with the positive x-axis. Using the previous exercise, and the matrix for Proj_L, deduce the formula for Ref_L.

In the next exercise, we observe properties of reflections in the algebra of reflection matrices.

Exercise 25

Let L be the line through the origin making an angle of ϕ with the positive x-axis.

a Let $P = (x, y)$ be a point on L. Show that

$$\begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and hence} \quad \mathrm{Ref}_L(P) = P.$$

b Show that

$$\begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}^2 = \mathrm{Id} \quad \text{and hence show} \quad \mathrm{Ref}_L \circ \mathrm{Ref}_L = I.$$

We have seen that reflection in a line passing through the origin is linear. But, as with projections, if L does *not* pass through the origin, then Ref_L is not linear.

Exercise 26

Let d be a real number, and let L be the line $x = d$. Show that $\mathrm{Ref}_L(x, y) = (2d - x, y)$.

(Hint: This can be shown directly, or by finding the projection and using exercise **??**.)

Dilations

> [R]ound the neck of the bottle was a paper label, with the words ‘DRINK ME’ beautifully printed on it in large letters...
>
> Alice ventured to taste it, and finding it very nice, (it had, in fact, a sort of mixed flavour of cherry-tart, custard, pine-apple, roast turkey, toffee, and hot buttered toast,) she very soon finished it off...
>
> ‘What a curious feeling!’ said Alice; ‘I must be shutting up like a telescope.’
>
> And so it was indeed: she was now only ten inches high
>
> – Lewis Carroll, *Alice’s Adventures in Wonderland*

The plane transformation of a *dilation* has a similar effect to Alice’s “DRINK ME” potion or “EAT ME” cake: it stretches or shrinks objects in the plane.

As discussed previously, given a line L and a real number k, we can consider the dilation from L by factor k. We write[8] this transformation as

$$\mathrm{Dil}_{L,k} : \mathbb{R}^2 \to \mathbb{R}^2.$$

The effect of such a dilation is to stretch lengths by a factor of k in the direction perpendicular to L. If k is negative, then points are flipped to the other side of L.

Here are some basic properties of dilations:

1 Dilating from L with factor m, and then k, gives a dilation from L with factor km.

$$\mathrm{Dil}_{L,k} \circ \mathrm{Dil}_{L,m} = \mathrm{Dil}_{L,km}$$

[8] Again, not a standard notation.

2 If $k \neq 0$, then dilation from L with factor k is bijective. Its inverse is given by dilation from L with factor $1/k$.

$$(\mathrm{Dil}_{L,k})^{-1} = \mathrm{Dil}_{L,\frac{1}{k}}$$

3 If P is a point on L, then for any k, $\mathrm{Dil}_{L,k}(P) = P$.

These facts are true even for dilation by negative factors.

We will analyse dilations similarly to projections and reflections. First, dilation from the x-axis with factor k leaves the x-coordinate unchanged, and multiplies the y-coordinate by k: $\mathrm{Dil}_{x\text{-axis},k}(x, y) = (x, ky)$. Similarly for the y-axis, $\mathrm{Dil}_{y\text{-axis},k}(x, y) = (kx, y)$. Hence

$$\mathrm{Dil}_{x\text{-axis},k}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathrm{Dil}_{y\text{-axis},k}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus a dilation from either axis is a linear transformation.

In fact, for any line L passing through the origin and any real k, $\mathrm{Dil}_{L,k}$ is a linear transformation. Letting ϕ denote the angle L makes with the positive x-axis, we have

$$\mathrm{Dil}_{L,k}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1+(k-1)\sin^2\phi & (1-k)\sin\phi\cos\phi \\ (1-k)\sin\phi\cos\phi & 1+(k-1)\cos^2\phi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

You can check that $\phi = 0$ or $\pi/2$ gives the matrices above for dilations in the axes.

Exercise 27 **a** Show that $\mathrm{Dil}_{L,k} = \mathrm{Rot}_{\mathbf{0},\phi} \circ \mathrm{Dil}_{x\text{-axis},k} \circ \mathrm{Rot}_{\mathbf{0},-\phi}$.

b Hence show that $\mathrm{Dil}_{L,k}$ is a linear transformation, with matrix as claimed above.

Example

Find a formula for dilation from the line $y = -x$ with factor 3.

Solution

This line L has angle $\phi = -\pi/4$, so $\sin\phi = \frac{-1}{\sqrt{2}}$ and $\cos\phi = \frac{1}{\sqrt{2}}$. Hence

$$\begin{bmatrix} 1+(k-1)\sin^2\phi & (1-k)\sin\phi\cos\phi \\ (1-k)\sin\phi\cos\phi & 1+(k-1)\cos^2\phi \end{bmatrix} = \begin{bmatrix} 1+2\cdot\frac{1}{2} & -2\cdot\frac{-1}{2} \\ -2\cdot\frac{-1}{2} & 1+2\cdot\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

so the dilation is given by $\mathrm{Dil}_{L,3}(x, y) = (2x + y, x + 2y)$.

Exercise 28

What is the dilation of the point $(3,0)$ from the line $y = 2x$ by factor 5?

Dilation matrices are more complicated than rotation, reflection or projection matrices. Nonetheless, geometric properties of dilations are reflected in the algebra of dilation matrices.

Exercise 29

Suppose $k \neq 0$ and L passes through the origin. Show that the matrices for dilation with factor k and $\frac{1}{k}$ from L are inverses. Hence conclude that $\left(\mathrm{Dil}_{L,k}\right)^{-1} = \mathrm{Dil}_{L,\frac{1}{k}}$.

Now for some particular values of k, dilation from a line L by factor k reduces to some transformations we have seen previously.

1. Dilation from L with factor $k = -1$ flips every point to the corresponding point on the opposite side of L: it is *reflection* in L.
2. Dilation from L with factor $k = 0$ is just *projection* onto L. "Stretching from L by factor zero" simply projects points onto L.
3. Dilation from L with factor $k = 1$ leaves every point where it is.

In this sense, dilations form a *generalisation* of reflections and projections. We have

$$\mathrm{Dil}_{L,0} = \mathrm{Proj}_L, \quad \mathrm{Dil}_{L,-1} = \mathrm{Ref}_L, \quad \mathrm{Dil}_{L,1} = I.$$

Exercise 30

a Show that a dilation matrix with $k = 0$ is a projection matrix.

b Show that a dilation matrix with $k = -1$ is a reflection matrix.

Exercise 31

Use the fact that $\mathrm{Dil}_{L,k} \circ \mathrm{Dil}_{L,m} = \mathrm{Dil}_{L,km}$ to show the following.

a A reflection is an involution, i.e. $(\mathrm{Ref}_L)^2 = I$.

b A projection is an idempotent, i.e. $\left(\mathrm{Proj}_L\right)^2 = \mathrm{Proj}_L$.

Exercise 32

We can characterise dilation, projection and reflection matrices as follows. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that the linear transformation corresponding to M is:

a a projection onto a line if and only if M is symmetric (i.e. M equals its transpose M^T), $ad - bc = 0$, and $a + d = 1$;

b a reflection in a line if and only if M is symmetric, $a+d=0$, and $ad-bc=-1$;

c a dilation if and only if M is symmetric and $a+d=1+ad-bc$.

Spiral symmetries

A *spiral symmetry* is another interesting type of plane transformation. The spiral symmetry about a point P with factor r and angle θ is given by by dilating the whole plane by a factor r from P, followed by a rotation by θ around P.

A spiral symmetry about the origin thus dilates a point (x,y) to (rx,ry) and then rotates it by θ, to give the result

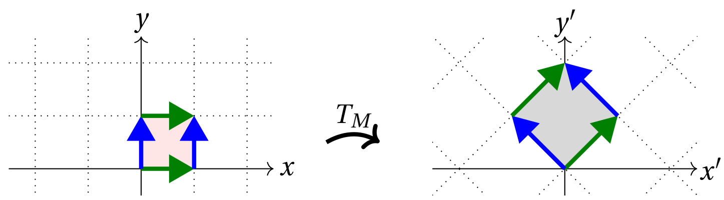
$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}rx\\ ry\end{bmatrix}=\begin{bmatrix}rx\cos\theta-ry\sin\theta\\ rx\sin\theta+ry\cos\theta\end{bmatrix}=\begin{bmatrix}r\cos\theta & -r\sin\theta\\ r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}.$$

Example

Let $M=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$. Describe the linear transformation T_M geometrically.

Solution

Form the parallelogram spanned by $T_M(1,0)=(1,1)$ and $T_M(0,1)=(-1,1)$, and view T_M as in previous examples. We see T_M sends the unit square to a square of side length $\sqrt{2}$, rotated by $\pi/4$.



Thus T_M is a spiral symmetry about the origin with factor $r=\sqrt{2}$ and angle $\theta=\pi/4$; indeed, M agrees with the matrix above for these values of r and θ:

$$M=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\sqrt{2}\cos\frac{\pi}{4} & -\sqrt{2}\sin\frac{\pi}{4}\\ \sqrt{2}\sin\frac{\pi}{4} & \sqrt{2}\cos\frac{\pi}{4}\end{bmatrix}=\begin{bmatrix}r\cos\theta & -r\sin\theta\\ r\sin\theta & r\cos\theta\end{bmatrix}.$$

One reason spiral symmetries are interesting is because they arise naturally with *complex numbers*. Consider a complex number w in polar form $w=r\ \mathrm{cis}\theta=r(\cos\theta+i\sin\theta)$, where $r = |w|$ and $\theta = \arg w$. As the complex numbers form the Argand plane $\mathbb{C} \cong \mathbb{R}^2$, we may regard multiplication by w as a plane transformation $\mathrm{Mult}_w : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\mathrm{Mult}_w(z) = wz$. That is, Mult_w multiplies complex numbers by w. If $z = x + yi$ then

$$wz = r(\cos\theta + i\sin\theta)(x + yi) = \big(rx\cos\theta - ry\sin\theta\big) + i\big(rx\sin\theta + ry\cos\theta\big).$$

If we write out the real and imaginary parts as components of a vector, this means

$$\mathrm{Mult}_w \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} rx\cos\theta - ry\sin\theta \\ rx\sin\theta + ry\cos\theta \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This is exactly a spiral symmetry! Hence, the effect of multiplication by a complex number w on the complex plane is a spiral symmetry about the origin 0 with factor $|w|$ and angle $\arg\theta$.

Transformations of graphs

A common type of problem asks you to transform a graph $y = f(x)$, and find the equation of the transformed graph. We illustrate with an example.

Example

Consider the graph $y = x^2$. Suppose the graph is dilated from the x-axis by a factor of 2, and then translated 3 units to the right. What is the equation of the resulting graph?

Solution

The transformation given is $\mathrm{Trans}_{(3,0)} \circ \mathrm{Dil}_{x\text{-axis},2}$. Denote by (x', y') the image of a point (x, y) under this transformation. A point (x, y) is sent to $(x, 2y)$ by the dilation, and then to $(x+3, 2y)$ by the translation, so $(x', y') = (x+3, 2y)$.

The original graph consists of points (x, y) such that $y = x^2$. Then it is transformed so that $(x, y) \mapsto (x', y') = (x+3, 2y)$. We must find what equation is satisfied by x' and y'.

Since $x' = x + 3$ and $y' = 2y$, we have $x = x' - 3$ and $y = y'/2$. Thus $y = x^2$ implies

$$\frac{y'}{2} = (x'-3)^2 \quad \text{or equivalently} \quad y' = 2(x'-3)^2.$$

Thus, after the transformation, the graph consists of points (x, y) satisfying $y = 2(x-3)^2$. This is the equation of the transformed graph.

Exercise 33

In the above example, suppose the translation was applied first, then the dilation. Would the answer have been different?

The method in this example can be used generally to find an equation for a graph subject to a transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$.

- Write $(x', y') = F(x, y)$, and invert this equation to write x and y as expressions in terms of x' and y'.
- Substitute these expressions for x and y into the equation of the graph to obtain an equation in x' and y'.
- The transformed graph then consists of points (x', y') satisfying this equation in x' and y'. Dropping the primes, we have an equation in x and y for the transformed graph.

Example

Consider the graph $y = e^x$. Suppose the graph is dilated from the y-axis by a factor of 3, reflected in the x-axis, and then translated 1 unit to the left and 2 units down. What is the equation of the resulting graph?

Solution

A point (x, y) maps to $(3x, y)$ under the dilation, then to $(3x, -y)$ under the reflection, then to $(3x-1, -y-2)$ under the translation. So $(x', y') = (3x-1, -y-2)$. Inverting, we obtain $x = \frac{x'+1}{3}$ and $y = -y'-2$. Substituting these expressions into $y = e^x$ gives $-y'-2 = e^{\frac{x'+1}{3}}$. So the equation of the transformed graph is

$$y = -2 - e^{\frac{1}{3}(x+1)}.$$

Exercise 34

Suppose that in the above example the transformations were applied in the reverse order: translations, then reflection, then dilation. Is the answer any different?

Commonly, a graph $y = f(x)$ is transformed by a combination of dilations from the axes, and a translation. The following theorem gives a formula for the resulting graph.

Theorem

Suppose the graph of $y = f(x)$ is transformed by a dilation of factor k from the x-axis, factor $h \neq 0$ from the y-axis, and then translated by the vector (c, d) (i.e. c units to the right and d units up). The resulting graph has equation

$$y = kf\left(\frac{1}{h}(x-c)\right) + d.$$

Note that if there is a reflection in the x-axis, then we can take k to be negative; and if there is a reflection in the y-axis, we can take h to be negative.

Proof

The point (x, y) is mapped to (hx, ky) under the dilations, then to $(hx + c, ky + d)$ under translation, so $x' = hx + c$ and $y' = ky + d$. Inverting these equations (assuming $h, k \neq 0$) gives

$$x = \frac{1}{h}\big(x' - c\big) \quad \text{and} \quad y = \frac{1}{k}\big(y' - d\big).$$

The graph with equation $y = f(x)$ is thus taken under the transformation to points (x', y') such that

$$\frac{1}{k}\big(y' - d\big) = f\left(\frac{1}{h}\big(x' - c\big)\right), \quad \text{which simplifies to} \quad y' = kf\left(\frac{1}{h}\big(x' - c\big)\right) + d.$$

We may now drop the primes and we obtain the desired equation.

There is also the possibility $k = 0$. In this case dilation projects the graph to $y = 0$, then translation takes it to $y = d$, agreeing with the formula. □

While questions involving translations and dilations in the axes are most common, the same method applies to general transformations.

Example

The hyperbola given by $y = 1/x$ is rotated by 45° clockwise about the origin. What is the equation of the resulting curve?

Solution

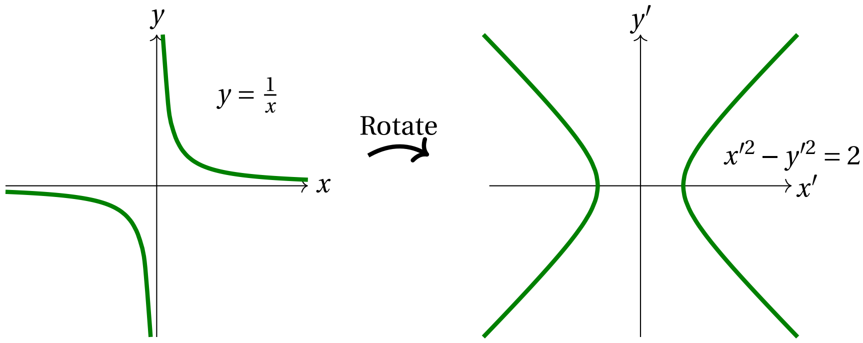
The point (x', y') is obtained by rotating (x, y) by $-\frac{\pi}{4}$, and (x, y) is obtained by rotating (x', y') by $\frac{\pi}{4}$ about the origin. Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x' - y' \\ x' + y' \end{bmatrix}.$$

Hence the curve $y = 1/x$, or $xy = 1$, is mapped to points (x', y') such that

$$\frac{1}{2}(x' - y')(x' + y') = 1, \quad \text{that is,} \quad x'^2 - y'^2 = 2.$$

Thus the equation of the rotated hyperbola is $x^2 - y^2 = 2$.



Exercise 35

Take the graph $y = e^x$. Consider *either* translating it k units to the left, *or* dilating it by e^k from the x-axis. Show that either transformation yields the same graph.

Exercise 36

The unit circle $x^2 + y^2 = 1$ is dilated by a factor of 2 from the x-axis, then rotated by $\pi/4$ about the origin. What sort of curve results? What is its equation?

Exercise 37

Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function. Show that after reflection in the line $y = x$, the graph of $y = f(x)$ is sent to the graph of $y = f^{-1}(x)$.

Links Forward

Areas and determinants

The *determinant* of a 2×2 matrix is given by

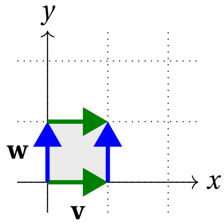
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant has a nice geometric interpretation in terms of *areas*, and this interpretation can also be applied to linear transformations.

We discussed earlier (in "The effect of a linear transformation") how two vectors $\mathbf{v}$ and $\mathbf{w}$ can be used to construct a parallelogram, the parallelogram *spanned* by $\mathbf{v}$ and $\mathbf{w}$. We discussed how, if the matrix $M = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ has $\mathbf{v}$ and $\mathbf{w}$ as its columns, then the linear transformation T_M sends the unit square to this parallelogram. We also discussed the *orientation* of this parallelogram and how T_M can *preserve* or *reverse* orientation.

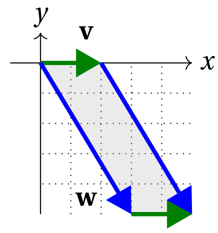
We now ask: what is the area of this parallelogram? It is closely related to the *determinant* of the matrix M.

For instance, if $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (0, 1)$, then $\mathbf{v}$ and $\mathbf{w}$ span the unit square, and these two vectors form the matrix



$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{which has determinant } 1.$$

Similarly, if $\mathbf{v} = (2, 0)$ and $\mathbf{w} = (3, -5)$, then we obtain a parallelogram with base length 2 and height 5, hence with area 10; and the corresponding matrix is



$$M = \begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}, \quad \text{which has determinant } -10.$$

The determinant equals the *negative* area. But in this case the parallelogram is *negatively oriented* and the linear transformation *reverses orientation*.

In fact, the determinant of $M = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ gives the *signed area* of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$, i.e. positive or negative area according to orientation.

The linear transformation T_M sends the unit square to this parallelogram, and in fact takes the tessellation of the plane by unit squares by a tessellation by these parallelograms. So unit squares of area 1 are taken to parallelograms of signed area $\det M$. Thus T_M *expands areas* by a factor of $|\det M|$, and preserves or reverses orientation accordingly as $\det M$ is positive or negative. For any region R in the plane, its image $T_M(R)$ under T_M has area $|\det M|$ times the area of R.

Exercise 38

In this exercise we prove that, if $M = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$, then $\det M$ is signed area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$. Let $A(\mathbf{v}, \mathbf{w})$ be this signed area.

a Show that for any real number k, $A(\mathbf{v}, \mathbf{w}) = A(\mathbf{v}, \mathbf{w} + k\mathbf{v})$ and $A(\mathbf{v}, \mathbf{w}) = A(\mathbf{v} + k\mathbf{w}, \mathbf{w})$.

b Show that $A(\mathbf{v}, \mathbf{w}) = -A(\mathbf{w}, \mathbf{v})$.

c Show that $A\left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) = ad$ and $A\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}\right) = -bc$.

d Show that, if $d \neq 0$, then $A\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) = A\left(\begin{bmatrix} a - \frac{bc}{d} \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) = ad - bc$.

e Conclude that $A(\mathbf{v},\mathbf{w}) = \det M$, where $M = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$.

Example

Find the determinant of the matrix corresponding to the rotation $\mathrm{Rot}_{\mathbf{0},\frac{\pi}{3}}$ of $\pi/3$ about the origin, and explain what this means for areas and orientations.

Solution

The corresponding rotation matrix (with $\phi = \pi/3$) is

$$M = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \text{which has} \quad \det M = 1.$$

Thus the linear transformation $\mathrm{Rot}_{\mathbf{0},\pi/3}$ preserves areas and orientations.

In fact, any rotation preserves areas, so *any* rotation matrix has determinant 1. Similarly, reflections preserve area, but reverse orientation, so reflection matrices have determinant -1. Projections collapse all areas down to zero, so have determinant zero.

Exercise 39

Prove these facts. Show any rotation matrix has determinant 1, any projection matrix has determinant 0, and any reflection matrix has determinant -1.

What about dilation? Dilation from a line L by factor k expands lengths by a factor of k in the direction perpendicular to L, but leaves lengths unchanged in the direction of L. Thus, areas expand by a factor of k.

Example

Let L be the line $y = x$. Find the determinant of the matrix corresponding to dilation from L by a factor of 3. Explain what this means for areas and orientations.

Solution

Using the formula for a dilation matrix, with $k = 3$ and $\phi = \pi/4$, $\mathrm{Dil}_{L,3}$ has matrix

$$\begin{bmatrix} 1+(k-1)\sin^2\phi & (1-k)\sin\phi\cos\phi \\ (1-k)\sin\phi\cos\phi & 1+(k-1)\cos^2\phi \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{which has determinant 3.}$$

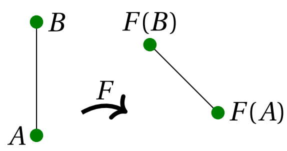
Hence $\mathrm{Dil}_{L,k}$ expands areas by a factor of 3, and preserves orientation.

Exercise 40

Show that the determinant of a general dilation matrix is k.

Isometries

An *isometry* is a plane transformation which *preserves lengths*. That is, a plane transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if, for any points A and B in the plane, the distance between A and B equals the distance between $F(A)$ and $F(B)$.



Isometries are important types of transformations, as they preserve geometry. All lengths must remain the same, and hence angles and areas must also remain the same.

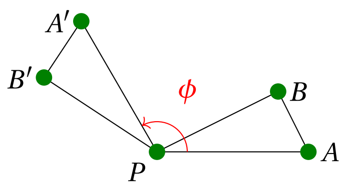
Some of the transformations we have seen are isometries, and others are not.

Lemma

Any rotation is an isometry. That is, for any point P and any angle θ, $Rot_{P,\theta}$ is an isometry.

Proof

Take two distinct points A and B. Let A', B' be their images under rotation. We must show $AB = A'B'$.



Suppose A, B are distinct from P. Consider triangles PAB and $PA'B'$. As A' is obtained from A by a rotation about P, $PA = PA'$. Similarly $PB = PB'$. Since A', B' are obtained by rotating A, B by ϕ about P, $\angle APB = \angle A'PB'$. Hence $\triangle PAB \cong \triangle PA'B'$, so $AB = A'B'$.

In the case $A = P$, we have $A' = P$ so $PB = PB'$ implies $AB = A'B'$. The case $B = P$ is similar. □

Exercise 41

Show that any translation or reflection is an isometry, but that any projection onto a line, or dilation from a line by factor $k \neq \pm 1$ is not an isometry.

An isometry must preserve areas, so if we have an isometry which is a linear transformation corresponding to a matrix M, then $\det M = \pm 1$. However, the converse is not true. There are transformations which preserve areas, but which are not isometries.

Exercise 42

Let $M = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Show that T_M preserves areas, but is not an isometry.

If you compose two isometries, the result must be another isometry. It is interesting to think abut what you get by composing various types of isometries.

For instance, composing a reflection in L with itself results in the identity. But what if we take two reflections in *different* lines L_1 and L_2?

Example

Let L_1 be the line $x = 0$, and L_2 the line $x = d$. Show that reflection in L_1 followed by L_2 is translation by $(2d, 0)$.

Solution

We note that $\mathrm{Ref}_{L_1}(x, y) = (-x, y)$ and $\mathrm{Ref}_{L_2}(x, y) = (2d - x, y)$ (see exercise **??**), so

$$\mathrm{Ref}_{L_2} \circ \mathrm{Ref}_{L_1}(x, y) = \mathrm{Ref}_{L_2}(-x, y) = (x + 2d, y) = (x, y) + (2d, 0),$$

which is translation by $(2d, 0)$.

In general, if L_1 and L_2 are parallel, distance d apart, then the composition $\mathrm{Ref}_{L_2} \circ \mathrm{Ref}_{L_1}$ is a translation by distance $2d$ in a direction perpendicular to L_1 and L_2.

What if L_1 and L_2 intersect at a point P? Then it turns out that reflection in L_1, followed by reflection in L_2, gives a *rotation*.

Example

Let L_1 be the x-axis and L_2 the line through the origin making an angle of ϕ with the x-axis. Show $\mathrm{Ref}_{L_2} \circ \mathrm{Ref}_{L_1}$ is a rotation, and find its angle.

Solution

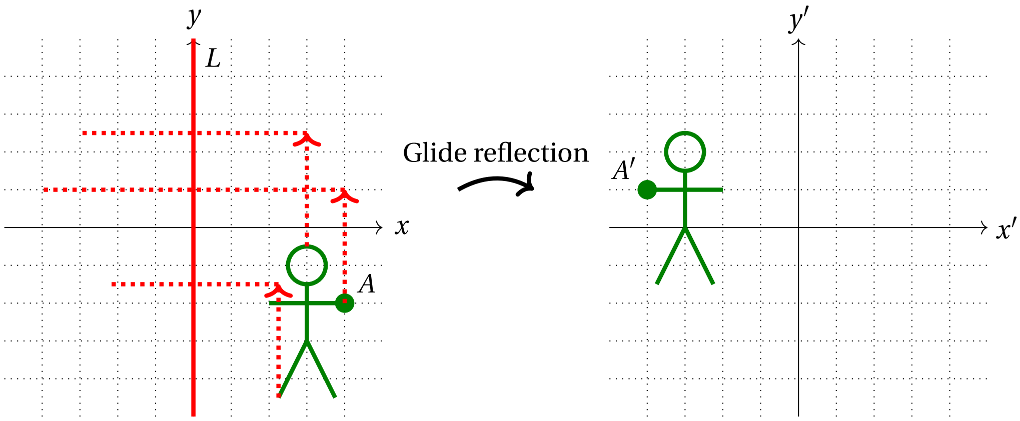
Denote points by polar coordinates (r, θ). Then reflection in L_1 sends a point (r, θ) to $(r, -\theta)$, and reflection in L_2 sends (r, θ) to $(r, 2\phi - \theta)$. Performing the two reflections in order sends $(r, \theta) \mapsto (r, -\theta) \mapsto (r, \theta + 2\phi)$. So the composition is a rotation by angle 2ϕ about the origin:

$$\mathrm{Ref}_{L_2} \circ \mathrm{Ref}_{L_1} = \mathrm{Rot}_{\mathbf{0}, 2\phi}.$$

Exercise 43

Give an alternative proof of this statement by showing that the product of the matrices for reflection in L_1 and L_2 is a rotation matrix of angle 2ϕ.

If you compose a translation of some distance along a line L, with reflection in L, you obtain a type of isometry known as a *glide reflection*.



We have now seen many examples of isometries: rotations, translations reflections, glide reflections, and the identity. We might ask if there are any others. The following theorem says that *there are not*: we have now seen them all.

Theorem (Classification of isometries of the plane)

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ *is an isometry. Then* F *is a rotation, translation, reflection, glide reflection or the identity.*

We will not prove this theorem. A proof can be found, for instance, in I. M. Yaglom, *Geometric Transformations I*.

Three-dimensional and higher-dimensional transformations

All of our discussion in this module has been about *two-dimensional* transformations — transformations of the plane. But much of our reasoning applies to three dimensions, and even to higher dimensions!

Just as a plane transformation is a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, a spatial transformation is a function $F : \mathbb{R}^3 \to \mathbb{R}^3$. Just as a linear plane transformation is given by a 2×2 matrix, a linear spatial transformation is given by a 3×3 matrix.

In three dimensions, we can consider translations by 3-dimensional vectors, rotations about a *line*, reflections in a *plane*, projections onto a *plane*, and dilations from a plane.

Similar methods apply as for the two-dimensional case. Just as there is a standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$, there is a standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. A linear transformation T_M takes the three standard basis vectors to the three columns of the matrix M. We can visualise T_M as taking the tessellation of $\mathbb{R}^3$ by unit cubes, to a tessellation of $\mathbb{R}^3$ by *parallelepipeds* (the 3-d version of parallelograms), where each parallelepiped is spanned by the three column vectors of M. The determinant of M now gives the *volume* expansion of T_M; its sign again tells us if T_M preserves or reverses orientation.

In essence, everything we have said in this module can be generalised to $\mathbb{R}^3$. And in principle it can be generalised to any dimension $\mathbb{R}^n$ — it's just a little harder to visualise!

Answers to exercises

Exercise 1

If X is on L then $X' = X$, and the closest point on L to X is X itself. So assume L is on X. For any point Y on L other than X', $XX'Y$ forms a triangle with a right angle at X', so $XY > XX'$. Hence X' is the closest point to X on L.

Exercise 2

Take any vector $\mathbf{v}$ and $c = 0$ in $F(c\mathbf{v}) = cF(\mathbf{v})$ to obtain $F(\mathbf{0}) = \mathbf{0}$.

Exercise 3

a Rotation $\pi/3$ anticlockwise about the origin.

b Dilation by factor 7 in all directions from the origin.

c Rotation by π about the origin.

d Dilation from the x-axis with factor 3, and dilation from the y-axis with factor 2.

e Reflection in the line $y = x$.

Exercise 4

We compute $f \circ f(x, y) = f(y, -x + y) = (-x + y, -x)$ and then $f^3(x, y) = f \circ f \circ f(x, y) = f(-x+y, -x) = (-x, -y)$. Now f^6 is obtained by composing f^3 with itself twice, so $f^6(x, y) = f^3(f^3(x, y)) = f^3(-x, -y) = (x, y)$. Thus f^6 is the identity.

Exercise 5

$$MM^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{Id}$$

The computation for $M^{-1}M$ is similar.

Exercise 6

Note that $x' = x+2y$, $y' = 2x+4y$ implies $y' = 2x'$, so the image of F lies on the line $y' = 2x'$. So F is not surjective: for instance, there is no (x, y) such that $F(x, y) = (0, 1)$; if there were, then $x+2y = 0$ and $2x+4y = 1$, but then $2x+4y = 2(x+2y) = 0$, a contradiction. Alternatively, F is not injective as, for instance, $F(0,0) = F(-2,1) = (0,0)$.

Exercise 7

Let M be the desired matrix, so $M\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$. Thus $M = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 7 \\ 7 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. Hence $T_M(x, y) = (y, x-y)$.

Exercise 8

The x and y-axes are sent to the y-axis and negative x-axis respectively, so a point (x, y) is mapped to $(-y, x)$.

Exercise 9

The point $(0,1)$ is on the unit circle at an angle of $\pi/2$ from the positive x-axis. Rotating by ϕ gives the point $(\cos(\phi+\frac{\pi}{2}), \sin(\phi+\frac{\pi}{2})) = (-\sin\phi, \cos\phi)$.

Exercise 10

We verify

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

which by sine and cosine addition formulas is the rotation matrix for angle $\theta+\phi$. Hence the composition of rotations by θ and ϕ is rotation by $\theta+\phi$.

Exercise 11

If ϕ is not a multiple of 2π, then a rotation of ϕ about P sends no point to itself except P.

So if $P \neq \mathbf{0}$ then $\mathrm{Rot}_{P,\phi}(\mathbf{0}) \neq \mathbf{0}$. By exercise **??** a linear transformation must send the origin to the origin, so $\mathrm{Rot}_{P,\phi}$ cannot be linear.

Exercise 12

Translations by $\mathbf{v}$ and $-\mathbf{v}$ undo each other; they are inverse transformations. Thus $\mathrm{Trans}_{\mathbf{v}}$ is bijective, with inverse $\mathrm{Trans}_{-\mathbf{v}}$.

Exercise 13

Let $\mathbf{x}$ be a point in the plane. Translation by $-\mathbf{v}$ takes P to the origin and carries $\mathbf{x}$ along to $\mathbf{x}-\mathbf{v}$. Then rotating by ϕ about $\mathbf{0}$ preserves $\mathbf{0}$ and rotates $\mathbf{x}-\mathbf{v}$ around it. Then translating by $\mathbf{v}$ moves the origin back to P and takes our point to the point rotated by ϕ about P. Thus

$$\begin{aligned}\mathrm{Rot}_{P,\phi}\begin{bmatrix} x \\ y \end{bmatrix} &= \mathrm{Trans}_{\mathbf{v}} \circ \mathrm{Rot}_{\mathbf{0},\phi} \circ \mathrm{Trans}_{-\mathbf{v}} \begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{Trans}_{\mathbf{v}} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \\ &= \mathrm{Trans}_{\mathbf{v}} \begin{bmatrix} (x-a)\cos\phi - (y-b)\sin\phi \\ (x-a)\sin\phi + (y-b)\cos\phi \end{bmatrix} = \begin{bmatrix} (x-a)\cos\phi - (y-b)\sin\phi + a \\ (x-a)\sin\phi + (y-b)\cos\phi + b \end{bmatrix}.\end{aligned}$$

Exercise 14

a Let Y be a point not on L. Then there is no X such that $\mathrm{Proj}_L(X) = Y$, so Proj_L is not surjective.

b Let L' be a line perpendicular to L, intersecting L at a point Y. Then for any point X on L', $\mathrm{Proj}_L(X) = Y$. Hence Proj_L is not injective.

c If P is on L, then the perpendicular to L through P intersects L at P, so $\mathrm{Proj}_L(P) = P$.

d For any point X, $\mathrm{Proj}_L(X)$ lies on L, so using the previous part with $P = \mathrm{Proj}_L(X)$ we have $\mathrm{Proj}_L\big(\mathrm{Proj}_L(X)\big) = \mathrm{Proj}_L(X)$.

Exercise 15

The line through (x_0, y_0) with gradient -1 has equation $y - y_0 = -(x - x_0)$, or equivalently $y = -x + x_0 + y_0$. At the intersection with the line $y = x$ we have $y = -x + x_0 + y_0 = x$ so $x = \frac{1}{2}(x_0 + y_0)$ and then $y = x = \frac{1}{2}(x_0 + y_0)$ as desired.

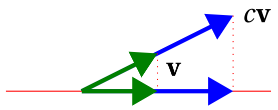
Exercise 16

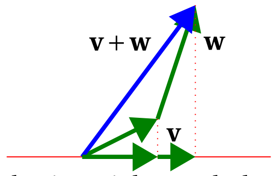
The line L given by $y = 2x$ makes an angle ϕ with the positive x-axis such that $\tan\phi = 2$, hence $\sin\phi = \frac{2}{\sqrt{5}}$ and $\cos\phi = \frac{1}{\sqrt{5}}$. Thus

$$\mathrm{Proj}_L \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{28}{5} \end{bmatrix}.$$

Exercise 17

a A proof may be given by explaining the following pictures.





b Dropping a perpendicular from $(1,0)$ to L as shown we obtain a right-angled triangle with hypotenuse 1 and angle ϕ; so the projection of $(1,0)$ to L is a distance of $\cos\phi$ along L from the origin. Lying on a ray at angle ϕ at a distance $\cos\phi$ from the origin, then, $\mathrm{Proj}_L(1,0) = \cos\phi(\cos\phi, \sin\phi) = (\cos^2\phi, \sin\phi\cos\phi)$. Similarly $\mathrm{Proj}_L(0,1)$ lies at distance $\sin\phi$ from the origin on L, so equals $(\sin\phi\cos\phi, \sin^2\phi)$.

c The first part shows Proj_L is linear, hence given by a matrix. The second part finds the columns of this matrix. So Proj_l is as claimed.

Exercise 18

a If $P = (x, y)$ is on L, then $(x, y) = (s\cos\phi, s\sin\phi)$ for some s. Hence

$$\mathrm{Proj}_L(P) = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} s\cos\phi \\ s\sin\phi \end{bmatrix} = \begin{bmatrix} s\cos\phi(\cos^2\phi+\sin^2\phi) \\ s\sin\phi(\cos^2\phi+\sin^2\phi) \end{bmatrix} = \begin{bmatrix} s\cos\phi \\ s\sin\phi \end{bmatrix} = P.$$

b We compute

$$\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}^2 = \begin{bmatrix} \cos^2\phi(\cos^2\phi+\sin^2\phi) & \sin\phi\cos\phi(\cos^2\phi+\sin^2\phi) \\ \sin\phi\cos\phi(\cos^2\phi+\sin^2\phi) & \sin^2\phi(\cos^2\phi+\sin^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}.$$

As this is the matrix for Proj_L, we conclude $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$ as desired.

Exercise 19

Under Proj_L, points on L are sent to themselves, but any point not on L is sent to a different point. So if L does not pass through the origin $\mathbf{0}$, then $\mathrm{Proj}_L(\mathbf{0}) \neq \mathbf{0}$. By exercise **??**, a linear transformation must send $\mathbf{0}$ to $\mathbf{0}$. Hence Proj_L is not linear.

Exercise 20

a Rotating by $-\phi$ about the origin brings L to the x-axis, carrying along a point P. Then reflecting in the x-axis flips the point over the x-axis; rotating back by ϕ sends the x-axis back to L, and carries our point to the reflection of P in L. Hence Ref_L is given by the desired composition of transformations.

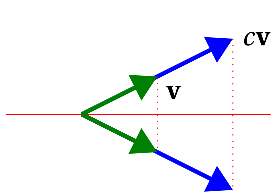
b As $\mathrm{Rot}_{\mathbf{0},\phi}$, $\mathrm{Ref}_{x\text{-axis}}$ and $\mathrm{Rot}_{\mathbf{0},-\phi}$ are all linear, and we know their matrices, we can multiply them together to find a matrix for Ref_L, which must also be linear. The matrix is

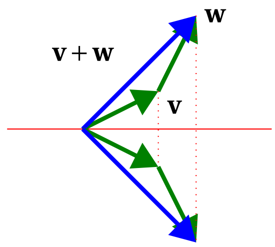
$$\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi\\ 2\sin\phi\cos\phi & \sin^2\phi-\cos^2\phi\end{bmatrix} = \begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}.$$

Exercise 21

a A proof may be given by explaining the following diagrams.





b Since $(1,0)$ is distance 1 from the origin on an angle of 0, reflection in a line at angle ϕ sends $(1,0)$ to a point distance 1 from the origin on an angle of 2ϕ. Thus $\mathrm{Ref}_L(1,0) = (\cos(2\phi), \sin(2\phi))$. Similarly one can show $\mathrm{Ref}_L(0,1) = (\sin(2\phi), -\cos(2\phi))$.

c The first part shows Ref_L is linear; the second part calculates the columns of the corresponding matrix, which is as claimed.

Exercise 22

The line L given by $y = 2x$ passes through the origin on an angle ϕ such that $\tan\phi = 2$, hence $\sin\phi = \frac{2}{\sqrt{5}}$ and $\cos\phi = \frac{1}{\sqrt{5}}$. Hence $\cos(2\phi) = \cos^2\phi - \sin^2\phi = \frac{-3}{5}$ and $\sin(2\phi) = 2\sin\phi\cos\phi = \frac{4}{5}$. So reflection in L is given by

$$\mathrm{Ref}_L\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\frac{-3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}.$$

Exercise 23

The vector from $\mathrm{Proj}_L(\mathbf{x})$ to $\mathbf{x}$, and the vector from $\mathrm{Proj}_L(\mathbf{x})$ to $\mathrm{Ref}_L(\mathbf{x})$, have the same length and point in opposite directions. Hence

$$\mathbf{x} - \mathrm{Proj}_L(\mathbf{x}) = -\left(\mathrm{Ref}_L(\mathbf{x}) - \mathrm{Proj}_L(\mathbf{x})\right),$$

which rearranges to $\mathrm{Ref}_L(\mathbf{x}) = 2\mathrm{Proj}_L(\mathbf{x}) - \mathbf{x}$, as desired.

Exercise 24

From the previous exercise, the matrix for Ref_L is given by twice the matrix for Proj_L, minus the identity matrix, which is

$$2\begin{bmatrix}\cos^2\phi & \sin\phi\cos\phi\\ \sin\phi\cos\phi & \sin^2\phi\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2\cos^2\phi-1 & 2\sin\phi\cos\phi\\ 2\sin\phi\cos\phi & 2\sin^2\phi-1\end{bmatrix}=\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}.$$

Exercise 25 **a** If $P=(x,y)$ lies on L then $(x,y)=(s\cos\phi, s\sin\phi)$ for some s, so

$$\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}\begin{bmatrix}s\cos\phi\\ s\sin\phi\end{bmatrix}$$

$$=\begin{bmatrix}s\cos(2\phi-\phi)\\ s\sin(2\phi-\phi)\end{bmatrix}=\begin{bmatrix}s\cos\phi\\ s\sin\phi\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

b We verify $\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}$ is diagonal, with both diagonal entries given by $\cos^2(2\phi)+\sin^2(2\phi)=1$, hence is the identity matrix.

Exercise 26

We note that projection onto L is given by $\mathrm{Proj}_L(x,y)=(d,y)$. Using exercise **??** then $\mathrm{Ref}_L(x,y)=2\mathrm{Proj}_L(x,y)-(x,y)=2(d,y)-(x,y)=(2d-x,y)$.

Exercise 27 **a** Rotating by $-\phi$ about the origin brings L to the x-axis, and brings a point P along with it; then dilating from the x-axis and rotating back by ϕ brings our line back to L and sends our point to its dilation from L.

b The three transformations in the previous part are all linear; multiplying them will give a matrix for $\mathrm{Dil}_{L,k}$, which must be linear.

$$\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}1&0\\0&k\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -k\sin\phi & k\cos\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\phi+k\sin^2\phi & (1-k)\sin\phi\cos\phi\\ (1-k)\sin\phi\cos\phi & \sin^2\phi+k\cos^2\phi\end{bmatrix}=\begin{bmatrix}1+(k-1)\sin^2\phi & (1-k)\sin\phi\cos\\ (1-k)\sin\phi\cos\phi & 1+(k-1)\cos^2\end{bmatrix}$$

Exercise 28

The line L given by $y=2x$ makes angle ϕ with the x-axis, where $\tan\phi=2$, so $\sin\phi=\frac{2}{\sqrt{5}}$ and $\cos\phi=\frac{1}{\sqrt{5}}$. Hence

$$\mathrm{Dil}_{L,5}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}1+4\cdot\frac{4}{5} & -4\cdot\frac{2}{5}\\ -4\cdot\frac{2}{5} & 1+4\cdot\frac{1}{5}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}\frac{21}{5} & \frac{-8}{5}\\ \frac{-8}{5} & \frac{9}{5}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}\frac{63}{5}\\ \frac{-24}{5}\end{bmatrix}.$$

Exercise 29

We first note that $\mathrm{Dil}_{L,k}$ has determinant

$$\big(1+(k-1)\sin^2\phi\big)\big(1+(k-1)\cos^2\phi\big)-(1-k)^2\sin^2\phi\cos^\phi = 1+(k-1)\sin^2\phi+(k-1)\cos^2\phi = k.$$

Hence the inverse of the matrix for $\mathrm{Dil}_{L,k}$ is

$$\frac{1}{k}\begin{bmatrix} 1+(k-1)\cos^2\phi & (k-1)\sin\phi\cos\phi \\ (k-1)\sin\phi\cos\phi & 1+(k-1)\sin^2\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{k}+\big(1-\frac{1}{k}\big)\cos^2\phi & \big(1-\frac{1}{k}\big)\sin\phi\cos\phi \\ \big(1-\frac{1}{k}\big)\sin\phi\cos\phi & \frac{1}{k}+\big(1-\frac{1}{k}\big)\sin^2\phi \end{bmatrix}.$$

Now $\frac{1}{k}+(1-\frac{1}{k})\cos^2\phi = \frac{1}{k}(1-\cos^2\phi)+\cos^2\phi = \frac{1}{k}\sin^2\phi+1-\sin^2\phi = 1+\big(\frac{1}{k}-1\big)\sin^2\phi$ and, similarly, $\frac{1}{k}+\big(1-\frac{1}{k}\big)\sin^2\phi = 1+\big(\frac{1}{k}-1\big)\cos^2\phi$. Hence this matrix is the matrix for dilation by factor $1/k$ from L.

Exercise 30

a When $k=0$, the matrix for $\mathrm{Dil}_{L,k}$ becomes

$$\begin{bmatrix} 1-\sin^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & 1-\sin^2\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \cos^2\phi \end{bmatrix},$$

which is the matrix for projection onto L.

b When $k=-1$ we have

$$\begin{bmatrix} 1-2\sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & 1-2\cos^2\phi \end{bmatrix} = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

which is the matrix for reflection in L.

Exercise 31

a A reflection is a dilation by factor -1, so applying it twice gives a dilation with factor $(-1)^2=1$, i.e. the identity.

b A projection is a dilation by factor 0, so applying it twice gives a dilation with factor $0^2=0$, i.e. the same projection.

Exercise 32

a A projection matrix clearly has the desired properties. For the converse, suppose M is symmetric ($b=c$), $ad=b^2$ and $a+d=1$. Since a and d have sum and product ≥ 0, both are ≥ 0; since they sum to 1, both lie in $[0,1]$. Hence we have $a=\cos^2\phi$ and $b=\sin^2\phi$ for some ϕ. Then $b^2=\cos^2\phi\sin^2\phi$ so $b=\sin\phi\cos\phi$ or $-\sin\phi\cos\phi$. In the first case we have the matrix for projection onto a line at angle ϕ; in the second case for projection onto a line at angle $-\phi$.

b A reflection matrix clearly has the desired properties. For the converse, suppose M is symmetric ($b=c$), $a=-d$ and $ad-b^2=-1$. Substituting $a=-d$ then gives $-a^2-b^2=-1$, so (a,b) lies on he unit circle and hence $a=\cos(2\phi)$, $b=\sin(2\phi)$ for some ϕ. We then have a reflection matrix.

c A dilation matrix can be verified to have the desired properties. For the converse, suppose $b = c$ and $a + d = 1 + ad - b^2$. Let $ad - b^2 = k$, so $a + d = k + 1$. Since $(a - 1) + (d - 1) = k - 1$ and $(a-1)(d-1) = ad - a - d + 1 = b^2$, both $a - 1$ and $d - 1$ have the same sign as $k - 1$. Since they sum to $k - 1$, they lie between 0 and $k - 1$. Thus we may write $a - 1 = (k-1)\sin^2\phi$ and $d - 1 = (k-1)\cos^2\phi$ for some ϕ. We then have $b^2 = ad - k = [1 + (k-1)\sin^2\phi][1 + (k-1)\cos^2\phi] - k = (k-1)^2\sin^2\phi\cos^2\phi$. Thus $b = \pm(1-k)\sin\phi\cos\phi$. Replacing ϕ with $-\phi$ if necessary, we have a dilation matrix.

Exercise 33

Applying the translation we have $(x, y) \mapsto (x + 3, y)$, then applying the dilation we have $(x + 3, y) \mapsto (x + 3, 2y) = (x', y')$. So the transformation is the same, and the answer is the same.

Exercise 34

Applying the translation we have $(x, y) \mapsto (x + 1, y + 3)$; then applying the reflection we have $(x + 1, y + 3) \mapsto (-x - 1, y + 3)$; then applying the dilation we have $(-x - 1, y + 2) \mapsto (-3x - 3, y + 2)$. The result is a different transformation, giving a different answer.

Exercise 35

Translating k units to the left gives $y = e^{x+k}$. Dilating by e^k form the x-axis gives $y = e^k e^x$. These are the same graph since $e^{x+k} = e^k e^x$.

Exercise 36

From (x, y), dilating and then rotating gives an ellipse; the transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and hence, inverting,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x' + y' \\ \frac{-1}{2}x' + \frac{1}{2}y' \end{bmatrix}.$$

Thus the expression $x^2 + y^2$ becomes

$$\frac{1}{2}(x' + y')^2 + \frac{1}{2}\left(-\frac{1}{2}x' + \frac{1}{2}y'\right)^2 = \frac{5}{8}x'^2 + \frac{5}{8}y'^2 + \frac{3}{4}x'y'.$$

The equation of the transformed circle is thus $\frac{5}{8}x^2 + \frac{5}{8}y^2 + \frac{3}{4}xy = 1$, which is an ellipse.

Exercise 37

Reflection in the line $y = x$ takes (x, y) to $(x', y') = (y, x)$. The equation $y = f(x)$ becomes $x' = f(y')$, and hence the transformed graph is given by $x = f(y)$, or equivalently, $y = f^{-1}(x)$.

Exercise 38

a The parallelograms spanned by $\mathbf{v}, \mathbf{w}$ and $\mathbf{v}, \mathbf{w} + k\mathbf{v}$ both have base $\mathbf{v}$ the same height above this base, and the same orientation, so have the same signed area. The same applies to $\mathbf{v}, \mathbf{w}$ and $\mathbf{v} + k\mathbf{w}, \mathbf{w}$ above a base of $\mathbf{w}$.

b If we reverse the order of $\mathbf{v}$ and $\mathbf{w}$ the parallelogram is unchanged but has its orientation reversed, hence its signed area changes sign.

c The vectors $(a, 0)$ and (b, d) span a parallelogram with base a and perpendicular height d, hence area $|ad|$. The parallelogram is positively oriented if a and d have the same sign, and negatively oriented otherwise, so the signed area is ad. A similar argument applies to (a, c) and $(b, 0)$.

d Applying the first part to (a, c) and (b, d), the signed area they span is equal to the signed area spanned by $(a, c) - \frac{c}{d}(b, d)$ and (b, d).

e For any vectors $\mathbf{v} = (a, c)$ and $\mathbf{w} = (b, d)$, if $d \neq 0$ then the previous part applies and gives the signed area as $\det M$. If $d = 0$ then part c applies and gives the signed area as $-bc = ad - bc = \det M$.

Exercise 39

We compute $\det \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \cos^2\phi + \sin^2\phi = 1$, $\det \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} = \cos^2\phi\sin^2\phi - \cos^2\phi\sin^2\phi = 0$ and $\det \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = -\cos^2(2\phi) - \sin^2(2\phi) = -1$.

Exercise 40

Compute $\det \begin{bmatrix} 1 + (k-1)\sin^2\phi & (1-k)\sin\phi\cos\phi \\ (1-k)\sin\phi\cos\phi & 1 + (k-1)\cos^2\phi \end{bmatrix} = \left[1 + (k-1)\sin^2\phi\right]\left[1 + (k-1)\cos^2\phi\right] - (1-k)^2\sin^2\phi\cos^2\phi$, which simplifies to $1 + (k-1)(\sin^2\phi + \cos^2\phi) + (k-1)^2\sin^2\phi\cos^2\phi - (1-k)^2\sin^2\phi\cos^2\phi = 1 + (k-1) = k$.

Exercise 41

Let A, B be points in the plane and let $A'B'$ be their images under transformation; to demonstrate an isometry we show $AB = A'B'$.

Under a translation, $ABB'A'$ forms a parallelogram so $AB = A'B'$.

For a reflection, let X be a point on the line of reflection. Then triangles XAB and $XA'B'$ are congruent, so $AB = A'B'$.

For a projection, take A, B to be distinct points projecting onto the same point. Then $A' = B'$ so $A'B' = 0 < AB$. As $AB \neq A'B'$, a projection is not an isometry.

For a dilation, take A to be a point on the line of dilation and B a point which projects onto A. Then $A'B' = |k|AB \neq AB$. Hence this transformation is not an isometry.

Exercise 42

Since $\det M = 1$, T_M preserves areas. But, for instance, let $A = (0,0)$ and $B = (1,0)$, so $A' = (0,0)$ and $B' = (2,0)$. Then $AB = 1$ but $A'B' = 2$ so T_M is not an isometry.

Exercise 43

We compute the matrix for the composition $\mathrm{Ref}_{L_2} \circ \mathrm{Ref}_{L_1}$ as

$$\begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix},$$

which is the matrix for rotation by 2ϕ about the origin.

Years

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