VCAA-AMSI maths modules

A guide for teachers – Years 11 and 12

Geometry

Spherical Geometry

Years 11 & 12

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Spherical Geometry – A guide for teachers (Years 11–12)

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Spherical Geometry

Assumed knowledge

- Familiarity with circle measurement formulae, including circumference and area
- Familiarity with trigonometric calculations, including basic SOHCAHTOA, sine and cosine rules
- Familiarity with area of triangles using sine area rule
- Familiarity with circle geometry.

Motivation

- What have circles to do with measuring distances on the surface of the Earth ?
- Do I have to use radians for measuring any angles ?
- Will it make much difference to the distance calculated if I use an incorrect method ?
- Why is the straight-line distance between two points on a sphere not necessarily the shortest distance between those two points ?
- How does a spherical model of the Earth help me to understand the different time zones that have been established ?

Content

A circle is formed by a point moving around a fixed point (the centre) at a constant distance.

A sphere is a three-dimensional form of a circle, where all points on the surface of the sphere are at a constant distance from the centre of the sphere.

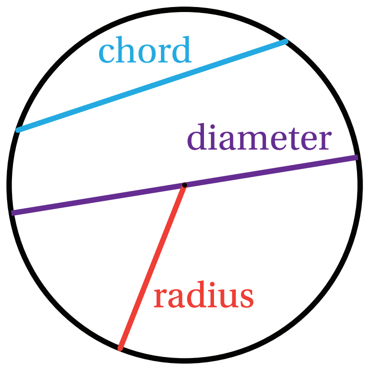
The Earth is not exactly a sphere. A grapefruit is a better analogy, as it is flatter at the top and bottom (an oblate sphere). The diameter as measured through the North and South Poles is about 40 km less than the diameter using two points on the Equator. The radius of the Earth is usually given as 6 370 kilometres.

For the purposes of modelling key aspects of travel and time on the Earth, for a course such as Further Mathematics in the Victorian Certificate of Education, treating the Earth as a true sphere with a radius of 6 400 kilometres will provide reasonable answers from a conceptually simple framework.

Circular Geometry

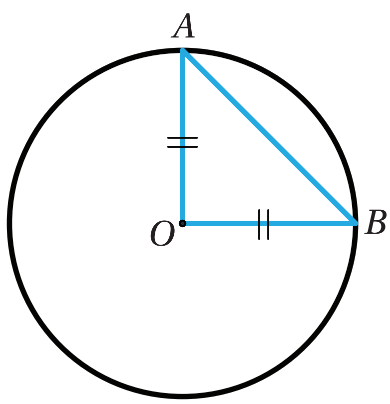
Radii and chords

We begin by revisiting the definition of a circle and some of the language used for describing the geometry of circles.

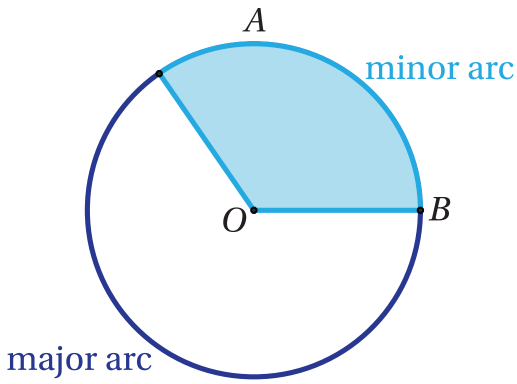


- A **circle** is the set of all points in the plane that are a fixed distance (the radius) from a fixed point (the centre).
- Any interval joining a point on the circle to the centre is called a **radius**. By the definition of a circle, any two radii have the same length.
- An interval joining two points on the circle is called a **chord**.
- A chord that passes through the centre is called a **diameter**. Since a diameter consists of two radii joined at their endpoints, every diameter has length equal to twice the radius.

Let AB be a chord of a circle not passing through its centre O. The chord and the two equal radii OA and BO form an isosceles triangle whose base is the chord. The $\angle AOB$ is called the angle at the centre subtended by the chord.



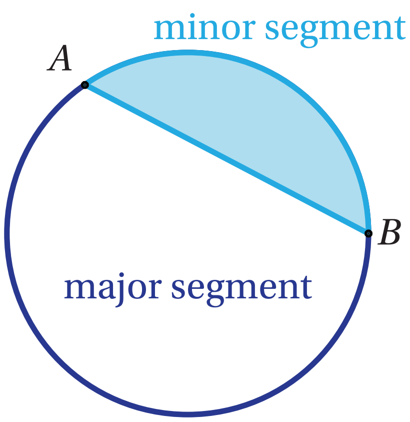
Let A and B be two different points on a circle with centre O. These two points divide the circle into two opposite arcs. If the chord AB is a diameter, then the two arcs are called **semicircles**. Otherwise, one arc is longer than the other. The longer arc is called the **major arc** AB and the shorter arc is called the **minor arc** AB.



Now join the radii OA and OB. The reflex angle AOB is called the **angle subtended** by the major arc AB. The non-reflex angle AOB is called the angle subtended by the minor arc AB, and is also the angle subtended by the chord AB. The two radii divide the circle into two sectors, called correspondingly the major sector OAB and the minor sector OAB.

Segments

A chord AB of a circle divides the circle into two segments. The two segments are called a **major segment** (the larger one) and a **minor segment** (the smaller one).



Subtend

The word 'subtend' literally means 'holds under', and is often used in geometry to describe an angle. The phrase 'opposite to' could also be used, since, in the case above, the angle subtended by an arc is located opposite to that arc in the diagram.

Calculation Notes

Unless it is specifically requested, the π value obtained by accessing the appropriate function in a scientific or graphic calculator should be used in any calculation involving π.

ALL angles in this unit will be specified in degrees.

Measuring arc length

In any circle, the length of an arc (l) is proportional to the angle (θ) it creates (subtends) at the centre. For a circle of radius r, the circumference will be 2π units.

Thus,

$$\frac{\text{the length of the arc}}{\text{circumference}} = \frac{\theta}{360^\circ},$$

so $\dfrac{l}{2\pi r} = \dfrac{\theta}{360^\circ}$.

$$\begin{aligned}\therefore\ l &= \frac{\theta}{360^\circ} \times 2\pi r\\ &= \frac{\theta}{180^\circ} \times \pi r\end{aligned}$$

Example

A circle has a radius of 30 cm. Find the length of an arc subtending an angle of 75° at the centre, correct to one decimal place.

Solution

$$\begin{aligned}\text{Length of arc} &= \frac{r\pi}{180^\circ} \times \theta\\ &= \frac{30\pi}{180^\circ} \times 75^\circ\\ &= 39.2699\ldots\\ &\approx 39.3 \text{ cm}\end{aligned}$$

Example

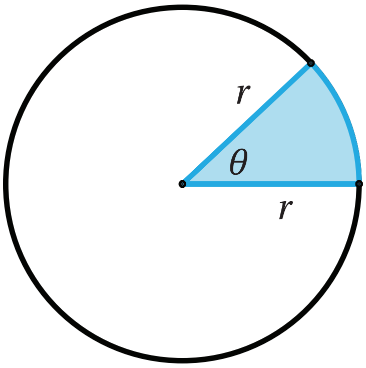
An arc of a circle has a length of 30 cm. If the radius of the circle is 25 cm, what angle, correct to the nearest whole degree, does the arc subtend at the centre ?

Solution

$$\begin{aligned}30 &= \frac{r\pi}{180^\circ} \times \theta\\ &= \frac{25\pi}{180^\circ} \times \theta\\ \theta &= \frac{30 \times 180^\circ}{25 \times \pi}\\ &= 68.754\ldots \approx 69^\circ\end{aligned}$$

Sector

The area of a sector depends not only on the radius of the circle involved, but also the angle between the two straight edges.



We can see this in the following table :

Angle	Fraction of circle area	Area rule
$180°$	$\frac{180°}{360°} = \frac{1}{2}$	$A = \frac{1}{2} \times \pi r^2$
$90°$	$\frac{90°}{360°} = \frac{1}{4}$	$A = \frac{1}{4} \times \pi r^2$
$45°$	$\frac{45°}{360°} = \frac{1}{8}$	$A = \frac{1}{8} \times \pi r^2$
$30°$	$\frac{30°}{360°} = \frac{1}{12}$	$A = \frac{1}{12} \times \pi r^2$
θ	$\frac{\theta}{360°}$	$A = \frac{\theta}{360} \times \pi r^2$

Example

Find the area of a sector with a radius of 40 mm, that contains an angle of $60°$, correct to the nearest square millimetre.

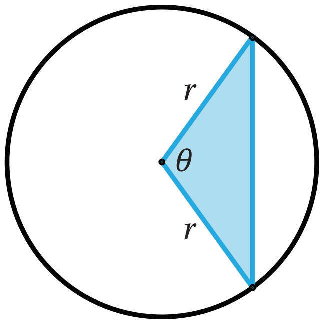
Solution

$$A = \frac{\theta}{360°} \times \pi r^2 = \frac{60°}{360°} \times \pi 40^2 = 923.628\ldots \approx 924 \text{ mm}^2$$

Areas of segments

Where a chord subtends an angle θ at the centre of a circle of radius r, the area of the minor segment is given by :

$$\text{Area} = \frac{1}{2} \times r^2 \left(\frac{2\pi\theta}{360°} - \sin\theta \right)$$



The area of the sector that includes the segment is given by $A = \dfrac{\theta}{360°} \times \pi r^2$

or rewritten as

$$A = \frac{1}{2} \times r^2 \times \frac{2\pi\theta}{360°}.$$

The area of the triangle whose boundaries are the two radii and the chord is given by $\dfrac{1}{2} \times r^2 \times \sin\theta$ $\left(\text{from the Sine Area Rule } \dfrac{1}{2} \times ab\sin C\right)$.

Hence the area of the segment (minor) can be calculated by subtracting the area of the triangle from the area of the sector.

The area of the major segment can be calculated by taking the area of the minor segment from the total area of the circle.

Example

Find the area, correct to two decimal places, of the minor segment in a circle of radius 10 cm where the angle subtended at the centre of the circle by the chord is 85°.

Solution

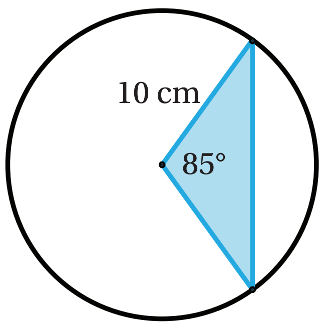
Area

$$= \frac{1}{2} \times r^2 \left(\frac{2\pi\theta}{360°} - \sin\theta \right)$$

$$= \frac{1}{2} \times 10^2 \left(\frac{2 \times \pi \times 85°}{360°} - \sin 85° \right)$$

$$= 24.366\ldots$$

$$\approx 24.37\text{cm}^2$$



Example

Find the area, correct to one decimal place, of the minor segment in a circle, of radius 15 cm where the chord length is also 15 cm.

Solution

If the chord length is the same as the radius of the circle, then the triangle that is formed will be equilateral (all three sides are 15 cm!), and the angle subtended at the centre by the chord will be 60° (the angle inside an equilateral triangle).

$$\text{Area} = \frac{1}{2} \times r^2 \left(\frac{2\pi\theta}{360^\circ} - \sin\theta \right) = \frac{1}{2} \times 15^2 \left(\frac{2 \times \pi \times 60^\circ}{360^\circ} - \sin 60^\circ \right) = 20.381\ldots \approx 20.4 \text{ cm}^2$$

Example

Find the area, correct to two decimal places, of the minor segment in a circle, of radius 13 cm where the chord length is 21 cm.

Solution

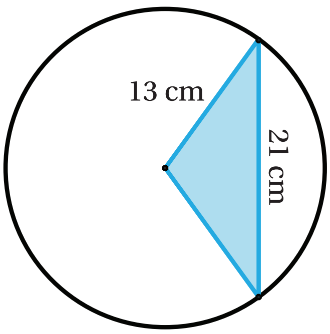
To find the angle subtended at the centre by the chord, we will have to use the Cosine Rule

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc} \right)$$

where $b = c =$ radius

$a =$ chord length,

$A =$ angle subtended at centre of circle.



$$\text{So A} = \cos^{-1}\left(\frac{13^2 + 13^2 - 21^2}{2 \times 13 \times 13} \right)$$

$$= 107.74^\circ$$

$$\text{Area} = \frac{1}{2} \times r^2 \left(\frac{2\pi\theta}{360^\circ} - \sin\theta \right)$$

$$= \frac{1}{2} \times 13^2 \left(\frac{2 \times \pi \times 107.74^\circ}{360^\circ} - \sin 107.74^\circ \right)$$

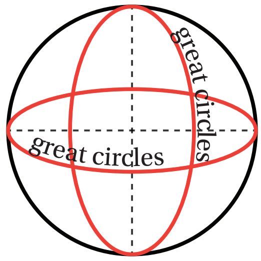
$$= 78.409\ldots$$

$$\approx 78.41 \text{ cm}^2$$

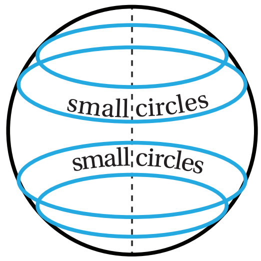
The Earth as a sphere

In this topic, the Earth will be considered to be a perfect sphere, with a radius of 6 400 kilometres. (see Introductory Notes)

There are two types of circles that can be drawn on the surface of a sphere.



A **great circle** is the largest circle that can be drawn on the surface of a sphere, as the centre of the great circle is also the centre of the sphere. The radius of a great circle on Earth will therefore be 6 400 kilometres which is the same as the radius of the Earth sphere. As shown above, the equator and a circle drawn through both the North and South poles are great circles. The meridians of longitude are all great circles.



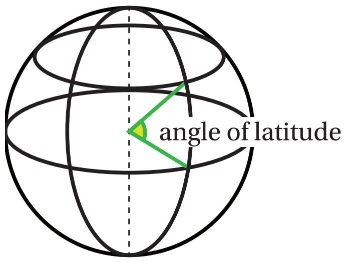
A great circle track is the shortest distance between two points on the surface of a sphere and forms the basis of many navigational activities for long distance travel by air or sea. A key calculation as part of this process is the finding of waypoints; these are points that the traveller should pass through to maintain their journey on the great circle.

A **small circle** does not have its centre at the centre of the sphere. The radius of a small circle will be LESS than the radius of the sphere. For Earth, it means that the radius of a small circle will be less than 6 400 kilometres. The parallels of latitude (apart from the Equator) are all small circles.

Latitude and Longitude

Latitude

Latitude is a measure of the position of a point on the earth's surface in terms of degrees north or south of a baseline - the equator. These values vary from 90°S to 90°N. The latitude of the Equator is 0°.



Every plane perpendicular (at right angles) to the earth's axis cuts the surface of the earth in a circle called a parallel of latitude.

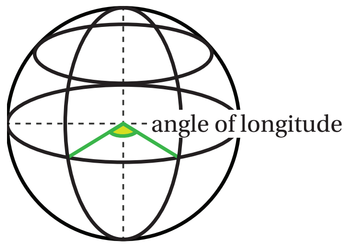
The parallel of latitude formed by a plane passing through the centre of the earth and equidistant from both the North and South Poles is called the **equator**. All other parallels of latitude are small circles, each having a radius smaller than that of the earth, i.e. less than 6400 km. As the parallels run east and west, places having the same latitude will be due east or due west of each other.

Distance north or south of the equator is calculated from the arc length using the angle between two radii - one going to the equator and the other going to the point whose latitude is required.

Latitude is determined by measuring the angle of elevation of the sun at noon, and adding or subtracting an allowance for the time of year. Using the angle only is correct for just two days of the year - the spring and autumnal equinoxes. In midwinter we need to subtract 23.45° and at midsummer we need to add 23.45°. These allowances are due to the fact that the earth is tilted at this angle as it orbits the sun. Books of correction factors for every day of the year (almanacs) were used for many years until global positioning satellites were used.

Longitude

In the same way that latitude fixes a place north or south of the equator, **longitude** fixes a place east or west of the prime meridian - also known as the Greenwich meridian which passes through Greenwich near London, UK.



Each great circle passing through the north and south poles has the earth's axis as its diameter. Each great circle is divided by the poles into two semi-circles called **meridians of longitude.** As the meridians run north and south, places having the same longitude will be due or due south of each other.
Distance east or west of Greenwich is measured as an angle between the plane containing the Greenwich meridian and the plane containing the meridian passing through the point whose longitude is required.

Longitude varies from 180° W of Greenwich to 180° E of Greenwich.

The two meridians on a particular great circle will have values which sum to 180°, but one will have a designation W and the other a designation E. For example, 60°W is on the same great circle as 120°E.

The accurate determination of longitude was one of the great technological challenges of the eighteenth century, since a number of shipping disasters had shown up the inadequacies of the methods being used at the time. A number of European nations offered financial prizes for the development of a reliable and accurate method for determining longitude.

The Earth rotates at a rate of 360° per day, or 15° per hour, so there is a direct relationship between time and longitude. If a navigator knew the time at a particular fixed reference point when the local time could be determined at the ship's location, the difference between the reference time and the apparent local time would give the ship's position relative to the fixed location.

Two methods were in common use.

One involved measuring the lunar distance - the angle between the Moon and another celestial object (a star or the sun) - which would lead to determination of the position of the measurer through calculation based on reference values. This method required the compilation of tables of reference values (an almanac) for every day of every year well in advance of their required use - a task which required a lot of human calculating time in a pre-electronic computer era.

The second, and simpler, method involved taking a chronometer, which could keep very accurate time at sea, and calculate position almost immediately by comparing the chronometer time for the fixed reference point with local time. It took until the 1850's to build accurate chronometers cheap enough for their widespread use.

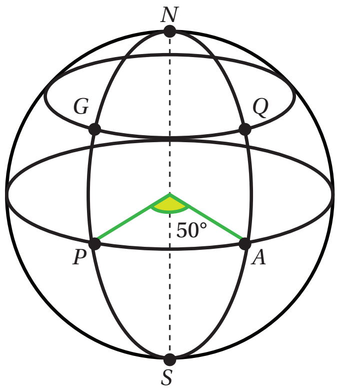
The advent of radio in the early twentieth century enabled navigators to verify the accuracy of their chronometers with time signals broadcast from known locations. Global positioning satellite systems and radar have added to the tools available to the modern navigator.

Locating positions on Earth

The position of any point on the earth's surface is given uniquely by the intersection of the circles of latitude and longitude.

By convention, the latitude is given first when giving the position of any point.

Example



In the diagram , the meridian through $NGPS$ is the Greenwich meridian. Points G and Q are on the 40° N parallel of latitude. Points P and A are on the equator. Write down the positions of points G, Q, A and P

Solution

$$G(40^\circ N, 0^\circ E (or W))$$

$$Q(40^\circ N, 50^\circ E)$$

$$A(0^\circ N (or S), 50^\circ E)$$

$$P(0^\circ (N or S), 0^\circ E)$$

Distances along meridians

Example

Paris (France) is at 48.67° N and 2.33° E. How far is it from Paris to the North Pole and the Equator travelling along the meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Paris and that of the Equator is 48.67°. The angle between the latitude of Paris and the North Pole is 90.00° - 48.67° = 41.33°.

From Paris to the Equator	From Paris to the North Pole
$l = \frac{r\pi}{180^\circ} \times \theta$	$l = \frac{r\pi}{180^\circ} \times \theta$
$= \frac{6400\pi}{180^\circ} \times 48.67^\circ$	$= \frac{6400\pi}{180^\circ} \times 41.33^\circ$
$= 5436.491\ldots$	$= 4616.605\ldots$
≈ 5436 km	≈ 4617 km

Note: The metre was originally defined as one ten-millionth of the distance from the

North Pole to the Equator travelling along the meridian through Paris. Owing to some errors in estimating the shape of the earth, the defined metre was about one-fifth of a millimetre shorter than the actual distance, meaning that the actual circumference of the earth through the poles is 40 007 863 m rather than the expected 40 000 000 m.)

Example

Melbourne (Victoria) is at 37.82° S and 144.97° E. How far is it from Melbourne to the South Pole, the Equator and the North Pole travelling along the meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km.

From Melbourne to the Equator

The angle between the latitude of Melbourne and that of the Equator is 37.82°.

$$
\begin{aligned}
&= \frac{r\pi}{180^\circ} \times \theta \\
&= \frac{6400\pi}{180^\circ} \times 37.82^\circ \\
&= 4224.53\ldots \\
&\approx 4225 \text{ km}
\end{aligned}
$$

From Melbourne to the South Pole The angle between the latitude of Melbourne and the South Pole is $90.00^\circ - 37.82^\circ = 52.18^\circ$

$$
\begin{aligned}
l &= \frac{r\pi}{180^\circ} \times \theta \\
&= \frac{6400\pi}{180^\circ} \times 52.18^\circ \\
&= 5828.56\ldots \\
&\approx 5829 \text{ km}
\end{aligned}
$$

From Melbourne to the North Pole The angle between the latitude of Melbourne and the North Pole is $90.00^\circ + 37.82^\circ = 127.82^\circ$

$$
\begin{aligned}
l &= \frac{r\pi}{180^\circ} \times \theta \\
&= \frac{6400\pi}{180^\circ} \times 127.82^\circ \\
&= 14277.63\ldots \\
&\approx 14278 \text{ km}
\end{aligned}
$$

Distances between two places

Finding the distance between two places on the surface of the Earth implies finding either the great circle distance (the shortest distance between any two points on the surface of a sphere) or the small circle distance (travelling along a parallel of latitude). To complete these calculations, we need to know the location (latitude and longitude) of each place.

There are four types of calculations that can be performed for this kind of problem.

- The first is finding the great circle distance between two places on the same meridian of longitude, or two places on the Equator.
- The second is finding the small circle distance between two places on the same parallel of latitude.
- The third is finding the great circle distance between two places on the same parallel of latitude.
- The fourth is finding the great circle distance between any two places.

The first two are part of the Further Mathematics course, and the third is a reasonable extension of this, and so the first three will be covered below. The fourth is not part of the Further Mathematics course, but will be covered in the Extension Activity at the end of these notes.

Distances between places with the same longitude

Example

Both Torrens Creek (Queensland) and Kyabram (Victoria) are on the 145° E meridian of longitude, but Torrens Creek is at 20.77° S whereas Kyabram is at 36.32° S How far is it from Torrens Creek to Kyabram travelling along the 145° E meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Torrens Creek and that of Kyabram is 36.32° - 20.77° = 15.55°.

(We find the difference since *both* places are on the *same* side of the Equator!)

From Torrens Creek to Kyabram

$$\begin{aligned} l &= \frac{r\pi}{180^\circ} \times \theta \\ &= \frac{6400\pi}{180^\circ} \times 15.55^\circ \\ &= 1736.95\ldots \\ &\approx 1737 \text{ km} \end{aligned}$$

Example

Both Cooktown (Queensland) and Kyabram (Victoria) are on the 145° E meridian of longitude, but Cooktown is at 15.47° S whereas Kyabram is at 36.32° S. How far is it from Cooktown to Kyabram travelling along the 145° E meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Cooktown and that of Kyabram is 36.32° - 15.47° = 20.85°.

(We find the difference since *both* places are on the *same* side of the Equator!)

From Cooktown to Kyabram

$$\begin{aligned} l &= \frac{r\pi}{180°} \times \theta \\ &= \frac{6400\pi}{180°} \times 20.85° \\ &= 2328.967\ldots \\ &\approx 2329 \text{ km} \end{aligned}$$

Example

Both Broken Hill (NSW) and Morioka (Japan) are on the 141° E meridian of longitude, but Broken Hill is at 31.95° S whereas Morioka is at 39.70° N. How far is it from Broken Hill to Morioka travelling along the 141° E meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Broken Hill and that of Morioka is 31.95° + 39.70° = 71.65°.

(We find the sum since both places are on *different* sides of the Equator!)

From Broken Hill to Morioka

$$\begin{aligned} l &= \frac{r\pi}{180°} \times \theta \\ &= \frac{6400\pi}{180°} \times 71.65° \\ &= 8003.31\ldots \\ &\approx 8003 \text{ km} \end{aligned}$$

Example

Both Shellharbour (NSW) and Magadan (Russia) are on the 151° E meridian of longitude, but Shellharbour is at 34.58° S whereas Magadan is at 59.57° N. How far is it from Shellharbour to Magadan travelling along the 151° E meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Shellharbour and that of Magadan is $34.58° + 59.57° = 94.15°$.

(We find the sum since both places are on *different* sides of the Equator!)

From Shellharbour to Magadan

$$\begin{aligned} l &= \frac{r\pi}{180°} \times \theta \\ &= \frac{6400\pi}{180°} \times 94.15° \\ &= 10516.655\ldots \\ &\approx 10517 \text{ km} \end{aligned}$$

Example

Both Perth (WA) and Baoding (China) are on the 115° E meridian of longitude, but Perth is at 31.93° S whereas Baoding is at 38.78° N. How far is it from Perth to Baoding travelling along the 115° E meridian, correct to the nearest km ?

Solution

Each meridian is a great circle, with a radius of 6400 km. The angle between the latitude of Perth and that of Baoding is $31.93° + 38.78° = 70.71°$.

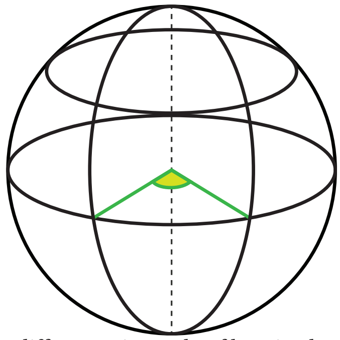
(We find the sum since both places are on *different* sides of the Equator!)

From Perth to Baoding

$$\begin{aligned} l &= \frac{r\pi}{180°} \times \theta \\ &= \frac{6400\pi}{180°} \times 70.71° \\ &= 7898.38\ldots \\ &\approx 7898 \text{ km} \end{aligned}$$

Distance between places on the Equator

The Equator is a great circle, and hence finding the distance between two points on the Equator uses the same calculation process as for finding the distance between two points on the same meridian of longitude.



difference in angle of longitude

Example

Both Libreville (Gabon) and Kismanyo (Somalia) are on the Equator on opposite sides of the African continent. Libreville is at 9.27° E and Kismanyo is at 42.32° E. How far is it from Libreville to Kismanyo travelling along the Equator, correct to the nearest km ?

Solution

The Equator is a great circle, with a radius of 6400 km. The angle between the longitude of Libreville and that of Kismanyo is 42.32° ? 9.27° = 33.05°.

(We find the difference since both places have the same longitude direction (E))

From Libreville to Kismanyo

$$\begin{aligned} l &= \frac{r\pi}{180^\circ} \times \theta \\ &= \frac{6400\pi}{180^\circ} \times 33.05^\circ \\ &= 3691.720\ldots \\ &\approx 3692 \text{ km} \end{aligned}$$

Example

Both the Galapagos Islands and the island of Naura are on the Equator, but the Galapagos Islands are at 90.30° W whereas the island of Nauru is at 166.56° E. How far is it from the Galapagos Islands to Nauru travelling over the Pacific ocean along the Equator, correct to the nearest km ?

Solution

The angle between the longitude of the Galapagos Islands and that of Nauru is $90.30° + 166.56° = 256.86°$. We find the sum since these places have different longitude directions, but this is the major arc, and the minor arc will be $360° - 256.86° = 103.14°$.

We could also find the angle between these two places by recognising that both are close to 180°E/W. We could find the angle between the Galapagos Islands and 180°E/W, the island of Nauru and 180°E/W, and then add these two angles together.

Angle between Galapagos Islands and 180°E/W = 180° - 90.30° = 89.70°.

Angel between Nauru island and 180°E/W = 180° - 166.56° = 13.44°

Total angle between Galapagos Islands and Nauru = 89.70° + 13.44° = 103.14°

From the Galapagos Islands to Nauru

$$\begin{aligned} l &= \frac{r\pi}{180°} \times \theta \\ &= \frac{6400\pi}{180°} \times 103.14° \\ &= 11520.848\ldots \\ &\approx 11521 \text{ km} \end{aligned}$$

Distances between places with the same latitude

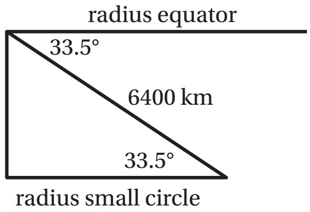
Example

Both Sydney (New South Wales) and Margaret River (Western Australia) are on the 33.5° S parallel of latitude, but Sydney is at 151.13° E whereas Margaret River is at 115.04° E. How far is it from Sydney to Margaret River travelling along the 33.5° S parallel, correct to the nearest km ?

Solution

First we need to calculate the radius of the small circle that forms the 33.5° S parallel of latitude.

In the right-angle triangle shown right, the hypotenuse is the radius of the Earth (6400 km), the angle is 33.5° and the radius of the small circle is the side adjacent to the angle.



We can use cosine relationship in standard trigonometry to find the required value.

For the small circle,

$$r = 6400 \times \cos 33.5^\circ$$

$$= 5336.869\ldots$$

$$\approx 5337 \text{ km}$$

Longitude angle difference for Sydney and Margaret River is $151.13^\circ - 115.04^\circ = 36.09^\circ$.

(We find the difference since BOTH places are on the SAME side of the Prime Meridian)

From Sydney to Margaret River

$$l = \frac{r\pi}{180^\circ} \times \theta = \frac{5337\pi}{180^\circ} \times 36.09^\circ = 3361.719\ldots \approx 3362 \text{ km}$$

Example

Both Hamilton (Western Victoria) and Orbost (Gippsland) are on the 37.4° S parallel of latitude, but Hamilton is at 142.02° E whereas Orbost is at 148.27° E How far is it from Hamilton to Orbost travelling along the 37.4° S parallel, correct to the nearest km ?

Solution

For the small circle, $r = 6400 \times \cos 37.4^\circ = 5084.253\ldots \approx 5084$ km

Longitude angle difference for Hamilton and Orbost is $148.27^\circ - 142.02^\circ = 6.25^\circ$.

From Hamilton to Orbost

$$l = \frac{r\pi}{180^\circ} \times \theta = \frac{5084\pi}{180^\circ} \times 6.25^\circ = 554.578\ldots \approx 555 \text{ km}$$

Example

Both Sydney (Australia) and Cape Town (South Africa) are on the 33.5° S parallel of latitude, but Sydney is at 151.13° E whereas Cape Town is at 18.22° E How far is it from Sydney to Cape Town travelling along the 33.5° S parallel, correct to the nearest km ?

Solution

For the small circle, $r = 6400 \times \cos 33.5° = 5336.869\ldots \approx 5337$ km

Longitude angle difference for Sydney and Cape Town is $151.13° - 18.22° = 132.91°$.

From Sydney to Cape Town

$$l = \frac{r\pi}{180°} \times \theta = \frac{5337\pi}{180°} \times 132.91° = 12380.330\ldots \approx 12380 \text{ km}$$

Example

Both Launceston (Tasmania, Australia) and Puerto Monti (Chile, South America) are on the 41.3° S parallel of latitude, but Launceston is at 147.08° E whereas Puerto Monti is at 72.57° W. How far is it from Launceston to Puerto Monti travelling over the southern Pacific ocean along the 41.3° S parallel, correct to the nearest km ?

Solution

For the small circle, $r = 6400 \times \cos 41.3° = 4808.090\ldots \approx 4808$ km

The angle between the longitude of Launceston and that of Puerto Monti is $147.08° + 72.57° = 219.65°$. We find the sum since these places are on different sides of the Prime Meridian, but this is the major arc, and the minor arc will be $360° - 219.65° = 140.35°$.

From Launceston to Puerto Monti

$$l = \frac{r\pi}{180°} \times \theta = \frac{4808\pi}{180°} \times 140.35° = 11777.530\ldots \approx 11778 \text{ km}$$

Great circle distance between places with the same latitude

Finding the great circle distance between two places on the same latitude takes advantage of the fact that both places are on the circumference of the same small circle.

Knowing the radius of that small circle and the angle between the small circle radii to each place (from the difference in longitudes), we can calculate the straight line distance between the two places using the cosine rule. Then using this distance and the radius of the earth (6400 km), we can again use the cosine rule, this time to calculate the angle between the two earth radii to each place.

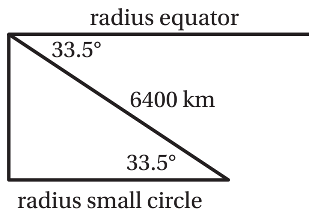
(This angle will be different to the difference in longitudes due to the curvature of the earth - the closer to the poles we go, the greater the difference between this angle and the difference in longitudes.)

Finally, using this angle and the radius of the earth, we can calculate the great circle distance.

Example

Both Sydney (Australia) and Cape Town (South Africa) are on the 33.5° S parallel of latitude, but Sydney is at 151.13° E whereas Cape Town is at 18.22° E How far is it from Sydney to Cape Town travelling the great circle route, correct to the nearest km ?

Solution

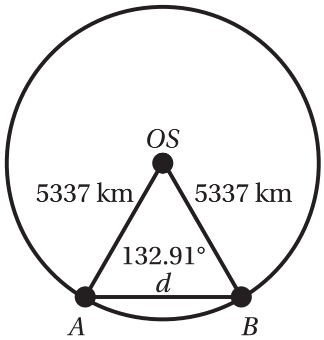


For the small circle, $r = 6400 \times \cos 33.5°$

$$= 5336.869\ldots$$

$$\approx 5337 \text{ km}$$

Longitude angle difference for Sydney and Cape Town is $151.13° - 18.22° = 132.91°$.



Straight line distance from Sydney to Capetown

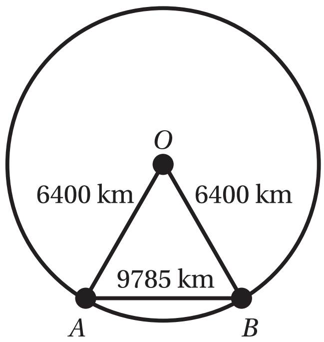
$$d = \sqrt{5337^2 + 5337^2 - 2 \times 5337 \times 5337 \times \cos 132.91°}$$

$$= 9785 \text{ km}$$

Angle between earth radii

$$\theta = \cos^{-1}\left(\frac{6400^2 + 6400^2 - 9785^2}{2 \times 6400 \times 6400}\right)$$

$$= 99.72^\circ$$



From Sydney to Cape Town via great circle route

$$= \frac{r\pi}{180^\circ} \times \theta$$

$$= \frac{6400\pi}{180^\circ} \times 99.72^\circ$$

$$= 11138.831\ldots \approx 11139 \text{ km}$$

We can see that, compared to the distance around the 33.5° parallel of latitude which was 12380 km, the great circle route at 11139 km is some 1241 km shorter.

Example

Both Launceston (Tasmania, Australia) and Puerto Monti (Chile, South America) are on the 41.3° S parallel of latitude, but Launceston is at 147.08° E whereas Puerto Monti is at 72.57° W. How far is it from Launceston to Puerto Monti travelling the great circle route, correct to the nearest km ?

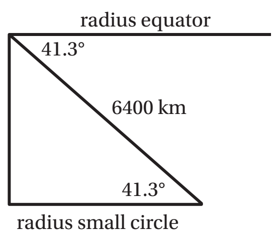
Solution

For the small circle,

$$r = 6400 \times \cos 41.3^\circ$$

$$= 4808.090\ldots$$

$$\approx 4808 \text{ km}$$

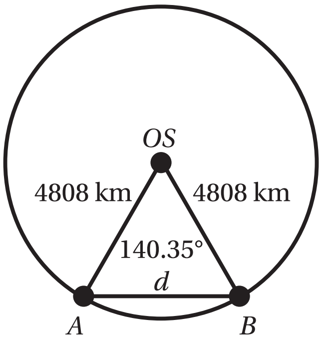


The angle between the longitude of Launceston and that of Puerto Monti is $147.08^\circ + 72.57^\circ = 219.65^\circ$. We find the sum since these places are on different sides of the Prime Meridian, but this is the major arc, and the minor arc will be $360^\circ - 219.65^\circ = 140.35^\circ$.

Straight line distance from Launceston to Puerto Monti

$d = \sqrt{4808^2 + 4808^2 - 2 \times 4808 \times 4808 \times \cos 140.35°}$

$= 9046$ km



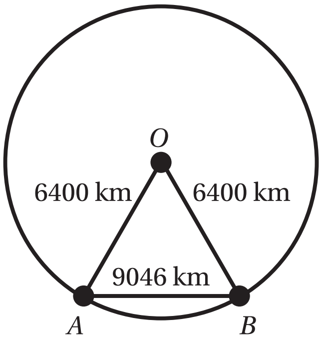
Angle between earth radii

$$\theta = \cos^{-1}\left(\frac{6400^2 + 6400^2 - 9046^2}{2 \times 6400 \times 6400}\right) = 89.94°$$

From Launceston to Puerto Monti via great circle route $= \dfrac{r\pi}{180°} \times \theta$

$= \dfrac{6400\pi}{180°} \times 89.94°$

$= 10046.394\ldots \qquad \approx 10046$ km



We can see that, compared to the distance around the 41.3° parallel of latitude which was 11778 km, the great circle route at 10046 km is some 1732 km shorter.

Time zones and travel times

The earth spins through 360 degrees in 24 hours, meaning that in one hour the earth spins through $(360 \div 24 =)15$ degrees. This means that, technically, as you travel around the earth, each 15 degrees of longitude difference in the location of places means that they should have a time difference of one hour.

Greenwich Mean Time (GMT) was established at the Royal Observatory at Greenwich (just outside of London) in 1675 to assist mariners in calculating their longitude at sea. This was at a time when every town and city in England kept its own time, based on the local noon. These differences became a problem when railway networks were built across the country. The English railway companies were the first to adopt a standard time in the 1840's, based on GMT and managed with portable chronometers, but GMT did not become the standard time for the whole of England until 1880.

New Zealand, at the time a British colony, was the first country to adopt a standard referenced time (New Zealand Mean Time) in 1868 when it set its clocks to 11 hours and 30 minutes ahead of GMT.

Standard time zones based on hourly differences were running in most countries by 1930, but it took until 1986 for all countries to set their time zones with reference to GMT. However, a number of countries, including Australia (for South Australia and the Northern Territory), have time zones offset by half-hours or quarter-hours to better suit local noon time. Other countries, such as India and China, have a single time zone for the entire country despite the fact that the extent of their borders exceeds 15 degrees of longitude.

Calculating journey times incorporates the time taken for the journey based on given local times plus a time factor allowing for the difference in longitudes. Where the longitudes are in the same direction, use the difference in values. Where the longitudes are in different directions (one East and one West) add the values together.

Example

A direct flight from Melbourne to Perth leaves Melbourne at 5.00 pm local time and arrives in Perth at 7.10 pm local time. If Melbourne is on the 145° E meridian and Perth is on the 115° E meridian, calculate the total journey time.

Solution

Difference in longitude = $145° - 115° = 30°$

Time difference due to longitude = $30° \div 15° = 2$ hrs.

Total journey time = 7.10 pm - 5.00 pm + 2 hrs = 4 hrs 10 min.

Example

The Australian Cricket team is flying from Perth to Johannesburg (South Africa) for a series of matches. Their flight leaves Perth at 6.05 am (local time) and arrives in Johannesburg at 8.50 pm (local time). If Perth is on the 115° E meridian and Johannesburg is on the 28° E meridian, calculate the total journey time.

Solution

Difference in longitude = $115° - 28° = 87°$

Time difference due to longitude=$87° \div 15° = 5.8 \approx 6$ hrs.

Total journey time = 8.50 pm - 6.05 am + 6 hrs =14 hrs 45 min + 6 hrs = 20 hours 45 min.

Example

Australian Eastern Standard Time (which includes Melbourne and Sydney) is based on the 150° E meridian and New York (USA) time is based on the 75° W meridian. If a flight leaves Melbourne at 10.30 am and arrives in New York at 10.15 pm on the same day, calculate the total journey time.

Solution

Difference in longitude = 150° + 75° = 225°

(Add values since different directions)

Time difference due to longitude =225° ÷ 15° = 15 hours

$$\begin{aligned}\text{Total journey time} &= 10.15\text{ pm} - 10.30\text{ am} + 15\text{ hours}\\ &= 11\text{ hr }45\text{ min} + 15\text{ hours}\\ &= 26\text{ hours }45\text{ min.}\end{aligned}$$

Example

Australian Eastern Standard Time (which includes Melbourne and Sydney) is based on the 150° E meridian and London (UK) time is based on the 0° W meridian. If a flight leaves Melbourne at 2.20 am and arrives in London at 4.45 am on the next day, calculate the total journey time.

Solution

Difference in longitude = 150° + 0° = 150°

(Add values since different directions)

Time difference due to longitude =150° ÷ 15° = 10 hours

$$\begin{aligned}\text{Total journey time} &= 4.45\text{ am} - 2.20\text{ pm} + 10\text{ hours}\\ &= 14\text{ hr }25\text{ min} + 10\text{ hours}\\ &= 24\text{ hours }25\text{ min.}\end{aligned}$$

Using Pythagoras' theorem in spheres

Pythagoras' theorem can be used to find the radius of a small circle within a sphere where no angle information is given.

Example

A large pond in the shape of a hemisphere has a radius of 5 m and it is planned to fill the pond so that the water in the centre of the pond is 2 m deep. The pond's builders then want to attach a metal ring around the waterline which will be used to hold a cover for the pond in the winter months.

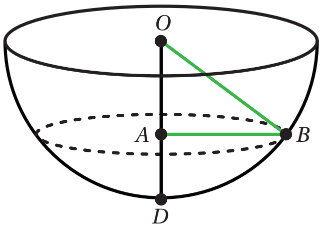
They want to know how long to make the metal ring, correct to the nearest mm, and what area of cover material will be required in square metres correct to two decimal places.

Solution

To find the length of the waterline and the area of the surface, we need to know the radius of the small circle as shown with the dotted line.

If AD represents the depth of water in the pond, then the air space above.

$$(OA) = OD - AD$$
$$= 5 - 2$$
$$= 3$$



OB is 5 m since that is the radius of the hemisphere.

We can find the radius of the small circle using Pythagoras' theorem.

$AB = \sqrt{5^2 ? 3^2} = \sqrt{25 ? 9} = \sqrt{16} = 4$

We find the length of the waterline (and hence the length of the metal ring) using $C = 2\pi r$.

$C = 2 \times \pi \times 4 = 8\pi = 25.13274\ldots \approx 25.133m$

The metal ring will need to be 25.133 m in circumference.

The area of the water surface can be found using $A = \pi r^2$

$A = \pi \times 4^2 = 16\pi = 50.2654\ldots \approx 50.27m^2$

The area of the cover will be 50.27 square metres.

Extension Activity

Distance between any two places on the earth's surface

It is possible to find the great circle distance between any two points on the surface of the Earth. As long as the two points are NOT directly opposite each other on the Earth's surface (antipodes), then there is a unique great circle that can be drawn through these two points. The length of the shorter arc between these two points is the great circle distance we are after.

There are three formulations that are commonly used for calculating these great circle distances.

The simplest method is based on the "Spherical Law of Cosines" which is a method for finding distances on the surface of a sphere. It can have difficulties calculating distances between two places that are very close together (less than 10 metres), but this is not really a practical calculation.

A more complex method, based on the "haversine" calculation method, forms the basis of a number of web-based processes for performing these calculations. It is accurate for most distances on a sphere, but has difficulties with "antipodal points" which are points at each end of a diameter .

A more complicated formula that is accurate for all distances is a special case of the Vincenty formula, which is generally used to calculate distances on ellipsoids. In this case, the sphere is an ellipsoid with equal major and minor axes.

In light of the fact that the "cosine" based method is easier to use, and is the preferred method of many geoscientists, it will be the method demonstrated here.

The calculation

The two key assumptions made in using this method in this activity are that the Earth is a sphere with a radius of 6400 km.

To carry out the calculation we require the latitude and longitude values for the two places concerned.

Following the terminology used in the development of this formulation, latitudes (North) and longitudes (East) are given positive values, and latitudes (South) and longitudes (West) are given negative values.

The full calculation is given by

$$d = \frac{6400 \times \pi}{180} \times \cos^{-1}\big((\sin\theta \times \sin\phi) + (\cos\theta \times \cos\phi \times \cos|c|)\big)$$

where θ and ϕ are the latitudes of the two places and c is the absolute difference in longitudes of the two places.

To clearly demonstrate why the sign applied to an angle is important, we will break the following example calculation into two parts, finding the cos value (D) first using sine and cosine values rounded to four decimal places, and then finding the distance (d).

Example

Find the great circle distance between Paris (France) (48.67° N, 2.33° E) and Melbourne (37.82° S, 144.97° E) correct to the nearest kilometre.

Solution

By the definition given above, the Melbourne latitude value will be negative.

The absolute difference in longitude is 144.97° - 2.33° = 142.64°.

$$D = \big((\sin\theta \times \sin\phi) + (\cos\theta \times \cos\phi \times \cos|c|)\big)$$

$$= \big((\sin 48.67^\circ \times \sin -37.82^\circ) + (\cos 48.67^\circ \times \cos -37.82^\circ \times \cos 142.64^\circ)\big)$$

$$= ((0.7509 \times -0.6132) + (0.6604 \times 0.7899 \times -0.7948)$$

$$= (-0.4605 - 0.4146)$$

$$= -0.8571$$

$$\text{distance} = \frac{6400 \times \pi}{180} \times \cos^{-1}(-0.8571) = 16642.623\ldots \approx 16643 \text{ km}$$

We can see from the details of this calculation that the signs allocated to each latitude value have a major impact on the calculation of the angle between the radii leading to the two places. The negative value for cosine leads to an obtuse angle, fairly reflecting the distance between the two places chosen.

Example

Find the great circle distance between Paris (France) (48.67° N, 2.33° E) and Melbourne (37.82° S, 144.97° E) correct to the nearest kilometre.

Solution

By the definition given above, the Melbourne latitude value will be negative.

The absolute difference in longitude is 144.97° - 2.33° = 142.64°.

Performing the angle calculation in one operation with a graphic calculator we get,

$$= \big((\sin 48.67^\circ \times \sin -37.82^\circ) + (\cos 48.67^\circ \times \cos -37.82^\circ \times \cos |142.64^\circ)\big)$$

$$= -0.857096106706\ldots$$

And hence finding the distance

$$\text{distance} = \frac{6400 \times \pi}{180} \times \cos^{-1}(-0.857096106706) = 16873.156\ldots \approx 16873 \text{ km}$$

This calculation can be performed in one operation with a graphic calculator,

$$\begin{aligned} d = \frac{6400 \times \pi}{180} \times \cos^{-1}((\sin 48.67 \times \sin -37.82) + (\cos 48.67 \times \cos -37.82 \times \cos 142.64)) \\ = 16873.156\ldots \\ \approx 16873 \text{ km} \end{aligned}$$

We can see that using the full decimal storage capabilities of the calculator in the latter two calculations has resulted in a larger answer by some 240 km to that obtained where the sine and cosine values were rounded to four decimal places. This is one of the issues raised in a number of discussions regarding the merit of the various methods for calculating great circle distances - that using lesser accuracy of values in the calculation affects the cosine method more than the haversine or Vincenty's formulations.

To cope with this, we recommend performing the calculations as in the last two instances, using the calculator to store the full decimal values.

Example

Find the great circle distance between San Francisco (USA) (37.783° N, 122.417° W) and Melbourne (37.82° S, 144.97° E) correct to the nearest kilometre.

Solution

By the definition given above, the Melbourne latitude value and the San Francisco longitude value will be negative.

The absolute difference in longitude is 144.97° - ($-122.417°$) = 267.387°.

Breaking the calculation into two parts, finding the cos value (D) first, and then finding the distance (d).

$$\begin{aligned} D &= \left((\sin\theta \times \sin\phi) + (\cos\theta \times \cos\phi \times \cos|c|)\right) \\ &= \left((\sin 37.783^\circ \times \sin -37.82^\circ) + (\cos 37.783^\circ \times \cos -37.82^\circ \times \cos 267.387^\circ))\right) \\ &= -0.404142814063\ldots \end{aligned}$$

$$\text{distance} = \frac{6400 \times \pi}{180} \times \cos^{-1}(-0.404142814063)$$

$$= 12715.762\ldots$$

$$\approx 12716 \text{ km}$$

Or in one operation with a graphic calculator,

$$d = \frac{6400 \times \pi}{180} \times \cos^{-1}((\sin 37.783 \times \sin -37.82) + (\cos 37.783 \times \cos -37.82 \times \cos 267.387))$$

$$= 12715.762\ldots$$

$$\approx 12716 \text{ km}$$

Comparing great circle distance with small circle distance

Example

Compare the small circle distance between Capetown (33.5° S, 18.22° E) and Sydney (33.5° S, 151.13° E) with the great circle distance.

Solution

From a previous calculation, the small circle distance between Capetown and Sydney was found to be 12380 km.

For the great circle calculation, both latitudes will have negative values and the absolute difference in longitudes will be 151.13° - 18.22° = 132.91°.

In one operation with a graphic calculator,

$$d = \frac{6400 \times \pi}{180} \times \cos^{-1}((\sin -33.5 \times \sin -33.5) + (\cos -33.5 \times \cos -33.5 \times \cos 132.91))$$

$$= 11138.642\ldots$$

$$\approx 11139 \text{ km}$$

From this we can see that the great circle distance is considerably shorter (by 1241 km) than the small circle distance.

If we put this value as the answer to our normal great circle calculation, we get

$$11139 = \frac{6400\pi}{180°} \times \theta\circ$$

$$\text{Giving } \theta = \frac{11139 \times 180^\circ}{6400\pi}$$

$$\theta = 99.72^\circ$$

So, due to the curvature of the earth at latitude 33.5°S, the angle between the radii to Capetown and Sydney is actually just under 100°.

Years 11 & 12