VCAA-AMSI maths modules

A guide for teachers – Years 11 and 12

Discrete Mathematics

Graph theory

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Years 11 & 12

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Graph theory – A guide for teachers (Years 11–12)

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Graph theory

An important note

This material is produced so that, with little nudges from time to time, the reader can discover the ideas and results for themselves. So the text provides pedagogical direction as well as content. Questions are sprinkled liberally throughout the text. These can be useful if you haven't taught graph theory before or if you are a little rusty. We leave it to you to decide how many of these you need to do in any great detail to bring you up to scratch. Hopefully there are enough questions for you to use in your teaching.

All of the answers to all of the questions are provided at the end.

Motivation and history

Why would anyone want to look at graph theory and the objects that it look at? How useful is graph theory? We want to look at these first of all from the point of view of the classroom and second from the point of view of the wider world.

In the classroom we can do something with graph theory that is often hard to do with other maths topics – we can show how mathematics develops and how mathematicians approach their subject. This is an important learning experience for students as they are usually given rules to follow and then regurgitate in some form of assessment. They very seldom get a chance to discover mathematical material for themselves. You will see how we attempt to do this as you go through this work.

In the wider world, the answer to the first question leads naturally into the second. As you will soon see as a result of The Walking Problem, graphs can be very easy and straightforward models of objects that simplify more complicated real situations. Once these models have been made they can be tackled more simply than reality and lead to solutions that may be missed or may be harder to find otherwise. The other day a friend of ours needed to get from their home in Parkville to the corner of Domain and Park Road South Yarra. They picked up their phone and got onto Google Maps. Immediately they saw a graph with several vertices and edges showing them a few ways that they could make the route. These graphs also included times along various edges. They were then able to take the best route to get them to their destination the best way for them.

Here then are two examples to consider but unfortunately the two graphs used aren't what we call **simple** graphs, which are the main subject of this site. But if you bear with us, once we get a few ideas out of the way, we'll pick up the application thread for simple graphs in the Applications section. These will include gene technology and assigning people to jobs.

But the history of graph theory is interesting too partly because it hasn't yet been around for 300 years. Euler's foray at Königsberg was published in 1736. This is considered to be the birth of the subject as well as being a precursor to topology.

Figure 1: Leonard Euler

In 1859, Hamilton (see below in the section on circuits) marketed a toy called the Icosahedron Game, where you had to find a path round a (simple) 'graph' that went through every 'vertex'. This was the beginning of Hamiltonian cycles. Things dribbled on for some time and it wasn't till 1936 that the first textbook on graph theory was published.

Figure 2: Four colours suffices for any map

Going back a little, in 1852 one Francis Guthrie, a schoolboy, conceived the idea of colouring maps so that no two adjacent countries had the same colour. This was the beginning of over 100 years during which many mathematicians and members of the public tried to show that you needed only 4 colours no matter how many colours your map had or how the relevant graph was arranged. It wasn't until 1976 that Appel and Haken produced a computer aided proof. It was partly because of the spinoff from research on the four colour 'theorem' that graph theory has prospered. But a number of other things were happening that raised the profile of graph theory and generated research in the subject. For more historic pointers, see http://en.wikipedia.org/wiki/Graph_theory.

Content

A walking problem

In the early eighteenth century the city of Königsberg was in Prussia. The city neatly straddled the River Pregel in which two large islands were connected to each other, and to the two banks of the river, by seven bridges (see Figure 3). The middle classes of the period created a problem for themselves. They wanted to start at some point of the city and walk over each bridge once and only once. But there was an extra condition. They also wanted to end up at the point where they had started.

Figure 3: The bridges of Königsberg

Questions

1. Were the fair people of Königsberg successful in walking over all the bridges once and getting back to where they started? (Model the situation using dots and lines joining them.)
2. Could the bridge walk be achieved if the walkers were content not to return to their starting point?
3. Can the round trip walk be achieved if one or more of the bridges were removed?
4. Think of a city with seven bridges arranged in some other way. Is it possible for the round trip walk to be made successfully? (Move towards a generalisation of the Königsberg problem.)
5. Repeat the last question with eight, nine and ten bridges. What can be said about the land masses in each case?
6. Is there a general result here that covers any number of bridges? What do you need to know about the way that the bridges are connected to the land masses?

The Königsberg Bridge Problem was solved by the Swiss mathematician Leonhard Euler (pronounced 'Oiler ') in 1736. It heralded the start of the study of two new areas of mathematics, graph theory and topology. For more details see
http://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg.
There are many situations that can be modelled using this simple dot and line model. You might think of the route maps found in various in-flight magazines, for example. But search out others. Look for social network situations, electrical networks, molecules or anything else that might use dots joined by lines.

Questions

7 Gas, electricity and water from three different sources in the street are to be connected to three different inlets in a new house. Can this be done underground so that no two of the connections cross in any way?

Simple graphs

The dots and line things of the last section are called graphs and from now on, we'll lift the tone of the discussion by using the more formal words 'vertex' for dots and 'edges' for lines.

Questions

8 How many graphs are there with just one vertex ? The possibilities are

A 0 **B** 1 **C** 547 **D** as many as you like.

At this point you should have a discussion with your neighbour or with the whole class as to which of these is correct. When you've done that, look at the answer.

9 How many graphs are there with just two vertices [1]? The possibilities are

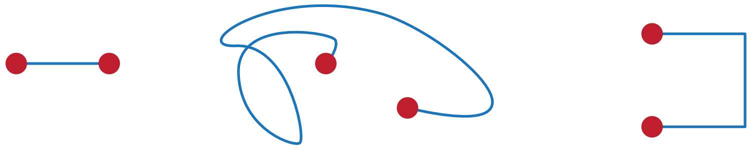
A 1 **B** 2 **C** 547 **D** as many as you like.

Again it is time to go into camera and work with a friend. Only then look at the answer.

The answers to Questions 8 and 9 force us to make a decision on what a graph actually is. First, any edge between two vertices can be drawn any way you like. It doesn't matter for the practical purposes that graphs are used for. Whatever shape you use it's the same edge.

[1] The plural of vertex is vertices

So we consider all of the graphs in the figure below to be the same.



What's more it doesn't matter whether we put the two vertices on the Moon or one on the Moon and one on Earth, or whatever, it's the same graph.

So we have two graphs on two vertices: one with no edges and one with one. How could we possibly have infinitely many graphs on two vertices? Well who said that there was only one edge between any two vertices? After all, when we were talking about Königsberg's bridges before, we sometimes had two lines joining two vertices. Edges like this are what are called **multiple edges** and we can use as many multiple edges as we like in a graph. However, such graphs are called **multigraphs**. See http://en.wikipedia.org/wiki/Multigraph

It's worth noting that loops are considered to be multiple edges and so multigraphs include loops. Do we allow multiple edges or not? But in fact here we won't talk about multiple edges very often. Instead we'll work mainly with graphs that have no loops and no multiple edges. These are called **simple graphs** . It turns out then, that there are only two simple graphs with two vertices. One has an edge and the other doesn't have any.

From here on, to make things less wordy, any time we use 'graph' we will mean simple graph. If we want to allow a graph to have loops or multiple edges we will specifically say so.

Questions

10 How many (simple) graphs are there with just three vertices? The possibilities are

A 2 **B** 3 **C** 4 **D** 8.

Before you start to think that this set of questions will go on forever, discuss Q10 with a friend. What do you need to know in order to decide this question? Then see what the answers say about it.

11 How many graphs are there with just four vertices? The possibilities are

A 8 **B** 9 **C** 10 **D** 11 **E** 12 **F** 21 **G** 22 **H** 23 **I** 24.

Isomorphism

The fundamental issue raised by Question 10 and 11 is, when are two graphs the same? This leads us to a fundamental idea in graph theory: **isomorphism**. If two graphs are the same they are **isomorphic**. But how do you tell if two graphs are isomorphic? Well the low brow way is to move the vertices of one onto the vertices of another in such a way that the edges of both overlap the edges of the other. If this is possible, then the two graphs are said to be the same, isomorphic. If you can't then they're not. Have a look at Figure 4.

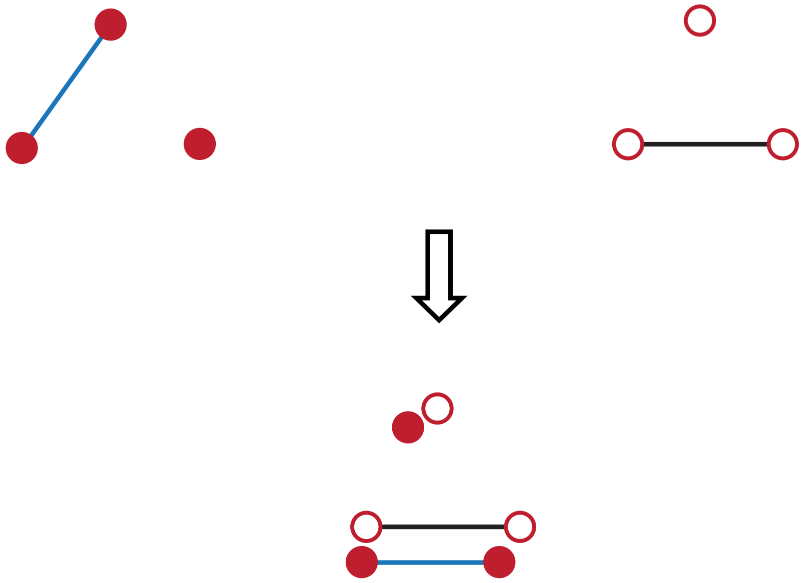


Figure 4: Showing the isomorphism of two graphs

The pictures show how to move the 'closed red' vertices onto the 'open red' ones so that the single edges of each graph line up. So the two graphs above are isomorphic. As a result the only graphs on three vertices are those with no edges, one edge, two edges and three edges. C in Question 11is the correct answer.

So two graphs that are the same are said to be **isomorphic**. But we have to get highbrow and formalise this matter of isomorphism. The rough idea from the answers to Question 11, means that vertices of one graph go to vertices of the other. What's more it shows that the edges of one graph go to the edges of the other and non-edges go to non-edges. We can do this formalisation by a function from the vertices of one graph to the other that sends edges to edges and non-edges to non-edges. We hope the following formal definition is never asked for in an exam because there are far more important things to worry about. But here it is. Two graphs G and H are **isomorphic** if there is a one-to-one onto function (a bijection) f between the vertices of G and H such that there is an edge between vertices u and v in G if and only if there is an edge between the vertices $f(u)$and

$f(v)$ in H. So first for every vertex of G there is a corresponding vertex of H and vice versa. And second, every edge of G corresponds to an edge of H and vice versa. That also means that non-edges go to non-edges. We write $G \cong H$ to indicate that G and H are isomorphic.

The isomorphism of graphs can trip you up, especially at the start. A particular graph can be drawn in more than one way so watch out for this. For instance, the apparently two different graphs shown in Figure 5 are actually isomorphic.

Figure 5: An isomorphic pair of graphs

Formally a bijection between vertices is $a \leftrightarrow u, b \leftrightarrow v, c \leftrightarrow x$, and $d \leftrightarrow w$ and between edges is $ab \leftrightarrow uv, ac \leftrightarrow ux, bd \leftrightarrow vw$. Using this bijection we can successfully 'sit' the right graph on top of the left graph so that edges sit on the top of edges.

Questions

12 Find another bijection between the two graphs in Figure 5.

13 You may also have found a Z-shaped graph on four vertices. That is also isomorphic to the graph in Figure 5. Write out a bijection that will show this. Show that the edges and non-edges of one graph go to edges and non-edges of the other.

While we are thinking about isomorphism, it turns to be extremely difficult to tell when two graphs with even a reasonable number of edges are isomorphic. This difficulty is called the Isomorphism Problem for Graphs. See
http://en.wikipedia.org/wiki/Graph_isomorphism _problemState_of_the_art).
Finding an isomorphism between two graphs is often very important.

Question

14 Have a look at the graphs G and H in Figure 6 and decide whether they are isomorphic or not.

Figure 6: Two potentially isomorphic graphs

In the answers we used an idea called 'edge-in numbers'. This is something that a student might invent. Now 'edge-in numbers' is ugly, awkward, lengthy and misleading (they are the same as the edge-out numbers after all). So let's introduce a more standard name for these things. These numbers are usually called **degrees**. So the degrees of the vertices of G are 1, 2, 3 and 2.

Questions

15 Find the degrees of all the vertices of all the graphs on 4 vertices. Beside them put the number of edges that each graph has. Notice anything? Is there some relationship? Does that relationship work for graphs with 5 vertices? Does that work in general? Can you prove it?

16 Will any sequence of numbers be the degrees of some graph? For instance, is it likely that there is a graph with degrees 1, 1, 1, 1, 1 or 1, 1, 1, 1, 1, 1?

17 At a meeting some people shake hands. What can be said about the situation when there is an even number of handshakes and when there are an odd number ofhandshakes?

(See http://en.wikipedia.org/wiki/Handshaking_lemma.)

18 Think about graphs with n vertices. Is it possible for such graphs to have vertices of degree 0? Is it possible for such graphs to have vertices of degree n?

19 Find all the graphs on 5 vertices all of whose vertices have the same degree.

20 What graph on n vertices has the most edges? What is the degree of each vertex? How many edges does it have?

21 Are all graphs on 8 vertices, all of whose vertices have degree 2, isomorphic?

22 Are all graphs on 4 vertices with degrees 1, 2, 2, 3 isomorphic?

From the work on Q15, students should have noticed that the sum of the degrees of a graph is always twice the number of edges. Did they manage to prove that? Did they have a better way than this?

Theorem

The sum of the degrees of a graph is equal to twice the number of edges.

Proof

Look at the edge that goes between vertices $u$and v. When we are considering degrees we count this edge once in the degree of u and once in the degree of v. So this edge is counted twice in the sum of the degrees. But the same is true for any edge. Hence the result follows.

□

Questions

23 What tips can you give for deciding when two graphs are isomorphic?

24 What is the largest number of edges that a graph on 3, 4, 5, 6, n vertices can have? Call this number N. Prove that your value of N is correct.

25 The graph on n vertices that has N edges is called the **complete graph** on n vertices and it is denoted by K_n. Draw K_4, K_5 and K_6.

26 Can you see any link between the graphs with e edges and those with $N-e$ edges? If so, what and why?

27 How many graphs are there on 5 vertices? (Use the idea of Q26 to be more efficient.)

In the light of the answers to Q11, and to the ideas that have come up in Q26 and Q27, we now want to define the complement of a graph G. The graph $\tilde{G}$ is the **complement** of G if when uv is an edge of G it isn't an edge of $\tilde{G}$ and vice versa. This is the systematic method that we used in the answer to Q11. Next to a graph in the first column there, in the second column we have listed the complement of that graph. The same thing holds in the third and fourth columns. The graph on its own at the bottom between the third and fourth columns is a special case. It's its own complement. So it's said to be **self-complementary.**

We have talked already about graphs all of whose vertices have the same degree. In this context another useful word is 'regular'. We say that a graph, with or without loops or multiple edges, is regular if every vertex has the same degree.

Questions

28 Find the complements of all the graphs on 3 vertices.

29 What is the relationship between $\tilde{\tilde{G}}$ and G?

30 Show that for a given G there is only one $\tilde{G}$.

31 What are the degree of the vertices in the complement of P? How many edges does it have?

32 Find all of the regular graphs on 8 vertices that are regular of degree 5.

33 Prove or disprove that the complement of a regular graph is regular. If it is true, what can be said about the degree of the complement? If it is false, show that it is false for an infinite number of regular graphs.

34 Find all of the graphs on 5 vertices by using the complement idea.

35 What graphs on 5 vertices are self-complementary?

The graph in Figure 7 is a regular graph of degree 3. This is the famous **Petersen graph**
See http://en.wikipedia.org/wiki/Petersen_graph
We'll call this graph P. This graph has an important place in graph theory as it turns out to be a counterexample for a lot of conjectures. It is so interesting to graph theorists that a book has been written about it[2]. Even that book doesn't contain everything that is important about the graph.

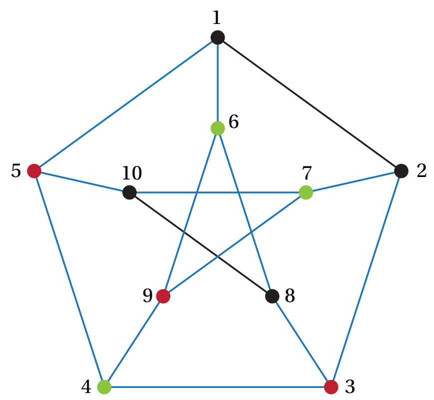


Figure 7: The Petersen graph P

We'll get back to P later.

[2] D. Holton and J Sheehan, 1993, Petersen Graph Theory, Cambridge: Cambridge University Press

Questions

36 What regular graphs are there on 1, 2, 3 and 4 vertices?

37 Find all of the regular graphs of degree 0 on n vertices. Find all of the regular graphs of degree 1 on n vertices.

38 Find all of the regular graphs of degree 2 on 8 vertices. Find all of the regular graphs of degree 3 on 9 vertices. Justify your answer.

39 What can be said about the regular graphs of odd degree on $2n+1$ vertices? Why? Generalise this as far as possible (i.e. not just for graphs on an odd number of vertices). Prove this result as a Corollary[3] of Theorem 1.

40 An amorous ant can only travel on edges of a graph. He and his girlfriend sit on vertices of a graph. On which of the graphs on 3, 4 and 5 graphs, can the ant be sure to be able to meet with his inamorata [4]?

Trees

Now Question 40 introduces the concept of connectivity in graphs. A graph is **connected** if it is possible to move from any vertex of the graph to any other vertex using edges of the graph.

Questions

41 Draw two non-isomorphic connected graphs on 6 vertices that have 6 edges.

42 A graph that is not connected is called **disconnected**. Find all of the disconnected graphs on 3 and 4 vertices.

43 Is it possible for a connected graph on n vertices to have a vertex of degree $n-2$? Is it possible for a disconnected graph on n vertices to have a vertex of degree $n-2$?

44 When is the complement of a disconnected graph disconnected? Is the following true or false? At least one of a graph or its complement is connected. If it is true prove it; if it is false provide a counterexample. Is it possible for a graph and its complement to both be connected?

---

[3] A corollary of a theorem is a result that follows easily from the theorem.

[4] girlfriend

Mathematicians often look at extremes to see what it tells them. As far as connectivity is concerned one sort of extremeness is about the number of edges. So let's look at the extremes of connectivity.

Questions

45 What connected graphs on n vertices have the most edges?

46 What connected graphs on 3, 4 and 5 vertices have the least number of edges?

47 On the basis of the last question what is the least number of edges a connected graph on n vertices has? Can you state this carefully as a conjecture? Can you prove your conjecture? Can you think of any other conjectures about connected graphs? Can you prove them?

The connected graphs that have the smallest number of edges seem to look a bit like trees. You could draw them all as having a root or roots and having branches. So we'll make a definition.

Definition

A graph on n vertices is called a **tree** if it is connected and has the fewest number of edges among all connected graphs on n vertices.

Questions

48 Draw some trees.

49 How many trees do you think there are on n vertices?

50 Remove some edges from a graph so that a tree is left. Can this be done for all graphs?

51 How many vertices of degree 1 have to exist in a tree on 2 or more vertices?

Subgraphs

This seems a good point to talk about subgraphs.

Let VH denote the set of vertices of graph H and VG the set of vertices of graph G.

Definition

A **subgraph** H of a graph G, is any graph such that VH is a subset of VG and all the edges of H are also edges of G. We also think of G itself as being a subgraph of itself.

Note that you can't just say that a subgraph of G consists of vertices and edges from G because you have to include among the vertices of a subgraph, vertices that your chosen edges are adjacent to.

Questions

52 What can be said about the subgraphs of K_4?

53 A maximal connected subgraph of a disconnected graph G is called a **component** of G. Find all of the graphs on 4 vertices that have two and three components.

54 Subgraphs that have the same set of vertices as the graph they are in, are called **spanning** subgraphs. What can be said about a graph G if it has K_n as a spanning subgraph?

55 What can be said about graphs that have a tree as a spanning subgraph? Make and prove a conjecture about this.

56 What can be said about the regular spanning subgraphs of P?

Right about now it's also useful to introduce the concept of a cycle. This is just what it sounds like: a series of vertices and edges that close up.

Definition

A **cycle** is a connected graph of degree 2. The cycle on n vertices is called C_n. Cycles appear as subgraphs of lots of graphs. For instance there are several subgraphs that are cycles in the graph of Figure 8.

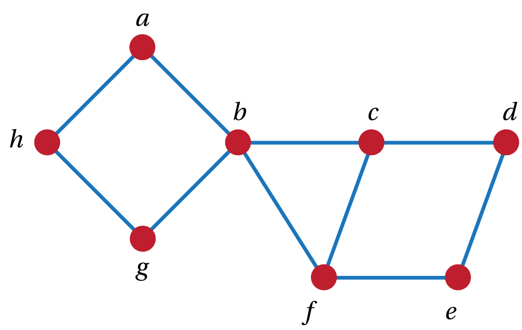


Figure 8: A graph with cycles

In Figure 8 $\{a, b, g, h\}$ defines a cycle with four edges ab, bg, gh, ha. In future, if we know the vertices we won't bother using the edges in saying what the cycle is.

Questions

57 Draw the graphs C_3, C_4 and C_6. Find all other cycles in Figure 8.

58 What cycles are there in K_n?

59 What cycles are there in a tree?

60 Show that there is a connected graph G on 6 vertices that has 6 edges and contains a C_5. Does there exist a disconnected graph H on 6 vertices that has 6 edges and contains a C_5?

61 What cycles are there in P? What cycles are there in the complement of P?

62 **Conjecture**: $\widetilde{C_n}$ is connected. If the conjecture is true, prove it. If the conjecture is false, correct it and prove that your correction is true.

Another useful kind of graph is a path.

Definition

A **path** is a sequence of s distinct vertices and the $s-1$ edges between them. We show the general idea in Figure 6. If a path has n vertices it is denoted by P_n.

Figure 9: A path

As you can see a path consists of two vertices of degree 1 and a number of vertices of degree 2.

Questions

63 Show that for each given value of i, for $i = 2, 3, 4, \ldots, n$ a tree on n vertices can be constructed that has P_i as a subgraph. Is this tree unique?

64 Is it true that a cycle has a spanning path? Is it true that a graph with a spanning path is a cycle?

65 Is it true that a graph with a spanning path is connected? Is it true that every connected graph has a spanning path?

66 Does P have a spanning path?

Now a given connected graph may not be a tree but it can be related to a tree?

Questions

67 Can you find trees in the connected graphs on 3, 4 and 5 vertices?

68 Can you find trees in the disconnected graphs on 3, 4 and 5 vertices? So how can trees distinguish between connected and disconnected graphs? Can you state these ways carefully as conjectures? Can you prove your conjectures?

Colouring vertices and edges is a favourite game of graph theorists because of both the applications and the nice mathematics that it produces. Let's think for a minute of colouring vertices. We do this so that adjacent vertices have different colours and call it a **vertex colouring** if the colouring contains the fewest number of colours

Questions

69 Take any tree on 3, 4 or 5 vertices. Produce a vertex colouring of this tree. What is the smallest number of colours that you need? Does this work for trees on n vertices?

70 How many different colours does it take to produce a vertex colouring of K_n?

71 How many different colours does it take to produce a vertex colouring of P_n?

72 How many different colours does it take to produce a vertex colouring of C_n?

73 How many different colours does it take to produce a vertex colouring of P?

Now is the time to put together all of the results we have about trees.

Theorem (T1)

A tree on n vertices has no cycles.

Proof

Suppose that a tree, T, has a cycle, C, with an edge uv. Since T is connected there is a path Q in T from any vertex a to any vertex b. Delete the edge uv from T. If Q contains the edge uv, then we can form another path Q' which doesn't have uv but does have other edges of C. Hence, since T has the minimal number of edges for a connected graph, it doesn't have a cycle. □

Corollary

In any tree there is a unique path between any two vertices.

Proof

Let $Q = xu_1u_2u_3 \cdots u_r y$ and $R = xv_1v_2v_3 \cdots v_s y$ be two paths in T between x and y. Moving along Q from x, let u be the last vertex that Q and R have in common and let v be the next vertex after u that Q and R have in common. Then there is a cycle in T formed by edges of Q and R, containing the vertices u and v. This contradicts the theorem, so a tree has a unique path between two given vertices.

□

Questions

74 Trees are usually defined as being acyclic[5]connected graphs. Show that this is equivalent to the definition that we have been using. We will use the ’acyclic connected’ definition from now on.

Theorem (1)

Let T be a tree with at least two vertices. Let $P_r = u_1u_2u_3 \cdots u_r$ be a longest path in T. Then u_1 and u_2 have degree 1 in T.

Proof

Suppose that u_1 has degree at least 2. Then there u_1v is an edge in T. If v is one of the vertices of P_r, then there is a cycle in T. This contradicts Theorem T1. So $vu_1u_2u_3 \cdots u_r$ is a path in T which is longer than P_r. This contradicts the assumption that P_r was the longest path in T. Hence there is no edge u_1v and degree of u_1 is 1. In the same way we can show that ur also has degree 1. □

Corollary

A tree on n > 1 vertices has at least two vertices of degree 1.

Proof

Such a tree has at least one edge, so it has a longest path. This longest path has two vertices of degree 1 by the theorem. □

Theorem (2)

A tree on n vertices has $n-1$ edges[6].

Proof

[5] ‘Acyclic’ means has no cycles.

We will use induction on n.

Step 1 The tree with 1 vertex has $n-1=0$ edges. So the result is true for $n=1$.

Step 2 Assume that a tree with k vertices has $k-1$ edges.

Step 3 We will prove that a tree with $k+1$ vertices has k edges. When this is done we will have proved the theorem.

Let T be a tree on $k+1$ vertices. Let v be a vertex in T of degree 1. (This exists by the Corollary to Theorem T2.) Let uv be the edge of T adjacent to v. If x and y are two vertices of T neither of which is v, then there is a path from x to y because T is connected. Further this path doesn't go through v because v has degree 1.

Consider $T-v$, the graph obtained by removing v and the edge uv from T. We want to show that $T-v$ is a tree so that we can use Step 2. From what we have said about the path from x to y, $T-v$ must be connected. $T-v$ cannot contain a cycle otherwise a cycle would already have existed in the tree T.

So $T-v$ is a tree and by the inductive Step 2, we Know that $T-v$ has $k-1$ edges. Since T has one more edge that this it has the k edges we were hoping for. □

In Conjecture 2 of the answer to Q47 we suggested that there were $n-2$ connected graphs on n vertices. But generally there are more than $n-2$ trees on n vertices. We show that the conjecture is false in Figure 10 by showing five different trees on 6 vertices. You should do a quick check to make sure that these graphs are not isomorphic.

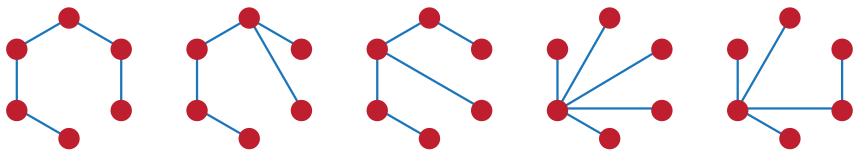


Figure 10: Five trees on 6 vertices

Theorem (3)

A graph is connected if and only if it has a spanning tree.

Proof

---

[6] We will assume here that a tree is acyclic and connected

Clearly if a graph has a spanning tree it is connected. Suppose that G is connected on n vertices. If $|E(G)| = n-1$, then G is a tree because it has the minimum number of edges possible in a connected graph. If $|E(G)| > n-1$, then G is not a tree. So G must contain a cycle. Choose any edge e on a cycle of G and remove it. Now $G-e$ is either a tree and so a spanning tree of G, or $G-e$ contains a cycle. We continue removing edges from cycles until we produce the spanning tree we are after.

□

The vertices of a tree can be coloured in two vertices so that no two vertices with the same colour are adjacent. We'll leave the result about the two colouring of the vertices of trees because that will be easier once we have a bigger result. In conclusion on trees we combine all that we have done so far in the following result.

Theorem (T4)

Let G be a graph on n vertices. The following statements are equivalent: G has the fewest edges of all connected graphs on n vertices; G is an acyclic connected graph; G is acyclic and has $n-1$ edges; G is connected and has $n-1$ edges.

Bipartite graphs

We'll now use the idea of the two colouring of the vertices of a graph to define a bigger class of graphs.

Definition

A graph whose vertices can be coloured in two colours is called a **bipartite graph**.

Questions

75 Find all of the graphs on 3, 4 and 5 vertices that are bipartite.

76 In what graphs are the vertices colourable with only one colour? Allow these graphs to be bipartite too.

77 Can you give an example of a graph other than P that is three colourable?

78 Find the bipartite graphs on 3, 4 and 5 vertices that have the most edges.

Another way of looking at two-colourable graphs is to first divide the vertex set of the graphs into two sets X and Y. Then a graph is bipartite if all the edges of the graph join a vertex in X to a vertex in Y. The **complete bipartite** is the graph with all possible edges drawn between X and Y. The notation for such graphs is $K_{|X|,|Y|}$.

Questions

79 Show that the X, Y definition and the two colourable definition of bipartite graphs are equivalent.

80 Draw $K_{2,2}, K_{2,3}, K_{2,5}$.

81 What is the complement of $K_{m,n}$?

82 How many edges does $K_{m,n}$ have?

83 List the sizes of all the cycles in $K_{2,2}, K_{2,3}, K_{2,5}, K_{3,4}, K_{3,6}, K_{1,5}, K_{4,4}$.

84 List the sizes of all the cycles in $K_{m,n}$.

85 In view of your results from the last Question, make some conjectures about the size of cycles in bipartite graphs. Can you prove these conjectures? How do your conjectures/theorems prove that trees are bipartite?

In Q85 you may have realised that bipartite graphs cannot have odd cycles. We prove that now.

Theorem

A graph is bipartite if and only if it has no odd cycles.

Proof

Suppose that B is bipartite and has an odd cycle. As we have seen in Question 72, an odd cycle requires 3 colours. So if B has an odd cycle it can't be bipartite.

Here's the curly part of the proof. It might help considerably if you draw some pictures to help you see what we're saying here. (In fact by now you should have drawn lots of pictures to help you see what is going on. Drawing pictures is an important tool in graph theory.) Suppose that G is a graph with no odd cycles. Consider any component of G. Let u be a vertex of this component. Let X be the set of all vertices w such that the shortest path from u to w is even. Note that u is in X. Let Y be the set of all vertices w such that the shortest path from u to w is odd.

Since $V(G) = X \cup Y$, then if we can show that there is no edge between vertices of X or vertices of Y, X and Y will be a bipartition of G. So we will assume that there is an edge between vertices u_1 and u_2 of X and produce a contradiction. A similar proof will show that no two vertices of Y are adjacent. Now let Q and R be shortest paths from u to u_1 and u to u_2, respectively. Suppose that z is the last vertex that Q and R have in common. Then the part of Q from u to z is one of the shortest paths from u to z. (Otherwise the paths Q from u to u_1 and R from u to u_2, would not be shortest paths from u to u_1 and u to u_2, respectively.) So the subpaths of Q and R from z to u_1 and from z to u_2 have the same length. These paths along with the edge u_1u_2 form an odd cycle. This is our contradiction. $\square$

Corollary

All trees are bipartite.

Proof

Trees are acyclic graphs so they have no odd cycles. They are therefore bipartite.

□

Planarity

Go back to Q7 for a moment. In that problem we wanted to know if it was possible to make some connections between three services and three houses so that the connections didn't cross. This produces a graph which is bipartite. Actually it is $K_{3,3}$. From what we saw there, we can't draw that graph in the plane so that none of its edges cross. However, many graphs can be drawn this way. For example, all trees and cycles. We call them planar. So a graph is **planar** if we can find some way to draw it in the plane so that no two edges cross.

Let's underline this. If a graph can be drawn in the plane so that no two edges cross it is planar. On the other hand, **non-planar** graphs can never ever be drawn in the plane without some edges crossing. However, it may be possible for a planar graph to be drawn with crossed edges, but it doesn't make it non-planar. We underline this in Figure 11. K_4 is first drawn with a crossing, and then without any crossings. So K_4 is planar. On the other, no matter how hard you try you will never be able to draw K_5 without a crossing.

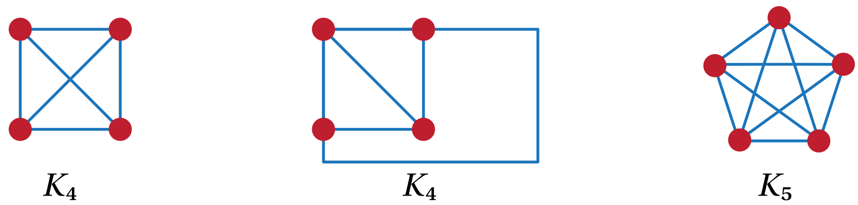


Figure 11: some drawings of planar and non-planar graphs

Questions

86 Now look back at all the graphs you have met so far. Which of them are planar? Particularly look at all the graphs on up to 5 vertices; complete graphs; complete bipartite graphs; regular graphs; self-complementary graphs; trees; and cycles. Make and prove conjectures, or find counter examples to, about all of these.

87 Show that K_n is non-planar for all $n \geq 5$. For what m and n is $K_{m,n}$ non-planar?

88 This last question leads to a characterisation of planarity by Kuratowski. See
http://en.wikipedia.org/wiki/Kuratowski%27s_theorem
but we have to be a bit loose in the statement we are going to make. A graph is non-planar if and only if it contains a K_5 or a $K_{3,3}$. Can you use this statement to show that P is non-planar?

Look up Kuratowski's Theorem to see precisely what it says.

Questions

89 If a planar graph is drawn so that no edges cross, it will not only have vertices and edges, it will also have faces. These are regions in the plane whose points can be joined by lines that do not cross an edge or meet a vertex of the graph. As such there is always one face 'outside' or 'around' the graph. Draw up a table for the numbers of vertices, edges and faces for all of the connected planar graphs in Q86. Is there a pattern?

90 One of the basic results of planar graph theory is Euler's Polyhedral Formula. Look this up on the web. Find a proof of it. Is this formula a necessary and sufficient condition for a connected graph to be planar? That means the two implications 'If a connected graph is planar then it satisfies the Euler Formula' and 'if a connected graph satisfies the Euler Formula then it is planar'.

91 Find all connected planar graphs with the following properties:

- The number of vertices is one more than the number of edges;
- The number of vertices is the same as the number of edges;
- All the faces have the same number of vertices and all the vertices have degree 3;
- All the faces have the same number of vertices and all the vertices have degree 4;
- All the faces have the same number of vertices and all the vertices have degree 5.

Questions

92 The graphs of Q91 dotpoint 3, dotpoint 4 and dotpoint 5 are called the Platonic graphs. These graphs have the property that they are all regular and all of the faces have the same number of vertices. Show that they are the only Platonic graphs.

93 How are the Platonic graphs related to the Platonic Solids?

94 How are the Platonic graphs related to the Platonic graphs? (Are they linked in some way? Try putting a vertex in a face. How might two vertices be joined?)

95 We have been careful to use the word 'connected' when talking about Euler's Formula. Why?

Can we get a formula for disconnected graphs?

Questions

96 Just in case you are not sure by now, Euler's Polyhedral Formula is $v - e + f = 2$, where v is the number of vertices, e the number of edges and f the number of faces. Call a graph toroidal if you can draw it on a torus (doughnut with a hole) so that no two edges cross. Does Euler have anything to say about these graphs in terms of v, e and f?

Circuits

In the Walking Problem at the start of this graph business, we looked at trying to find how to get across different bridges and get back to the start again. Figure 5 below is derived from Figure A1 in the answers. If the gentry followed $Ae_1Be_2Ae_5Ce_4Be_6De_3B$ they would have missed $e7$, but they would have certainly been on a walk. Probably for reasons similar to this, we define a **walk** on a (not necessarily simple) graph to be a sequence of vertices and edges, where, of course the edge between two given vertices in the walk actually joins these two vertices.

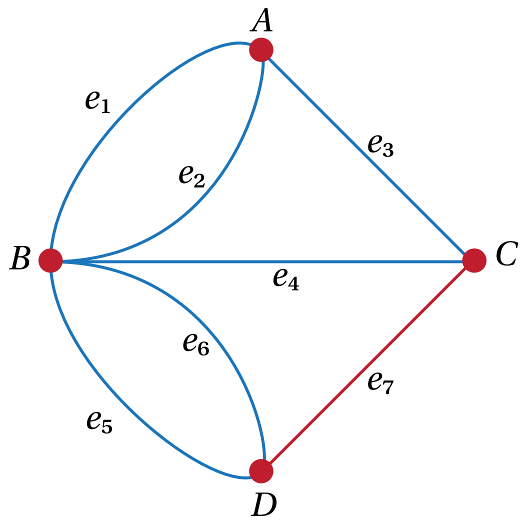


Figure 12: A labelled graph of the Königsberg problem

Questions

97 Find other walks on the graph of Figure 5.

98 Now there are all sorts of variations on this general definition where we make some sort of restriction. See if you can think of at least three variations on this theme. In what circumstances might these definitions be useful?

So Euler was interested in walks where every edge of the graph was used and the initial and final vertices were the same. He had no luck there. So he simplified things a bit to asking for a walk that used every edge but the first and last vertices were different. He failed there too. If he had been Hamilton, he would have looked for walks that went through every vertex once and only once whether or not the first and last vertex were the same. Now we know that Euler managed to settle The Walking Problem in general, and you know that we've pretty well solved this, but let's put it in formal words.

So let's define an **Euler trail** to be a walk in which every edge occurs exactly once and where the first and last vertices are different. On the other hand we'll call an **Euler tour** a walk in which every edge occurs exactly once and where the first and last vertices are the same. Let's think about this in the context of The Walking Problem. What the citizens of Königsberg really wanted was to find an Euler tour around their bridges. They might even have settled for an Euler trail. Euler resolved both these issues when he found the next two Theorems.

Theorem (Euler's Tour Theorem)

A connected graph has an Euler tour if and only if the degree of every vertex is even.

The proof of this is too long and involved for this pulication but it can be found in many places. You might try Clark and Holton, A *First Look at Graph Theory*, World Scientific, 1996 or some other graph theory text book.

Theorem (Euler's Trail Theorem)

A connected graph has an Euler trail if and only if it has two vertices of odd degree and the rest of even degree.

Note that both of these last two theorems apply equally to simple graphs and multi-graphs.

Questions

99 Which complete graphs have Euler tours and which have Euler trails? Repeat with complete bipartite graphs and the Platonic graphs. How about P?

100 Why do the two theorems above only talk about connected graphs?

101 Prove the Euler Trail Theorem assuming that the Euler Tour Theorem is true.

102 Do these theorems have any practical uses?

Euler's activity occurred in 1736. It took another 100 years or so before William Rowan Hamilton thought that it might be interesting to look at graphs that had a walk that went through every edge once and only once. So consider a **Hamiltonian** cycle to be a cycle that goes through every vertex of a graph. Similarly we define a **Hamiltonian path**. Incidentally Hamilton got his name attached to these objects because in 1857 he invented a game. His Icosahedron Game Game requires the player to find a Hamiltonian cycle around a dodecahedron. See http://en.wikipedia.org/wiki/Icosian_game)

Questions

103 Show that a dodecahedron has a Hamiltonian cycle.

104 Investigate complete graphs to see which of them have Hamiltonian cycles and which have Hamiltonian paths.

105 How many disjoint Hamiltonian cycles does a complete graph have? That is, how many Hamiltonian cycles do they have that have no vertices in common?

106 Does P have a Hamiltonian cycle or path?

107 Investigate graphs with and without Hamiltonian cycles to see if you can find some nice theorems.

108 A **Knight's tour** (see http://en.wikipedia.org/wiki/Knight%27s_tour) is a series of moves that takes a Knight around a chessboard from a given square visiting each square once and only once and returning to the original square. (In other words it's a Hamiltonian cycle on a particular graph defined by the moves of a Knight.) Show that such a tour exists on a 5×5 board.

109 For what m and n are Knight's tours possible on an $m \times n$ board?

As well as Euler, Hamilton has a famous connection with bridges. It was actually a piece of vandalism. He had been working on a problem for a long while and one day he went with his wife on a walk by a river, or maybe a canal, I'm not sure. Suddenly, near a bridge, out of the blue he solved his problem. He had invented quaternions. He was so excited that he carved the basic equations on the bridge!

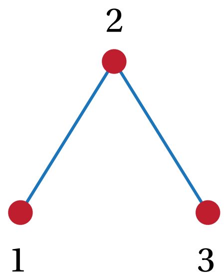
Alternative representations

In the text so far we have only given the dots and lines version of the representation of graphs. It is also possible to show a list or produce a matrix. There is not much to say about a list. It is just a collection of edges (or single vertices for vertices of degree zero). This would mean that P could be represented as

$$12\,,15\,,16\,,23\,,27\,,34\,,38\,,45\,,49\,,510\,,68\,,69\,,79\,,710,$$

where ab represents an edge between a and b. As you can see this does not give as much insight into a graph as does the pictorial relation we have used above.

A representation with a little more structure is given by what are known as adjacency matrices. An **adjacency matrix** is a matrix that shows which vertices are adjacent .For example if we number the rows by numbers 1 to 3 and columns in the same way, Figure 13 shows a graph and its adjacency matrix.



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 13: A graph in picture form and in matrix form.

It might be obvious, but the top left hand zero shows that 1 is not adjacent to 1, the 1 in the first row and second column shows that 1 is adjacent to 2. We get a 0 in row a and column b if a and b are not adjacent; we get a 1 in row c and column d if c and d are adjacent.

Questions

110 What are the adjacency matrices of the complete graphs and the complete bipartite graphs? How can you see that the adjacency matrix is the graph of a bipartite graph?

111 If M is the adjacency matrix of a graph, what is the adjacency matrix of its complement?

112 What does the adjacency matrix of a graph look like? How can you tell it from a matrix that is not an adjacency matrix of a graph?

113 Let A be the adjacency matrix of a graph. What does $2A$ and A^2 look like. What can we tell about the graph from A^2? What can you tell about a graph from A^k? (Work with some small graphs first to see what pattern develops.)

Some word problems

Questions

114 At a party of n people there are some people who initially don't know each other and some who do. It turns out that there are either three people who didn't know each other or three people who did. What's more if n had been any smaller this couldn't have happened. What is n?

115 In a group of nine people, one person knows two of the others, two people know each other, four each know five others, and the remaining two each Kknow six others. Show that there are three people who all know each other.

116 A man and his wife gave a formal party for four of their married friends. Various friends shook hands as they were introduced and naturally they didn't shake hands with their partners or more than once with the same person. Over drinks, the man asked everybody how many times they had shaken hands and discovered that no two people had shaken hands the same number of times. His wife then told him how many times he had shaken hands. What did the wife say?

Questions

117 A labelled graph is a graph where the 'names' of the vertices are important. For instance on flight maps it's important to know whether the edges joining two vertices is between the vertices Melbourne and Sydney or Brisbane and Adelaide. How many labelled graphs are there on n vertices?

118 Students, in a small class of 6, have to wait outside the maths class door in single file before their teacher lets them in. So that they get to know each other he tells them that they can't stand next to two classmates (in front or behind) more than once. How many classes go by before the teacher's condition is violated?

119 Twenty football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

See

http://www.math.cmu.edu{ ploh/docs/math /mop2009/graph-theory-intro.pdf after Tournament of the Towns 1986.

For more word problems see D Holton, 2010, *A First Step to Mathematical Olympiad*

Problems, Singapore: World Scientific.

Applications

In graph theory today people are using all kinds of graphs other than simple graphs. And there are many more ideas and techniques you have not met in this unit. There is a thing called matching where they are looking at pairing up adjacent vertices in some way. The first example of this was in World War II when the RAF had pilots and navigators available from the UK and Poland. Not all of the Poles spoke English nor did the UK pilots speak Polish. So a graph was drawn up with the pilots and navigators as vertices. Two vertices were joined if the corresponding pilots and navigators could speak the same language. Then someone tried to find the maximum number of edges that had no vertices in common. These pairings gave a flight crew for a plane. So these pairings became useful in assigning tasks in different situations.

Some graphs are called weighted graphs. They have numbers on the edges that provide information like the information on my Google map (see Motivation and History). Typically they may be routes for delivering material. For obvious reasons, you would like to be able to take the most efficient route on this graph. A lot of work has gone into this area of graph theory and there are algorithms that produce good results for small graphs. However, no one knows how to produce the best algorithm for every such graph. Indeed there is a good chance that such an algorithm doesn't exist.

The eigenvalues of an adjacency matrix are considered in spectral graph theory. These have important applications in quantum chemistry (see
http://en.wikipedia.org/wiki/Spectral_graph_theory)

Graph theory is also used in gene technology. So you have caught the person who you think has committed the murder and you have genetic evidence from the scene of the crime. How can you read the genetic evidence to show that the suspect and the evidence match up? Graph theory is involved in the sequencing techniques used in this exercise (see http://plus.maths.org/content/os/issue55/features/sequencing/index). We suggest that you look up further applications. These include computer tomography, data compression, communication networks, molecular structure and so on.

Answers to questions

Question 1

Unfortunately no. You can do this by trial and error on Figure 3. But it will help to alter Figure 3 slightly in the following way. Put a dot on every land mass and join two dots by a line for each bridge that connects them. So we get Figure A1 below. And then it would help to think a little more deeply than trial and error.

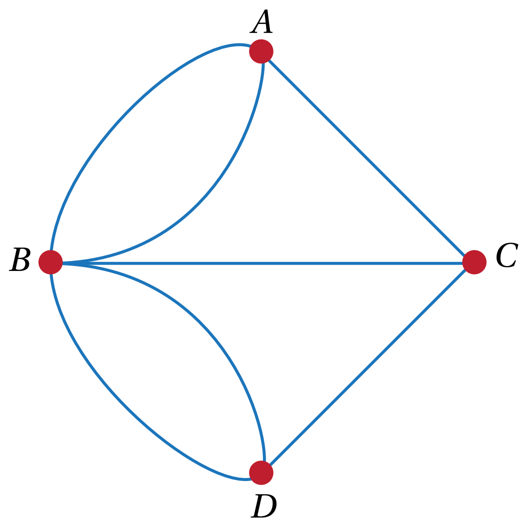


Figure A1: Königsberg reduced to dots and lines

Suppose that you could do the round trip walk. And suppose that you started at land mass A. Then you would have to go out from A on a bridge. So you start by using one bridge. At any time later that you came back to A you would use one bridge going in to A and one edge going out. So far you have used an odd number of bridges. Eventually you'll go back to A and use one final bridge to make the bridge count even. BUT! No land mass is attached to an even number of bridges. SO! There is no round trip walk.

Question 2

Apply the argument of Q1 to show that every land mass, except for the initial and final one, has to be attached to an even number of bridges. So can you see why the answer here is 'no' too?

Question 3

It turns out that, over time, Königsberg in Prussia has become Kaliningrad, Russia. In the process of time, the seven bridges have been reduced to five. If the current position of the bridges is as shown in Figure A2, then the walk (A, B, C, D, B, A) can be done.

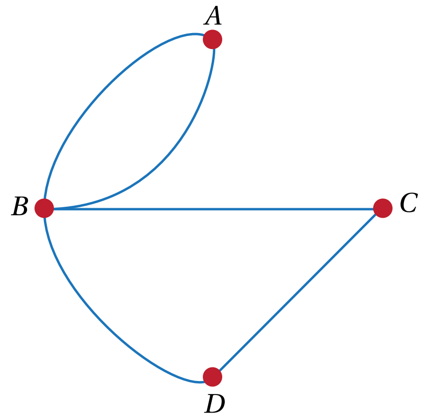


Figure A2: A walk with five bridges

How did you go with six bridges? Can you only find a walk that starts on one part of the city and ends up somewhere else? Why?

Question 4

Yes, put in two landmasses X and Y between C and D and then A, B, C, D, X, Y, BA will get you round. But can you be more imaginative than that?

Question 5

Well it just depends on how you deploy your bridges. Sometimes there is a round trip walk and sometimes there ain't. What is the important feature that will guarantee a walk?

Question 6

The key point is the even-ness of the number of bridges at each land mass. Is it possible to prove that if the number of bridges at each land mass is even, then there is a round trip walk? Is it possible to show that if there is a round trip walk, then the number of bridges at each land mass is even? So is it true that there is a round trip walk if and only if the number of bridges at each land mass is even? Can you prove this? Beware the lure of a simple result. Make sure that you carefully word your answer. (See Euler's Tour Theorem.)

Question 7

The answer is no, but we'll get back to this later when we've thought about 'planarity' and K_3 - whatever they are.

Question 8

Putting one dot on the page should give a graph with one vertex, so A is not a possibility. How could we have more than one graph though? Perhaps we could have differently shaped vertices? No, we will assume that however you draw the vertices they are the same vertex (though we usually draw them as small circles). So what else could we do? There is the possibility of edges. Could we draw an edge from a vertex to a vertex? There seems to be nothing in the rules to stop that. Such things are called **loops**. We show one in Figure A3.

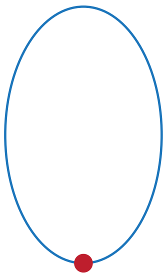


Figure A3: A graph with a loop

Those of you who wanted to include loops would have answered D to Q1. If you are against loops, though, you will have answered B.

Question 9

Because we are not allowing loops, it looks at the start as if there might only be two graphs on two vertices. But then look at Figure A4 below. Are these graphs the same or different?

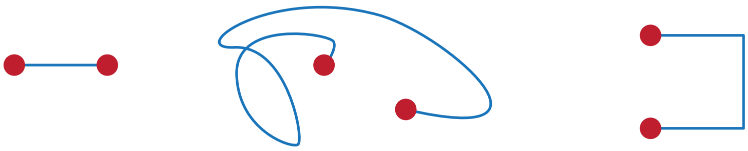


Figure A4: Graphs with different looking lines

Question 10

After some thought I'd guess that there are two schools of thought. Those who like C and those who like D. The problem again is 'when are two graphs the same?'. We will go for C, but why?

Question 11

There are 11 graphs on 4 vertices. We show them in Figure A5. But we show them systematically there. What system did we use? Why is the sixth one 'alone' with no partners?

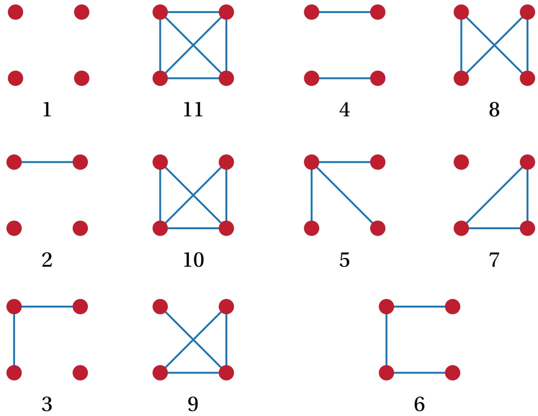


Figure A5: The simple graphs on four vertice

Question 12

Another way to do this is: is $a \leftrightarrow v, b \leftrightarrow u, c \leftrightarrow w$, and $d \leftrightarrow x$. This sends the edges $ab \leftrightarrow vu = uv, ac \leftrightarrow vw, bd \leftrightarrow ux$.

Question 13

Doing this simply, interchange the positions of c and d in Figure A6 and then rotate the graph .

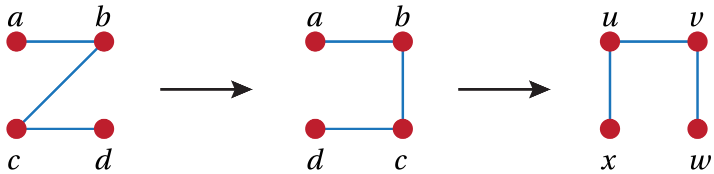


Figure A6: An isomorphic pair of graphs - from Z to Π

Using a bijection we have $a \leftrightarrow x, b \leftrightarrow u, c \leftrightarrow v$, and $d \leftrightarrow w$.

Question 14

One way to think about this is to see that there is a triangle in G, but not in H, and there is a square in H but not in G. But G and H are also not isomorphic because they have different numbers of edges. Let $E(G)$ be the number of edges of G. Then it's easy to see that $|E(G)| = 4$ and $|E(H)| = 5$. These two reasons should be enough to settle things. However, we can look at this in another way. Look at the number of edges that go into the different vertices. Starting at the bottom right and moving around clockwise these 'edge-in' numbers are for $G - 1,2,3,2$ and for $H - 3,2,3,2$. Surely these sequences would have to be the same if we had $G \cong H$!. So we have some tests for isomorphism.

Question 15

In order these are $(0,0,0,0)$, no edges; $(3,3,3,3)$, 6 edges; $(1,1,1,1)$, 2 edges; $(2,2,2,2)$, 4 edges; $(1,1,0,0)$, 1 edge; $(2,2,3,3)$, 5 edges; $(3,1,1,1)$, 3 edges; $(0,2,2,2)$, 3 edges;$(2,1,1,0)$, 2 edges; $(1,2,2,3)$, 4 edges; $(2,1,2,1)$, 3 edges. Make a conjecture about the relation between the degrees of a graph and its number of edges? Does that work for any graph? How would you prove it?

Question 16

There doesn't seem to be a way to get $1,1,1,1,1$. Why do you think that this is? However, the graph in Figure A7 has the sequence of degrees: 1, 1, 1, 1, 1, 1.

Figure A7: The graph with degree sequence 1, 1, 1, 1, 1, 1

Question 17

Experiment with this. What conjectures (guesses) do you have? Can you prove them? Does the Theorem (see after Q22.) help in any way?

Question 18

You can see from the simple graphs we have listed for $n = 3$ and$n = 4$ that it is possible for graphs to have every vertex of degree 0 or even just a few vertices of degree 0. But no vertex in a simple graph can have degree n because there are only $n - 1$ other vertices for a specific vertex to be joined to. This is not true for graphs that are not simple though. Find an example.

Question 19

Every vertex can have degree 0 (just five vertices and no edges); every vertex can have degree 2 (we'll see later that this is called the cycle C_5); every vertex can have degree 4 (put in all possible edges to get K_5 see Q25); but there are no graphs on 5 vertices where every vertex has degree 1 or 3 (why?).

Question 20

The one with each vertex joined to all of the other vertices. The degree of each vertex is $n-1$. The number of edges is $\frac{n(n-1)}{2}$ (why?). Make sure that you can prove that result in 2 ways. Why isn't it $n(n-1)$?

Question 21

No. There seem to be three non-isomorphic simple graphs with every vertex of degree 2. We've listed them in Figure A8.

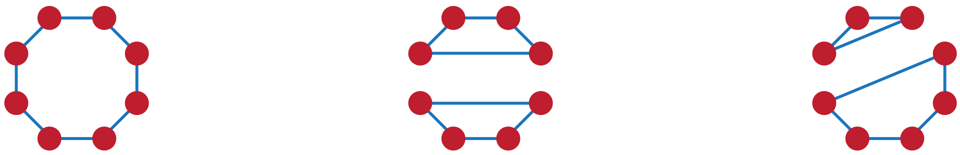


Figure A8: Three non-isomorphic simple graphs with the same degree sequence

Question 22

Yes. Look at the graphs of Q11.

Question 23

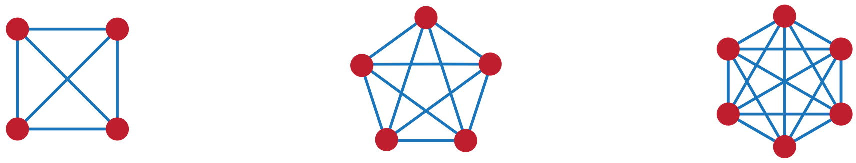
Does this have anything to do with the number of edges? Does it have anything to do with the degrees of vertices? Does Q16 have anything to do with this? All of these things help to simplify things, but in the last resort you have to go back to the definition to be sure if two graphs are isomorphic.How about the number of edges or the structure of the graphs?

Question 24

This is a repeat of Q.20. For 3 vertices the maximum number of edges is 3; for 4 it is 6; for 5 it is 10 and for 6 it is 15. For $n, N = n(n-1)/2$. There are two ways at least to prove this. We'll show you one way and then you find the another. To prove this result note that the first vertex has $n-1$ edges coming into it. The second vertex has $n-2$ edges that are different from the $n-1$ going to the first vertex. The third vertex has $n-3$ new edges. This continues until the last vertex has no new edges. So the total number of edges is $0+1+2+3+\cdots+(n-3)+(n-2)+(n-1)$. This is an Arithmetic Progression whose sum is $n(n-1)/2$.

Question 25

The graphs required are shown in Figure A9.



K_4 K_5 K_5

Figure A9: Some complete graphs

Question 26

For any graph with e edges there is a unique graph with $N - e$ edges. Just take away all of the e edges and add in all the edges that were not there in the first place. As a result if you put these two graphs on top of each other you would get K_n. So we have a one-to-one link here.

Question 27

34. And this will be easier if you use Q26. Or did you use it without knowing? Also refer forward to Q38.

Question 28

We have put the two complementary pairs together in Figure A10.

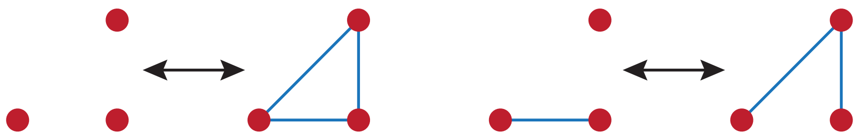


Figure A10: The complementary graphs on three vertices

Question 29

If you use the definition of the complement twice on G you get back to G again.

Question 30

There is only one way for the definition to work.

Question 31

P is regular of degree 6. It has 30 edges.

Question 32

These are just the complements of the graphs in the answer to Q21.

Question 33

Suppose that G is a regular graph of degree r. Then in the complement of G, every vertex has degree $n-1-r$. So the complement is regular.

Question 34

We'll list all of the graphs with 0, 1, 2, 3, and 4 edges. The graphs on 10, 9, 8, 7 and 6 edges follow by complementation. We'll then list all of the graphs with 5 edges (some of which are self-complementary.) This is done in Figure A11.

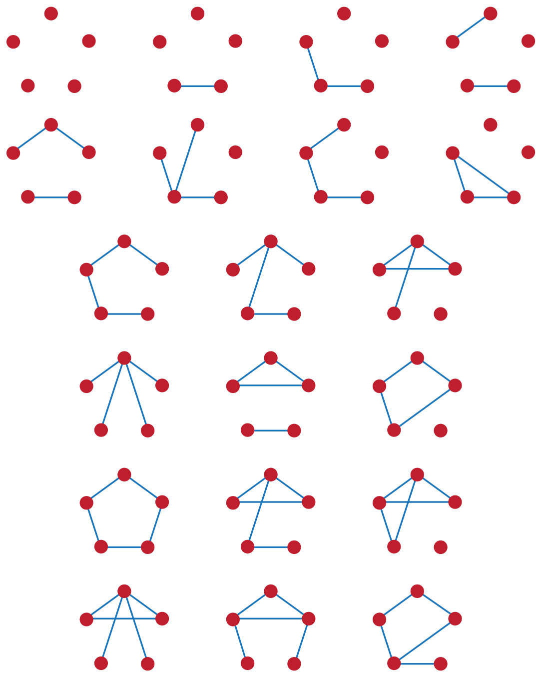


Figure A11: All the graphs on five vertices with less than or equal to five edges

Question 35

On 5 vertices: there are two of them; one with every vertex of degree 2 and the other with degrees of 1, 1, 2, 3, 3. On 6 vertices: there are none. Why not? Investigation: Let n be the number of vertices of a graph. What value must n have to possibly support a self-complementary graph? Can you show that for each of these values of n there is at least one self-complementary graph? If so you will have proved a conjecture that says There is a self-complementary graph on n vertices if and only if $n =$ whatever. This problem is a non-trivial exercise. Work on graphs on up to 13 vertices and see if you can generalise from there. You might find the graph in Figure A12 useful.

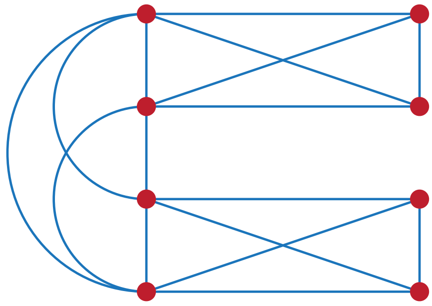


Figure A12: a graph on eight vertices

Question 36

The simple graph on 1 vertex is regular of degree 0. The graphs on 2 vertices are either regular of degree 0 or regular of degree 1. On 3 vertices there is one regular graph of degree 0 and one of degree 2. On 4 vertices there is one regular graph of degree 0, one of degree 1, one of degree 2, and one of degree 3.

Question 37

A graph with n vertices and no edges is regular of degree zero. Actually it is the complement of K_n. For $n = 2m$, the graph consists of m edges whose vertices are all distinct. For $n = 2m + 1$ no graph is possible.

Question 38

We have shown the regular graphs of degree 2 on 8 vertices in Q21; there are no others. There are no graphs that are regular of degree 3 on 9 vertices. Why? (How many edges would such a graph have?)

Question 39

There are no regular graphs of odd degree on $2n+1$ vertices. Suppose there were. Let the odd degree be $2k+1$. By Theorem 1, the number of edges is $(2n+1)(2k+1)/2$. But this is not a whole number. Can we go further than this? What can be said about the number of vertices of odd degree in any graph? We'll prove the general result here.

Corollary

The number of vertices of odd degree in a graph is even.

Proof

By the theorem, the sum of the degrees of all of the vertices is even. But this sum is also the sum of the even degree vertices and the sum of the odd degree ones. Now the sum of the even degree vertices is even. So the sum of the odd degrees has to be even too. This means that there have to be an even number of vertices of odd degree. □

Question 40

On 3 vertices the graphs with 2 or 3 edges will do the trick. On 4 vertices graphs 5, 6, 8, 9, 10, 11 in the answer to Q11. On the 5 vertices graphs show in Figure A12, the 9th, 10th, 12th, 15th, 16th, 18th, 19th and 20th in the answer to Q38. The complements of some of the graphs there are also connected. Which ones?

Question 41

There are several of these. We give three in Figure A13, but you might try to find them all. Be systematic!

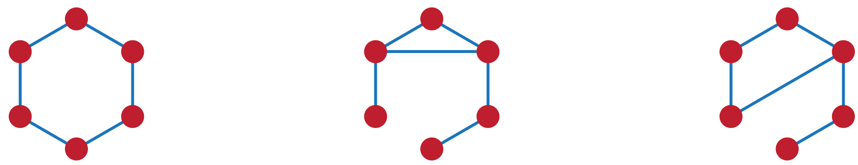


Figure A13: some connected graphs on 6 vertices with 6 edges

Question 42

On 3 vertices: the graphs with none and one edge. On 4 vertices: from Q 11, these are graphs 1, 2, 3, 4, and 7.

Question 43

Connected graph: take K_{n-1} and add a vertex to get a connected graph. Disconnected graph: the vertex of degree $n-2$ is joined to $n-2$ vertices. There is still a vertex not joined to any other vertex, so this graph is disconnected.

Question 44

Never. Suppose that G is disconnected, then there exists a vertex u that is not joined to at least one other vertex. If u is not joined to any other vertex, then$u$ is joined to every vertex in its complement. Suppose that v is adjacent to u in G. Let U be the set of vertices not connected to u directly or through other vertices of G. By definition v is not adjacent to a vertex in U. Let V be all the vertices adjacent to u in G. Then in $\tilde{G}, u$ is joined to every vertex in U and so is every vertex of V. Hence $\tilde{G}$, is connected. (If you don't see how this works, try the idea out on some specific disconnected graphs.) In general, at least one of a graph or its complement is connected. This follows from the argument above. All the graphs in Figure A13 have a connected complement. So a graph and its complement can both be connected

Question 45

Just K_n because it is clearly connected and has the most edges of any graph on n vertices. Can you do better than K_{n-1} plus an extra vertex of degree 0? Why? Why not?

Question 46

On 3 vertices: 1 graph with 2 edges; on 4 vertices: 2 graphs with 3 edges; on 5 vertices: 3 graphs with 4 edges. You can find them easily from the graphs in Q11 and Q38.

Question 47

Conjecture 1

The fewest edges a connected graph on n vertices has is $n-1$.

Note 1 No graph with $n-2$ edges is connected.

Note 2 Not all graphs with $n-1$ edges are connected.

Note 3 Not all graphs with n edges are connected.

Your proof has to take into account all the notes above. We'll give a proof of all the conjectures later in the text.

Conjecture 2

The number of connected graphs on n vertices is $n-2$.

Conjecture 3

In some graphs there is only one way to go from one given vertex to another given vertex.

Conjecture 4

Suppose we have a connected graph. If there is only one way to get from any vertex to any other then the graph has no cycles.

Check your conjectures out on graphs with 6 vertices to see if they are still true.

Question 48

You will have done all sorts of trees. In Figure A14 we give just three.

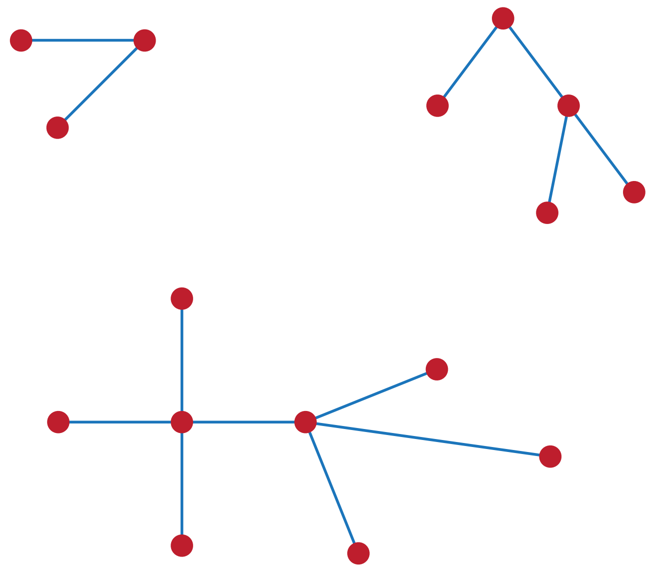


Figure A14: a random three trees

Question 49

There is no formula for the number of trees on n vertices –
see http://en.wikipedia.org/wiki/Tree_(graph_theory).

Question 50

This can be done for all connected graphs, but you have to be careful to take off edges that don't disconnect the graph. How do you know when to stop though?

Question 51

It looks as if they all have at least two vertices of degree 1. Are there some trees with only

2 vertices of degree 1? But how to prove these things? We'll take this up in the text later when we prove a series of results about trees.

Question 52

There are a lot of them (11). In fact all of the graphs on 4 vertices (see Q11) are subgraphs of K_4.

Question 53

From Q11, with two components: 3, 4, 7; and with three components: 2.

Question 54

It is K_n.

Question 55

Conjecture: They are connected. Proof: Since trees are connected, then so must be a graph with a tree as a subgraph. (Also: if the graph has n vertices it has to have at least $n-1$ edges.)

Question 56

They are connected, so they can't have degree 0 or 1; there is no spanning graph of degree 2 (see Q88); so every regular spanning graph of P is P.

Question 57

The answers are shown in Figure A15.

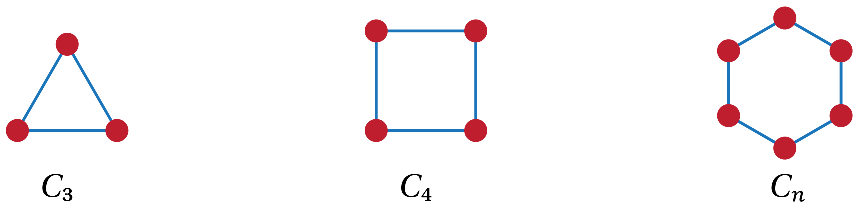


Figure A15: Some of the small cycles

Question 58

All the cycles from C_3 up to C_n.

Question 59

There are no cycles in any tree. Suppose there were. Then you could delete an edge from that tree and it would still be connected. But this would contradict the condition of trees having the minimum number of edges in any connected graph on a given number of vertices.

Question 60

Is the connected graph G in Figure A16 the only one possible?

Figure A16: the required graphs G and H

The graph H in Figure A16 has the required property. (It also has a C_3 and a C_4.)

Question 61

In P, you can find C_5 (on the vertices 1, 2, 3, 4, 5, for example), C_6 (2, 3, 4, 5, 10, 7), C8 (6, 8, 3, 2, 1, 5, 4, 9), and C9 (2, 3, 4, 5, 10, 8, 6, 9, 7), but no other cycles. In $\tilde{P}$ you can find lots of cycles of every type from C_3 up to C_10.

Question 62

The conjecture is false for $n = 3$ and 4. **Conjecture**: For $n > 4$, $\tilde{C_n}$ is connected **Proof** Number the vertices from 1 to n in order around the original cycle. $\tilde{C_n}$ contains tree with edges 1 to i for all $i \neq 2$ or $n-1$, and 4 adjacent to 2 and $n-1$.

Question 63

We'll illustrate this in Figure A17 using $i = 3$ and $n = 6$. The original P_3 has red edges; the remainder of the tree has dotted edges. There are clearly many more trees that have P_3 as a subgraph. In general start with the given P_i and add edges until a tree on n vertices is formed.

The answer to the uniqueness is 'sometimes'. For instance, there is only one tree on 4 vertices with P_3 as a subgraph. What do you think of the following conjecture?

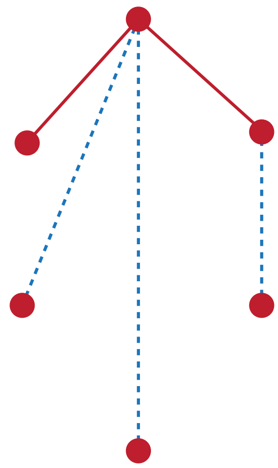


Figure A17: a tree on six vertices containing the subgraph P_3

Conjecture

Let $n > 4$. Suppose that we have a tree with P_i as a subgraph then the tree is unique if and only if $i = n$.

Question 64

Yes. Just delete an edge of the cycle. No. For example, any complete graph has a spanning path.

Question 65

Yes. If a graph has a spanning path, then it is possible to go from any vertex to any other vertex by edges of the graph. Hence the graph is connected. No. The graphs in Figure A14 (the answer to Q48) are connected graphs that don't have spanning paths.

Question 66

In Figure 4, the path 1, 2, 3, 4, 5, 10, 8, 6, 9, 7 is one possible spanning path. How many more are there? (See also Q106.)

Question 67

Yes, there are lots of them.

Question 68

Yes, again there are generally lots of them. The thing that distinguishes connected graphs from disconnected graphs is that connected graphs have spanning trees. So we'll conjecture that a graph is connected if and only if it contains a spanning tree.

Question 69

Two! Yes. Why?

Question 70

n, because every vertex is joined to every other vertex.

Question 71

2, just alternate the colours as you go from one end of the path to the other.

Question 72

2 if n is even and 3 if n is odd.

Question 73

First it can't take just 2 colours because P contains a C_5. We can colour it with 3 colours though. One way to do this it to colour the vertices 1, 4, 7, 8 with colour A; 2, 5, 6 with colour B; and 3, 9, 10 with colour C.

Question 74

Our original definition assumes that trees are connected and Theorem T1 shows that they are acyclic. If a graph is connected and acyclic we only have to prove that it has the fewest edges of any connected graph to show that it is a tree in our original definition. If the graph is acyclic, then the proof of the Corollary to Theorem T1 still holds and shows that there is a unique path between any two vertices. Hence if we remove an edge, some vertices will no longer be joined. So an acyclic connected graph has the fewest edges of all connected graphs.

Question 75

Can you tell by looking at the sorts of cycles they have?

Question 76

The only such graphs are $\tilde{K}_n$. This is because no vertex is adjacent to any other vertex so they only need one colour.

Question 77

Any odd cycle will do.

Question 78

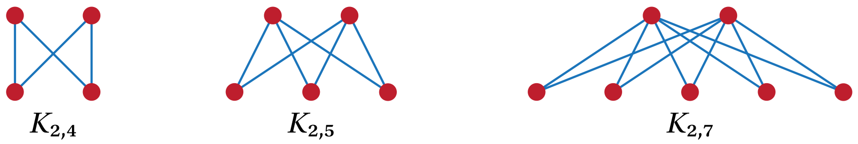


Figure A18: The bipartite graphs with most edges on 3, 4 and 5 vertices.

Question 79

Let the vertices of a bipartite graph be coloured in two colours, red and blue. By the definition of colouring, red vertices are only adjacent to blue vertices and vice versa. Let the red vertex set be X and the blue vertex set be Y. Then this satisfies the X, Y definition of bipartite.

Starting with theX, Y definition let the vertices of X be coloured red and the vertices of Y blue. This gives us the 2 colouring definition.

Question 80

The graphs are shown in Figure A19

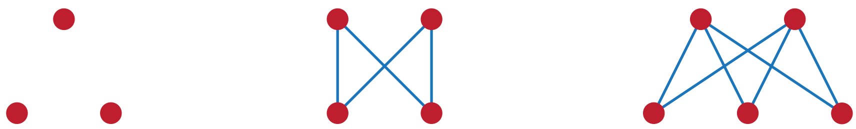


Figure A19: Three particular complete bipartite graphs

Question 81

The disjoint union of K_m and K_n (a copy of K_m together with a copy of K_n).

Question 82

mn because every vertex in the m set has degree n.

Question 83

In order these are 4; 4; 4; 4 and 6; 4 and 6; none; 4, 6 and 8.

Question 84

$2, 4, 6, \ldots, 2m$.

Question 85

Conjecture

A graph B is bipartite if and only if it has no odd cycles.
The proof is not straightforward. It is to be found in the next section of the text as is the answer to the tree's question.

Question 86

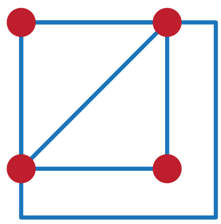
K_n is planar for $n \leq 4$ and non-planar otherwise. You may think that K_4 is non planar but it can be drawn as in Figure A18 where no edges cross. To prove that K_5 is non-planar requires the use of the Jordan Curve Theorem. You might want to look this up but it is beyond the scope of this course.

$K_{m,n}$ is planar for $2 \leq m \leq n$ and non-planar otherwise. Figure A18 shows a non-planar drawing of $K_{2,3}$. Again to show that all of the other complete bipartite graphs are non-planar requires the Jordan Curve Theorem.

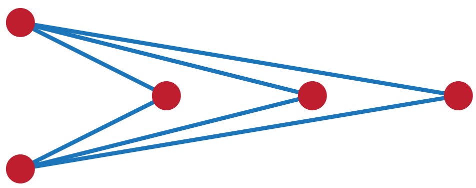
For regular graphs there is no order. True all regular of degree 2 graphs are planar but from there on some are and some aren't. For example, it is easy to draw a planar graph on 10 vertices that is regular of degree 3, but P is regular of degree 3 on 10 vertices and is not planar.

Self-complementary graphs are non-planar as soon as they get to have more than 8 vertices.

Trees and cycles are always planar.



A planar K_4



A planar $K_{3,3}$

Figure A20: Some examples of planar graphs

Question 87

If K_5 is non-planar, then so is K_n for $n > 5$, because K_5 is a subgraph of all of these. The same argument shows that if $K_{3,3}$ is non-planar so are $K_{m,n}$ for $3 \leq m \leq n$.

Question 88

We show a 'sort of' $K_{3,3}$ in P in Figure A21. The blue lines represent the edges of the $K_{3,3}$ and the red and green vertices are the two sets of vertices.

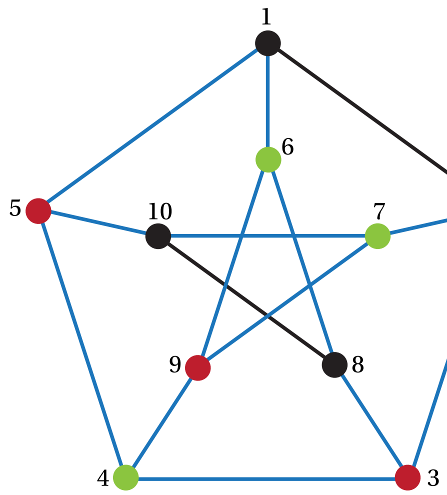


Figure A21: A K_3 in P

Question 89

We show some of the graphs in Table A1.

Table A1: A table to see any patterns in numbers of vertices, edges and faces

Graph	Vertices	Edges	Faces
A tree on n vertices	n	$n-1$	1
C_n	n	n	2
K_4	4	6	4
K_2	$n+2$	$2n$	n

Do you see any patterns in these numbers? Do they hold for every planar graph you can think of?

Question 90

The proofs that you will find assume that the graph being dealt with is connected and planar.

Now, why do we need connected? Can you find a disconnected planar graph that doesn't satisfy Euler's formula?

Does it make sense to think about faces for non-planar graphs?

Question 91

By the results in the tree section these can only be trees; Trees with one edge added (so they have one cycle). They are connected so they must have spanning trees. But we need to add one more edge. (iii), (iv), and (v) are shown in Figure A22.

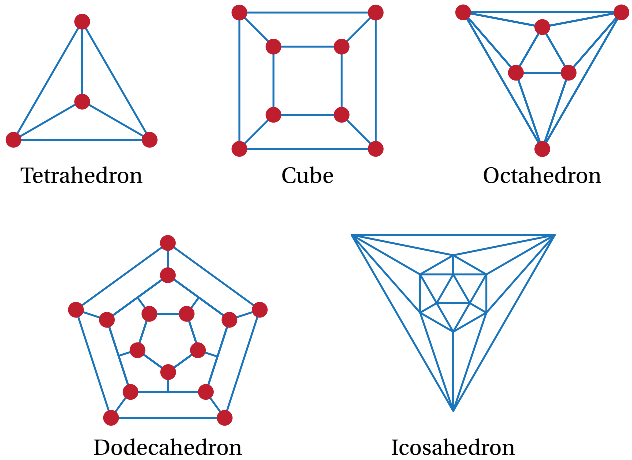


Figure A22: The graphs of the Platonic solids

(see http://www.bing.com/images/search?q=graphs+of+the+platonic+solids &qpvt=graphs+of+the+platonic+solids&FORM=IGREa)

Question 92

We'll do this for the graphs that have degree 3 and leave the others to you. Since $2e$ is the sum of the degrees, suppose we have n vertices then $2e = 3n$. If every face has r vertices, then rf counts each vertex 3 times and so $3n = rf$. Put these numbers into the Euler formula and we get $n - 3n/2 + 3n/r = 2$. Rearranging we get $4 = 6n/(n+4)$. To make life easier for ourselves rearrange to get $r = 6 - 24/(n+4)$. So $n+4$ divides 24 to give $n = 4, 8$ and 20 (no graphs exist for lower divisors of 24). Hence we get the right numbers for the Platonic graphs of degree 3. We omit showing that there is only one graph for each of these three sets of vertices, edges and faces.

Question 93

Look both of these up on the web and compare them. Take a face of one of the solids and make it the outer face of the graph. Then squash 'project' the solid onto the plane.

Question 94

Put a vertex in a face of a Platonic graph and join two vertices if they are in adjacent faces. What graphs do you get? What does the tetrahedron's new graph look like? How about the cube or the dodecahedron?

Question 95

It is easy to find a disconnected planar graph for which Euler's Polyhedral Formula doesn't hold.

For a formula for disconnected graphs see http://en.wikipedia.org/wiki/Euler_characteristicSurfaces.

Question 96

See http://en.wikipedia.org/wiki/Euler_characteristicSurfaces.

Question 97

There are lots of possibilities here, actually an infinite number because you can keep repeating any edge you like. So long as you go vertex, edge, vertex, etc. and the edge between two vertices joins the two vertices you have a walk.

Question 98

You might want to make sure that every edge of a walk appears only once - if you want to make sure that you go over every 'bridge' once and only once. You might want to make sure that every vertex of a walk appears only once - if you want to make sure that you get to each 'land mass' once and only once. You might want to make sure that the first and last vertex of a walk is the same - if you want to make sure that you get back to where you started. You might want to make sure that the first and last vertex of a walk are different - if you want to make sure that you don't get back to where you started.

Question 99

Complete graphs have Euler tours if and only if they have every vertex even. This means that the number of vertices is odd. So K_n has an Euler tour if and only if n is odd. You can only get an Euler trail if $n = 2$.

Complete bipartite graphs $K_{m,n}$ have Euler tours if and only if m and n are both even. On the other hand they have an Euler trail if and only if $m = 2$ and n is odd.

Platonic graphs are shown in Figure A22. Only one, the octahedron, has an Euler tour because it is the only one that is regular with an even degree. None of them have Euler trails.

The Petersen graph can't have either a tour or a trail because every vertex of P is odd (three). Clearly this applies to every regular graph of odd degree. On the other hand regular graphs with even degree al have Euler tours.

Question 100

Disconnected graphs don't have edges from one component to another. Hence no Euler tour or Euler trail.

Question 101

Add an edge between the two vertices of odd degree. (This may give you a multigraph but that is OK because the Euler Theorems apply to multigraphs.) This new graph has an Euler tour because every vertex has even degree. Now remove the added edge and the Euler tour we have just found becomes an Euler trail.

Question 102

What if a delivery van needs to drop something at every house in every street of a suburb? It would be good to know if the street graph has an Euler tour as then the van's work can be done efficiently. Even an Euler trail would make life easier.

Question 103

See http://en.wikipedia.org/wiki/Icosian_game.

Question 104

All complete graphs on n vertices have spanning cycles because they have all graphs on n vertices as spanning graphs. Likewise they all have Hamiltonian paths.

Question 105

K_3 has one Hamiltonian cycle; K_4 has one Hamiltonian cycle and two non-adjacent edges left; K_5 has two disjoint Hamiltonian cycles; and K_6 has two Hamiltonian cycles and three non-adjacent edges left over.

If you try to generalise this you might conjecture that (i) for n is odd K_n has $(n-1)/2$ edge disjoint Hamiltonian; and (ii) for n even K_n has $(n-2)/2$ edge disjoint Hamiltonian cycles and $n/2$ independent edges. Are these conjectures true?

Question 106

P has no Hamiltonian cycle.

Suppose then that it does have a Hamiltonian cycle. Therefore it has to use an even number of the edges 16, 27, 38, 49, 5 10, which we'll call the cross edges.

Step1 Without loss of generality assume that it uses four of these edges and the cross edge not used is 16. Colour the not used edge in blue as in Figure A20. This means that it must use the red edges in Figure A20. But the red edges form a cycle that is not a Hamiltonian cycle. So Step 1 can't apply.

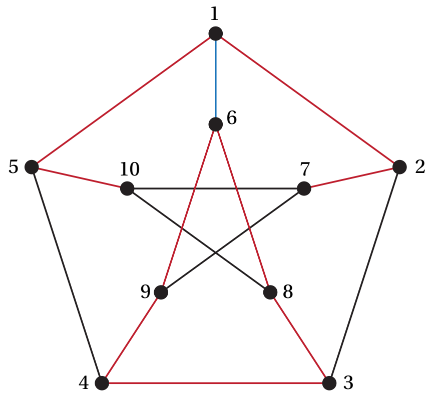


Figure A23: P is not Hamiltonian, Step 1

Step 2 Assume that only two of the cross edges are used. Without loss of generality we can assume that (a) 16 and 5 10 are used or that (b) 16 and 38 are used. If 16 and 5 10 are used, then 27, 38 and 49 are not used so we colour these in blue (see Figure A21). The red lines that have to be in the Hamiltonian cycle so far are coloured red. But there is already a cycle here so there can't be a Hamiltonian cycle.

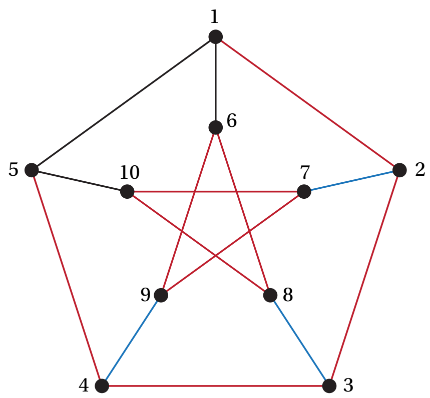


Figure A24: P is not Hamiltonian, Step 2a

Assume that 16 and 38 are used. This also leads to a contradiction, but we leave that for you to complete.

3, 8, 6, 1,2,7, 10, 5, 4, 9 is one of many Hamiltonian paths in P.

Question 107

Unfortunately there is no nice characterisation for Hamiltonian cycles like the Euler Tour Theorem. In fact even finding a Hamiltonian cycle in a given graph is NP complete (which means it's pretty hard to do!). There are some valuable results but they are not easy to state. Search the web to see what results are known.

Question 108

Draw a 5×5 board and put a vertex in the centre of each square. Then join two vertices if it is possible for a Knight to move between the corresponding squares. Show that this graph has a Hamiltonian cycle.

Question 109

The answer can be found at http://en.wikipedia.org/wiki/Knight%27s_tour.

Question 110

The adjacency matrix of a complete graph consists of ones except for the main diagonal which is all zeros.

For a bipartite graph with parts of size m and n, there is an $m \times m$ block of zeros in the top left hand corner of the adjacency matrix and an $n \times n$ block in the bottom right hand block. The other blocks are $m \times n$ and $n \times m$ and each is the transpose of the other.

Question 111

Replace all 1s by zeros and all zeros, except the ones on the main diagonal, by 1s.

Question 112

An adjacency matrix has to have the following properties:

- It is symmetric;
- Its entries are from the set $\{0, 1\}$; and
- The main diagonal consists only of zeros.

Show that these are necessary and sufficient conditions for a matrix to be an adjacency matrix or find all the necessary and sufficient conditions.

Question 113

The sum of an adjacency matrix with itself has a 2 when the graph has an edge and a zero otherwise. The square of an adjacency matrix tells you something about the number of walks of length two. What is that?

From $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\quad A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\quad A^3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

we can see that the entries have something to do with walks. It turns out that in general the i, j entry in the matrix A^k, tells you how many walks of length k there are from vertex i to vertex j.

Question 114

The answer is 6.

Question 115

Think of the 9 people as vertices of a graph and join those who know each other. Let u be adjacent to u_i for $i = 1, 2, 3, 4, 5, 6$. If v and w are the remaining vertices we know that one of u_i has degree at least five. Hence u_i knows u_j for some j and we have our triangle.

Question 116

The steps here are to note (i) that two people must have shaken hands the same number of times, so one of these is the male host; (ii) one person has to have shaken hands 8 times. Now remove the two people who have shaken hands the same number of times. Repeat everything above to reduce the number of people under consideration to 4 and so on. On a countback you'll see that the man and his wife had shaken hands with 4 people.

Question 117

Trying a few small cases shows that if $n = 1$, the answer is 1; $n = 2$ gives 2 (include the labelled graph with no edges); $n = 3$ gives 8; and higher values of n give powers of 2 as well. But what is the power here? It's the number of possible edges. To show this note that every edge is used or not used. So the number of edges is $2^{n(n-1)/2}$.

Question 118

Three times. The edges of K_6 can be divided into 3 Hamiltonian paths.

Question 119

This is a graph on 20 vertices whose edge set is a union of edges. So the degrees of all of the vertices are less than 3 and the graph is a disjoint union of even cycles. It is then possible to get an independent set of size 10.

Years

11 & 12