

Maths delivers!

A guide for teachers - Years 11 and 12

## Proofs by induction

*Maths delivers!*  
*Proofs by induction*

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For teachers of Secondary Mathematics

These notes are based in part on lecture notes and printed notes prepared by the author for use in teaching mathematical induction to first-year students at Latrobe University.

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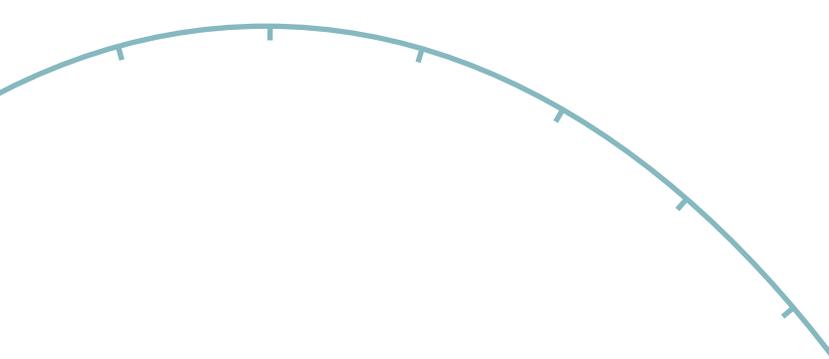
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## Proofs by induction

### The principle of mathematical induction

#### The idea of induction: falling dominoes

There are two steps involved in knocking over a row of dominoes.

**Step 1.** You have to knock over the first domino.

**Step 2.** You have to be sure that when domino  $k$  falls, it knocks over domino  $k + 1$ .



These are the same as the steps in a proof by induction. We have an infinite number of claims that we wish to prove: Claim(1), Claim(2), Claim(3), ..., Claim( $n$ ), .... If we can perform the following two steps, then we are assured that all of these claims are true.

**Step 1.** Prove that Claim(1) is true. That is, knock over the first domino.

**Step 2.** Prove that, for every natural number  $k$ , if Claim( $k$ ) is true, then Claim( $k + 1$ ) is true. That is, show that if the  $k$ th domino is knocked over, then it follows that the ( $k + 1$ )st domino will be knocked over.

The **principle of mathematical induction** says that, if we can carry out Steps 1 and 2, then Claim( $n$ ) is true for every natural number  $n$ .

### More formally

The principle of mathematical induction is based on the following fundamental property of the natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Indeed, this property is one of the **Peano postulates** that are used to define the natural numbers.

#### Peano's induction postulate

If  $S$  is a subset of  $\mathbb{N}$  such that

- 1 is in  $S$ , and
- if  $k$  is in  $S$ , then  $k + 1$  is in  $S$ , for each natural number  $k$ ,

then  $S = \mathbb{N}$ .

To see that the principle of mathematical induction follows from this postulate, let  $S$  be the set of all natural numbers  $n$  such that  $\text{Claim}(n)$  is true. Assume we have carried out Steps 1 and 2 of a proof by induction. Then  $\text{Claim}(1)$  is true, by Step 1, and so 1 is in  $S$ . Step 2 says precisely that, if  $k$  is in  $S$ , then  $k + 1$  is also in  $S$ , for each natural number  $k$ . Thus, by Peano's induction postulate, we have  $S = \mathbb{N}$ . That is,  $\text{Claim}(n)$  is true, for every natural number  $n$ .

### A little history

Informal induction-type arguments have been used as far back as the 10th century. The Persian mathematician al-Karaji (953–1029) essentially gave an induction-type proof of the formula for the sum of the first  $n$  cubes:  $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ .

The term *mathematical induction* was introduced and the process was put on a rigorous basis by the British mathematician Augustus De Morgan in 1838. For more details, see <http://www-history.mcs.st-andrews.ac.uk>.



Augustus De Morgan (1806–1871).

## A typical proof by induction

### Theorem

For every natural number  $n$ , we have  $2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$ .

### Proof

For each natural number  $n$ , let  $\text{Claim}(n)$  be the sentence:

$$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (n+1) \cdot 2^n = n \cdot 2^{n+1}.$$

**Step 1.**  $\text{Claim}(1)$  is the statement  $2 \cdot 2 = 1 \cdot 2^{1+1}$ , i.e.,  $4 = 4$ . Hence,  $\text{Claim}(1)$  is true.

**Step 2.** Let  $k$  be a natural number, and assume that  $\text{Claim}(k)$  is true, i.e., assume that

$$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (k+1) \cdot 2^k = k \cdot 2^{k+1}. \quad \text{Claim}(k)$$

We wish to use this to prove  $\text{Claim}(k+1)$ , i.e.,

$$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + ((k+1)+1) \cdot 2^{k+1} = (k+1) \cdot 2^{(k+1)+1}, \quad \text{i.e.,}$$

$$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (k+2) \cdot 2^{k+1} = (k+1) \cdot 2^{k+2}. \quad \text{Claim}(k+1)$$

When proving this, it will help to write out the last *and* the second-last term of the summation, so that we can see how to use our assumption,  $\text{Claim}(k)$ .

We have

$$\begin{aligned} \text{LHS of Claim}(k+1) &= \underbrace{2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (k+1) \cdot 2^k}_{= k \cdot 2^{k+1}} + (k+2) \cdot 2^{k+1} \\ &= \underbrace{k \cdot 2^{k+1}}_{\text{by Claim}(k)} + (k+2) \cdot 2^{k+1} \\ &= (k + (k+2)) \cdot 2^{k+1} \\ &= (2k+2) \cdot 2^{k+1} \\ &= (k+1) \cdot 2 \cdot 2^{k+1} \\ &= (k+1) \cdot 2^{k+2} \\ &= \text{RHS of Claim}(k+1). \end{aligned}$$

Thus,  $\text{Claim}(k+1)$  is true. We have proved that  $\text{Claim}(k) \Rightarrow \text{Claim}(k+1)$ , for every natural number  $k$ . By the principle of mathematical induction, it follows that  $\text{Claim}(n)$  is true, for all natural numbers  $n$ , i.e.,

$$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + (n+1) \cdot 2^n = n \cdot 2^{n+1},$$

for every natural number  $n$ . □

## Two non-proofs by induction

Where do the following two proofs go wrong?

### The colour of rabbits

#### “Theorem”

*All rabbits are the same colour.*

#### “Proof”

For each natural number  $n$ , let  $\text{Claim}(n)$  be the sentence:

*‘If  $X$  is a set consisting of  $n$  rabbits, then all rabbits in  $X$  are the same colour.’*

If we can prove  $\text{Claim}(n)$ , for every natural number  $n$ , then it will follow that all rabbits are the same colour.

**Step 1.**  $\text{Claim}(1)$  is ‘If  $X$  is a set consisting of one rabbit, then all rabbits in  $X$  are the same colour.’ This is clearly true.

**Step 2.** Let  $k$  be a natural number, and assume that  $\text{Claim}(k)$  is true, i.e., assume that ‘if  $X$  is a set consisting of  $k$  rabbits, then all rabbits in  $X$  are the same colour.’

We wish to use this to prove  $\text{Claim}(k+1)$ , i.e., ‘If  $X$  is a set consisting of  $k+1$  rabbits, then all rabbits in  $X$  are the same colour.’

Let  $X = \{r_1, r_2, \dots, r_{k+1}\}$  be a set consisting of  $k+1$  rabbits. Make a smaller set

$$A = \{r_1, r_2, \dots, r_k\}$$

by removing the last rabbit from the set  $X$ . The set  $A$  consists of  $k$  rabbits. By  $\text{Claim}(k)$ , which we have assumed to be true, all rabbits in the set  $A$  are the same colour.

Now put the last rabbit back into the set, and make another new set  $B$  by removing the first rabbit from the set  $X$ ; thus

$$B = \{r_2, r_3, \dots, r_k, r_{k+1}\}.$$

The set  $B$  consists of  $k$  rabbits. By  $\text{Claim}(k)$ , all rabbits in the set  $B$  are the same colour. But the ‘middle’ rabbits  $r_2, r_3, \dots, r_k$  (i.e., all but the first and the last) belong to both of the sets  $A$  and  $B$ , and so they have the same colour as rabbit  $r_1$  and rabbit  $r_{k+1}$ . So all the rabbits in the set  $X$  have the same colour! Thus,  $\text{Claim}(k+1)$  is true.

By the principle of mathematical induction, it follows that Claim( $n$ ) is true, for all natural numbers  $n$ , i.e., ‘If  $X$  is a set consisting of  $n$  rabbits, then all rabbits in  $X$  are the same colour.’  $\square$

This proof breaks down because the general argument in Step 2 should work for every value of the natural number  $k$ . Try using the general argument with  $k = 1$ , i.e., try to use the proof given above to show that, if all of the rabbits in a set consisting of one rabbit are the same colour, then all the rabbits in a set consisting of two rabbits are the same colour. (The problem is that the set  $\{r_2, r_3, \dots, r_k\}$  of ‘middle’ rabbits will be empty!)

### A formula that gives prime numbers

#### “Theorem”

For every natural number  $n$ , the number  $n^2 - n + 41$  is prime.

#### “Proof”

Let’s check these claims in turn:

- 1 Claim(1) is true, as  $1^2 - 1 + 41 = 41$  is a prime number.
- 2 Claim(2) is true, as  $2^2 - 2 + 41 = 43$  is a prime number.
- 3 Claim(3) is true, as  $3^2 - 3 + 41 = 47$  is a prime number.
- 4 Claim(4) is true, as  $4^2 - 4 + 41 = 53$  is a prime number.
- 5 Claim(5) is true, as  $5^2 - 5 + 41 = 61$  is a prime number.

⋮

Continuing in this way, we can see that the number  $n^2 - n + 41$  is prime, for every natural number  $n$ .  $\square$

A good exercise here is:

- 1 Use a spreadsheet to calculate  $n^2 - n + 41$ , for  $n = 1, 2, \dots, 100$ .
- 2 Find a list of the first 1000 prime numbers. (Google finds it immediately.)
- 3 Starting at  $n = 1$ , can you find a number  $n^2 - n + 41$  that is not prime?
- 4 Can you see from the formula  $n^2 - n + 41$  why this number is not prime?

(In fact, you can check whether a number is prime in Excel, using formulas found easily on the web, but I think it is a better exercise to do this by hand, as suggested above.)

Rather surprisingly, the number  $n^2 - n + 41$  is prime for  $n = 1, 2, \dots, 40$ . But Claim(41) is *false*, since  $41^2 - 41 + 41 = 41^2 = 1681$ , which is not prime, as it is obviously divisible by 41.

The lesson to take away is that *checking the first few claims (even the first 40 claims) never constitutes a proof.*

## Modified or strong induction

In **modified** (or **strong**) **induction**, we first establish Claim(1). We then assume that all the claims from Claim(1) up to Claim( $k$ ) are true, and use them to prove Claim( $k + 1$ ).

*Note.* Anything that can be proved by modified induction can also be proved by induction. You just need to have a smarter Claim( $n$ ).

### Interesting natural numbers

The following proof is one of my favourites. It usually leads to an interesting discussion of the meaning of the word *interesting* and of what was actually proved. It uses the modified principle of mathematical induction, and also requires a proof by contradiction.

#### Theorem

*All natural numbers are interesting.*

#### Proof

For each natural number  $n$ , let Claim( $n$ ) be the sentence:

*‘The number  $n$  is interesting.’*

**Step 1.** Claim(1) is *‘The number 1 is interesting.’* This is clearly true.

**Step 2.** Let  $k$  be a natural number, and assume that Claim(1), Claim(2),  $\dots$ , Claim( $k$ ) are true, i.e., assume that the numbers  $1, 2, \dots, k$  are interesting.

We wish to use this to prove Claim( $k + 1$ ), i.e., that the number  $k + 1$  is interesting.

Suppose, by way of contradiction, that the number  $k + 1$  is not interesting. By assumption, the numbers  $1, 2, \dots, k$  are interesting, so it follows that  $k + 1$  is the *first uninteresting natural number*, and that makes it very interesting, which is a contradiction!

Since our assumption that  $k + 1$  was uninteresting led to a contradiction, this assumption is false, and hence  $k + 1$  is interesting. Thus, Claim( $k + 1$ ) is true.

By the (modified) principle of mathematical induction, it follows that Claim( $n$ ) is true, for every natural number  $n$ , i.e., *every natural number is interesting.* □

What have we actually proved here? A reasonable interpretation is that, if we believe that 1 is an interesting natural number, then every natural number is interesting. The proof comes down to two statements of belief about what is, or is not, interesting:

- the natural number 1 is interesting
- being the smallest uninteresting natural number would make a number interesting.

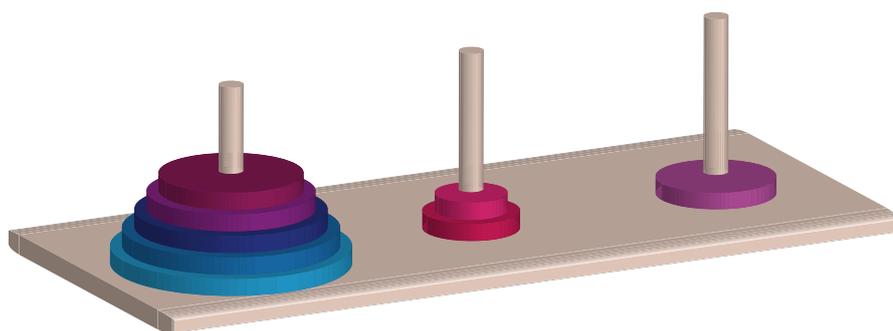
If you reject either of these claims, then the proof doesn't work. If you accept them, then you have to accept that all natural numbers are interesting.

## The Tower of Hanoi puzzle

The **Tower of Hanoi** is a famous puzzle, sometimes known as the *End of the World Puzzle*. Legend tells that, at the beginning of time, there was a Hindu temple containing three poles. On one of the poles was a stack of 64 gold discs, each one a little smaller than the one beneath it. The monks of the temple were assigned the task of transferring the stack of 64 discs to another pole, by moving discs one at a time between the poles, with one important proviso: a large disc could never be placed on top of a smaller one. When the monks completed their assigned task, the world would end!

How many individual moves are required for the monks to complete their task?

The modern version of the puzzle, usually with around eight discs, is made from wood. The picture below shows the puzzle with eight discs after four moves have been made. (You'll find a nice Java version of the puzzle at <http://www.mazeworks.com/hanoi/>.)



The Tower of Hanoi with eight discs.

For each  $n \in \mathbb{N}$ , let  $T_n$  be the minimum number of moves needed to solve the puzzle with  $n$  discs. We shall prove by induction that  $T_n = 2^n - 1$ , for all natural numbers  $n$ .

**Problem 1: A formula for  $T_n$** 

Guess a formula for  $T_n$  in terms of  $n$ . You should do this by first solving the puzzle with 1 disc, then 2 discs, then 3 discs, then 4 discs; each time recording the minimum number of moves required. (You can do this, for example, by using the Java version of the puzzle mentioned above.) Then you should guess the general relationship between  $T_n$  and  $n$ .

**Discussion**

When we solve the puzzle with 1, 2, 3 and 4 discs, we obtain the first and second columns of the following table.

$n$	$T_n$	$2^n$
1	1	2
2	3	4
3	7	8
4	15	16

After students have correctly calculated  $T_1, \dots, T_4$ , as in the table, they should be encouraged to guess the general formula. If they cannot do this, then the third column of the table should be added, and they should be asked to guess again. After comparing  $T_n$  with  $2^n$ , it would appear that a reasonable guess is that  $T_n = 2^n - 1$ .

**Problem 2: A formula for  $T_{k+1}$  in terms of  $T_k$** 

Let  $k$  be a natural number. By considering the way you go about solving the puzzle with  $k + 1$  discs, find a relationship between  $T_{k+1}$  and  $T_k$ .

**Discussion**

When we solve the puzzle with  $k + 1$  discs, we first transfer the top  $k$  discs to a new peg; this takes a minimum of  $T_k$  moves. We then move the largest disc to the free peg; this takes one move. We then transfer the top  $k$  discs onto the largest disc; this again takes a minimum of  $T_k$  moves. Thus the minimum number of moves required to shift the stack of  $k + 1$  discs equals  $T_k + 1 + T_k$ . Hence, we have

$$T_{k+1} = 2T_k + 1. \quad (*)$$

It is important to be convinced that this argument shows not only that the puzzle with  $k + 1$  discs can be solved in  $2T_k + 1$  moves, but also that the *minimum* number of moves required to solve the puzzle with  $k + 1$  discs is given by  $2T_k + 1$ .

Some students may try to argue as follows: We first move the smallest disc to a new peg; this takes one move. We then transfer the remaining  $k$  discs to the free peg (without moving the smallest disc); this takes a minimum of  $T_k$  moves. We then move the smallest disc to the top of the pile; this takes one move. Thus the minimum number of moves required to shift the stack of  $k + 1$  discs equals  $1 + T_k + 1$ . Hence, we have  $T_{k+1} = T_k + 2$ .

Even if no student suggests this strategy, it should be introduced. Then there should be a discussion as to why it doesn't work. (If you try to apply this strategy with three discs, you will quickly see that it fails, as you will be forced to move the smallest disc more than once.)

### Problem 3: Proving the formula for $T_n$

Use the equation

$$T_{k+1} = 2T_k + 1 \quad (*)$$

to prove by induction that  $T_n = 2^n - 1$ , for every natural number  $n$ .

#### Discussion

For each natural number  $n$ , let Claim( $n$ ) be the sentence:

$$T_n = 2^n - 1.$$

**Step 1.** It is clear that the minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1, i.e.,  $T_1 = 1$ . Since  $2^1 - 1 = 1$ , it follows that Claim(1) is true, i.e.,  $T_1 = 2^1 - 1$ .

**Step 2.** Let  $k$  be a natural number, and assume that Claim( $k$ ) is true, i.e., assume that  $T_k = 2^k - 1$ .

We wish to use this to prove Claim( $k + 1$ ), i.e., that  $T_{k+1} = 2^{k+1} - 1$ .

The remaining details are left as an easy exercise: Substitute  $T_k = 2^k - 1$ , which we have assumed to be true, into equation (\*), which we have proved to be true. Hence, show that  $T_{k+1} = 2^{k+1} - 1$ .

By the principle of mathematical induction, it follows that the minimum number of moves required to solve the Tower of Hanoi puzzle with  $n$  discs is  $2^n - 1$ .

*Note.* The discussion for Problem 2 above gives a recursive strategy for solving the Tower of Hanoi puzzle with  $n$  discs. But there is an easier way to remember this strategy, so as to be able to solve the puzzle very quickly:

- On the first move, and also on every odd-numbered move, shift the smallest disc one peg to the right. (Imagine the three pegs as being on a ‘circle’.)
- On every even-numbered move, shift one of the other discs (not the smallest). There will only ever be one possible move to make.

Can you give a proof by induction that this works?

### A little history

It is not uncommon for people to believe that the Tower of Hanoi puzzle is of ancient Chinese or Hindu origin. In fact, it was invented in 1883 by the French mathematician Edouard Lucas (1842–1891).



Edouard Lucas (1842–1891).

Lucas created the legend of the Hindu temple and the monks with their 64 gold discs. It is interesting to note that the number of moves of single discs that the monks must make in order to transfer the tower is

$$T_{64} = 2^{64} - 1 = 18\,446\,744\,073\,709\,551\,615.$$

If the monks worked day and night, making one move every second, it would take them slightly more than 585 billion years to accomplish their task! The world is probably safe.

Apart from inventing this puzzle, Lucas is also famous for giving the formula for the  $n$ th Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right). \quad (\dagger)$$

He also proved that  $2^{127} - 1$  is prime; this is the largest prime number ever discovered without the aid of a computer.

Lucas died aged 49 as the result of a freak accident at a banquet when a plate was dropped and a piece flew up and cut his cheek. The cut became infected and he died a few days later.

### Exercise 1

The Fibonacci numbers  $1, 1, 2, 3, 5, 8, 13, \dots$  are defined by

$$F_1 = 1, \quad F_2 = 1 \quad \text{and} \quad F_{k+1} = F_{k-1} + F_k, \quad \text{for } k \geq 2.$$

Use modified induction to prove that the  $n$ th Fibonacci number  $F_n$  is given by Lucas's formula (†). (In Step 1, you need to check that both Claim(1) and Claim(2) are true. Then in Step 2, you need to assume that  $k \geq 2$ .)

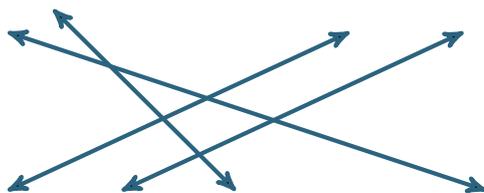
## A two-colour theorem

There are some nice colouring theorems that can be proved by induction. Here is one.

### Straight lines in the plane

#### Theorem

*Consider any finite collection of straight lines in the plane. Then the regions in the plane created by these lines can be coloured with two colours in such a way that any two regions sharing a common border have different colours.*



#### Proof

We will argue by induction on the number of lines. For each natural number  $n$ , let Claim( $n$ ) be:

*'The regions created in the plane by  $n$  straight lines can be coloured with two colours.'*

**Step 1.** It is clear that the regions created by a single straight line can be coloured with two colours, as there are only two regions.

**Step 2.** Let  $k$  be a natural number, and assume that  $\text{Claim}(k)$  is true, i.e., assume that the regions created in the plane by  $k$  straight lines can be coloured with two colours.

We wish to use this to prove  $\text{Claim}(k + 1)$ , i.e., that the regions created in the plane by  $k + 1$  straight lines can be coloured with two colours.

Assume that we have  $k + 1$  distinct straight lines in the plane. Choose any one of them, say  $\ell$ , and remove it. We are left with  $k$  straight lines, and so the resulting regions can be coloured with two colours, as we have assumed that  $\text{Claim}(k)$  is true. Hence, there is a two-colouring of this set of regions.

Take such a colouring of the regions, say with black and white, and add the line  $\ell$  back. Change the colour of each region on one side of the line  $\ell$  (so black becomes white and white becomes black) and leave the regions on the other side of  $\ell$  their original colour.

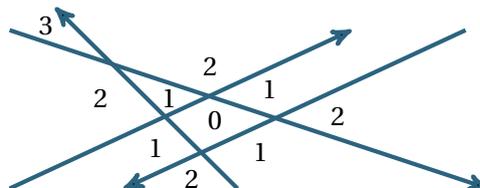
We leave it as an exercise to confirm that this is a proper colouring of the regions created by the original  $k + 1$  lines. (Consider two regions that share a border. There are two cases: either the border is part of the line  $\ell$  or it is not. In each case, convince yourself that the two regions are different colours.)

By the principle of mathematical induction, it follows that, for every natural number  $n$ , the regions created in the plane by  $n$  straight lines can be coloured with two colours.  $\square$

In fact, it is also possible to give a neat non-inductive proof of this colouring theorem, as follows.

### Non-inductive proof

Assume that we have a finite number of straight lines in the plane. Fix a direction on each line in any way at all. (An example is given by the arrowheads in the following diagram.) Now it makes sense to talk about the left side and the right side of each line. For each region  $R$ , count the number  $n_R$  of lines  $\ell$  such that the region  $R$  is on the left side of  $\ell$ . (The number  $n_R$  is given in each region  $R$  in the following diagram.)



Now colour a region  $R$  white if  $n_R$  is even, and black if  $n_R$  is odd.

Assume that  $R$  and  $S$  are regions that share a border that is part of the line  $\ell$ . We must prove that  $R$  and  $S$  have different colours. Since  $R$  and  $S$  share a border that is part of  $\ell$ , one of the regions is to the left of  $\ell$  and the other is to the right of  $\ell$ . Moreover, the regions  $R$  and  $S$  are on the same side of every other line. Thus, the numbers  $n_R$  and  $n_S$  must differ by 1, and so one of the numbers is even and the other is odd. Consequently, the regions  $R$  and  $S$  have different colours, as required.  $\square$

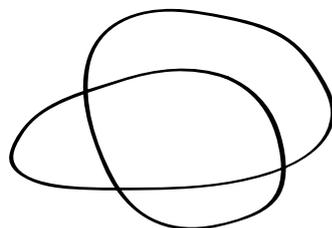
## A loop in the plane

Here is another two-colour theorem that can be proved by induction.

### Theorem

*Consider a continuous loop in the plane that may intersect itself a finite number of times. Then the regions in the plane created by this continuous loop, including the unbounded region outside the loop, can be coloured with two colours in such a way that any two regions sharing a common border have different colours.*

An example of such a loop is given below.



There is a subtle but deep assumption in the statement of this theorem, namely, that a continuous loop in the plane that only intersects itself a finite number of times divides the plane into a finite number of regions. While this might seem intuitively obvious, it requires proof, even in the case of a simple closed curve, i.e., where the loop does not intersect itself. In this case, there are just two regions — the *interior* and the *exterior* of the loop. The first proof that a simple closed curve divides the plane into two regions was published by Camille Jordan in his 1887 text *Cours d'analyse de l'École Polytechnique*. This result is now known as the **Jordan curve theorem**.

One proof of this two-colour theorem proceeds by induction on the number of crossings. The induction begins at 0 rather than 1. For each  $n \geq 0$ , let Claim( $n$ ) be:

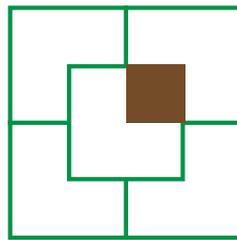
*'The regions created in the plane by a closed curve with  $n$  intersections can be coloured with two colours.'*

Step 1 is now to prove Claim(0), i.e., ‘The regions created in the plane by a closed curve with no intersections can be coloured with two colours.’ But this is precisely the Jordan curve theorem! A fully detailed proof of Step 2 is beyond these notes. Hand-waving versions can be found on the web, but they rely on our intuitive understanding of curves in the plane, and it requires quite a bit of careful analysis to make them completely rigorous.

## If you can't prove it, prove something harder!

### Tiling a courtyard

We wish to tile a  $2^n \times 2^n$  courtyard using L-shaped tiles: each tile is a  $2 \times 2$  square with a  $1 \times 1$  square removed from one corner. We wish to tile the courtyard in such a way that there is a single  $1 \times 1$  hole left in the ‘middle’ of the courtyard. (By the ‘middle’ we mean that one corner of the hole should be in the centre of the courtyard.) Here is an example of such a tiling in the case  $n = 2$ , i.e., for a  $4 \times 4$  courtyard.



A tiling of a  $4 \times 4$  courtyard.

Naturally, we try to prove that this is always possible by induction on  $n$ .

### A proof that goes wrong

Here is an attempt at a proof by induction.

Let Claim( $n$ ) be:

*‘It is possible to tile a  $2^n \times 2^n$  courtyard using L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left in the “middle” of the courtyard.’*

**Step 1.** It is clear that Claim(1) is true, as a  $2 \times 2$  courtyard can be tiled with a single L-shaped tile leaving a hole with one corner in the centre of the courtyard.

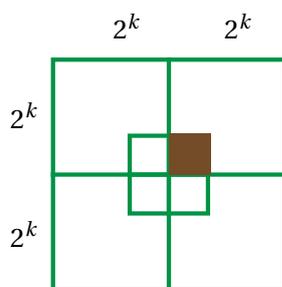


A tiling of a  $2 \times 2$  courtyard.

**Step 2.** Let  $k$  be a natural number, and assume that  $\text{Claim}(k)$  is true, i.e., assume that a  $2^k \times 2^k$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left in the ‘middle’ of the courtyard.

We wish to use this to prove  $\text{Claim}(k+1)$ , i.e., that a  $2^{k+1} \times 2^{k+1}$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left in the ‘middle’ of the courtyard.

Given a  $2^{k+1} \times 2^{k+1}$  courtyard, we need to break it up so that we can apply the inductive hypothesis. We can divide the courtyard up into four  $2^k \times 2^k$  areas. If we could guarantee that the holes were in the corners, we could arrange them as in the following diagram. Then inserting an extra L-shaped tile in the middle of the courtyard would produce the desired tiling.



Breaking up a  $2^{k+1} \times 2^{k+1}$  courtyard.

Unfortunately,  $\text{Claim}(k)$  only tells us that we can tile the four  $2^k \times 2^k$  areas with the hole in the ‘middle’, not in the corners. So this proof fails!

### A proof that goes right

The solution to this impasse is rather surprising. We shall prove something stronger than the result we were originally trying to prove, namely, we will prove that a  $2^n \times 2^n$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left *anywhere we like*.

Our original claim will certainly follow from this much stronger one. The reason this strategy can work is that, although we are trying to prove a stronger result, when we are establishing  $\text{Claim}(k+1)$  for the inductive step, the stronger assumption  $\text{Claim}(k)$  will give us more to work with.

### Theorem

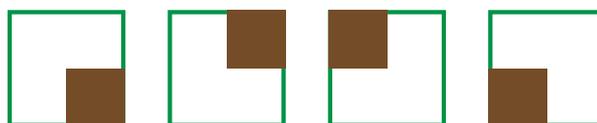
For each natural number  $n$ , a  $2^n \times 2^n$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left anywhere we like in the courtyard.

### Proof

For each natural number  $n$ , let Claim( $n$ ) be:

*'A  $2^n \times 2^n$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left anywhere we like in the courtyard.'*

**Step 1.** It is clear that Claim(1) is true, as there are just four places to have a  $1 \times 1$  hole in a  $2 \times 2$  courtyard, and each of these tilings can be achieved with a single tile.

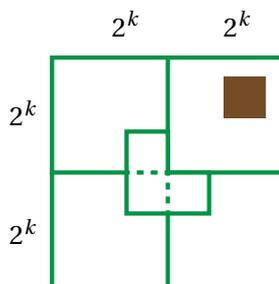


The four different tilings of a  $2 \times 2$  courtyard.

**Step 2.** Let  $k$  be a natural number, and assume that Claim( $k$ ) is true, i.e., assume that a  $2^k \times 2^k$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left anywhere we like in the courtyard.

We wish to use this to prove Claim( $k + 1$ ), i.e., that a  $2^{k+1} \times 2^{k+1}$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left anywhere we like in the courtyard.

Given a  $2^{k+1} \times 2^{k+1}$  courtyard with a specified  $1 \times 1$  hole to be left empty, we can divide the courtyard up into four  $2^k \times 2^k$  areas, and the hole will lie in one of the four resulting quadrants: it is in the first quadrant in our example below.



Breaking up a  $2^{k+1} \times 2^{k+1}$  courtyard.

By Claim( $k$ ), we can tile each of the four quadrants as shown in the diagram: in the first quadrant the hole is left exactly where we require it, while in the other three quadrants the hole is left in the corner closest to the centre of the  $2^{k+1} \times 2^{k+1}$  courtyard. Now inserting an extra L-shaped tile in the middle of the courtyard will produce the desired tiling.

By the principle of mathematical induction, it follows that, for every natural number  $n$ , a  $2^n \times 2^n$  courtyard can be tiled with L-shaped tiles in such a way that there is a single  $1 \times 1$  hole left anywhere we like in the courtyard.  $\square$

## Exercise 2

The proof of this theorem actually provides a recursive procedure for finding the tiling that we originally sought, with a hole in the 'middle' of the courtyard. Use this procedure to find such a tiling of a  $4 \times 4$ , an  $8 \times 8$  and a  $16 \times 16$  courtyard.

Since the total number of squares in a  $2^n \times 2^n$  courtyard is  $2^{2n}$ , and since the L-shaped tile consists of three such squares, it follows from the theorem that  $2^{2n} - 1$  is a multiple of 3. It is a good exercise to prove this directly.

## Exercise 3

Prove by induction that, for each natural number  $n$ , the number  $2^{2n} - 1$  is a multiple of 3 (i.e., there exists an integer  $m$  such that  $2^{2n} - 1 = 3m$ ).

# Answers to exercises

## Exercise 1

We can give a neater proof by noticing that  $a = \frac{1}{2}(1 + \sqrt{5})$  is the golden ratio, which satisfies the equation  $a^2 - a - 1 = 0$ . The number  $b = \frac{1}{2}(1 - \sqrt{5})$  is the other solution to this equation, that is,  $b^2 - b - 1 = 0$ .

Now, for each natural number  $n$ , let Claim( $n$ ) be the statement:

$$F_n = \frac{1}{\sqrt{5}}(a^n - b^n).$$

**Step 1.** We must prove Claim(1) and Claim(2). To check Claim(1), we calculate

$$\frac{1}{\sqrt{5}}(a - b) = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1.$$

So  $F_1 = 1 = \frac{1}{\sqrt{5}}(a - b)$ , and Claim(1) holds.

To check Claim(2), we calculate

$$\begin{aligned} \frac{1}{\sqrt{5}}(a^2 - b^2) &= \frac{1}{\sqrt{5}}((a+1) - (b+1)) && \text{as } a^2 - a - 1 = 0 \text{ and } b^2 - b - 1 = 0 \\ &= \frac{1}{\sqrt{5}}(a - b) \\ &= 1 && \text{as Claim(1) holds.} \end{aligned}$$

So  $F_2 = 1 = \frac{1}{\sqrt{5}}(a^2 - b^2)$ , and therefore Claim(2) holds.

**Step 2.** Let  $k \geq 2$  and assume that both Claim( $k-1$ ) and Claim( $k$ ) hold. So we have  $F_{k-1} = \frac{1}{\sqrt{5}}(a^{k-1} - b^{k-1})$  and  $F_k = \frac{1}{\sqrt{5}}(a^k - b^k)$ . Thus

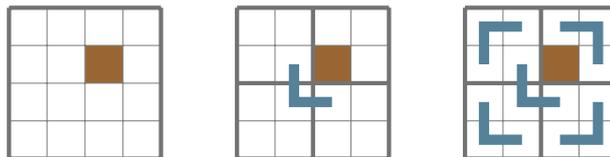
$$\begin{aligned} F_{k+1} &= F_{k-1} + F_k && \text{by the definition of } F_{k+1} \\ &= \frac{1}{\sqrt{5}}(a^{k-1} - b^{k-1}) + \frac{1}{\sqrt{5}}(a^k - b^k) && \text{by Claim}(k-1) \text{ and Claim}(k) \\ &= \frac{1}{\sqrt{5}}(a^{k-1} + a^k - b^{k-1} - b^k) \\ &= \frac{1}{\sqrt{5}}(a^{k-1}(1+a) - b^{k-1}(1+b)) \\ &= \frac{1}{\sqrt{5}}(a^{k-1}a^2 - b^{k-1}b^2) && \text{as } a^2 - a - 1 = 0 \text{ and } b^2 - b - 1 = 0 \\ &= \frac{1}{\sqrt{5}}(a^{k+1} - b^{k+1}). \end{aligned}$$

Hence, Claim( $k+1$ ) holds.

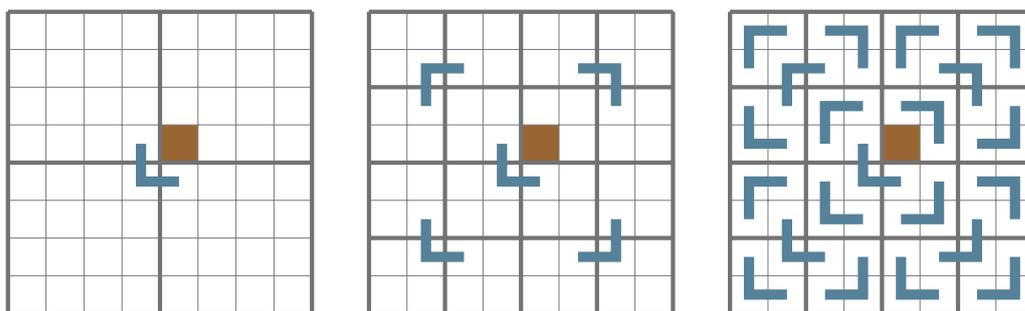
It follows by the (modified) principle of mathematical induction that  $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ , for every natural number  $n$ .

## Exercise 2

Tiling a  $4 \times 4$  courtyard:



Tiling an  $8 \times 8$  courtyard:



The  $16 \times 16$  courtyard is left to you.

### Exercise 3

For each natural number  $n$ , let Claim( $n$ ) be: 'The number  $2^{2^n} - 1$  is a multiple of 3.'

**Step 1.** We have  $2^{2 \times 1} - 1 = 4 - 1 = 3$ , which is a multiple of 3. So Claim(1) holds.

**Step 2.** Let  $k$  be a natural number, and assume that Claim( $k$ ) is true, i.e., assume that the number  $2^{2^k} - 1$  is a multiple of 3. So there is an integer  $m$  such that  $2^{2^k} - 1 = 3m$ .

We want to show that it follows that Claim( $k + 1$ ) is true, i.e., that  $2^{2^{(k+1)}} - 1$  is a multiple of 3. We need to show that there is an integer  $q$  such that  $2^{2^{(k+1)}} - 1 = 3q$ .

We have

$$\begin{aligned}
 2^{2^{(k+1)}} - 1 &= 2^{2^{k+2}} - 1 \\
 &= 4 \cdot 2^{2^k} - 1 \\
 &= 4(3m + 1) - 1 \quad \text{since } 2^{2^k} - 1 = 3m \\
 &= 12m + 3 \\
 &= 3(4m + 1).
 \end{aligned}$$

Since  $m \in \mathbb{Z}$ , we also have  $4m + 1 \in \mathbb{Z}$ . So we have shown that  $2^{2^{(k+1)}} - 1 = 3q$ , where  $q = 4m + 1 \in \mathbb{Z}$ . Thus  $2^{2^{(k+1)}} - 1$  is a multiple of 3, and therefore Claim( $k + 1$ ) holds.

It follows by the principle of mathematical induction that  $2^{2^n} - 1$  is a multiple of 3, for every natural number  $n$ .

0 1 2 3 4 5 6 7 8 9 10 11 12