Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Functions: Module 5

Functions I

Education Services Australia

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Functions I – A guide for teachers (Years 11–12)

Principal author: Associate Professor David Hunt, University of NSW

Peter Brown, University of NSW
Dr Michael Evans, AMSI
Associate Professor David Hunt, University of NSW
Dr Daniel Mathews, Monash University

Editor: Dr Jane Pitkethly, La Trobe University

Illustrations and web design: Catherine Tan, Michael Shaw

Full bibliographic details are available from Education Services Australia.

Published by Education Services Australia
PO Box 177
Carlton South Vic 3053
Australia

Tel: (03) 9207 9600
Fax: (03) 9910 9800
Email: info@esa.edu.au
Website: www.esa.edu.au

This publication is funded by the Australian Government Department of Education, Employment and Workplace Relations.

Supporting Australian Mathematics Project

Australian Mathematical Sciences Institute
Building 161
The University of Melbourne
VIC 3010
Email: enquiries@amsi.org.au
Website: www.amsi.org.au

Functions I

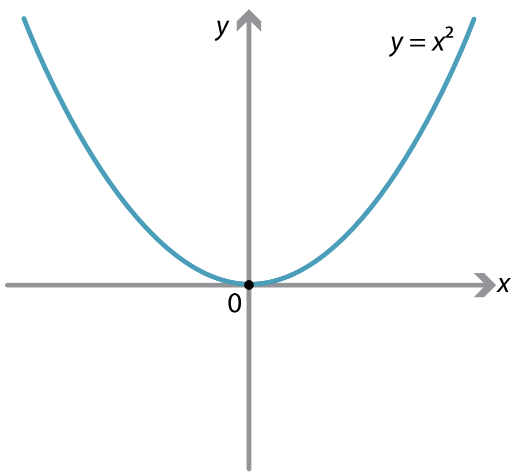
Assumed knowledge

- Familiarity with elementary set theory, as discussed in the TIMES module *Sets and Venn diagrams* (Years 7–8).
- Familiarity with the algebraic techniques described in the TIMES module *Formulas* (Years 9–10).
- The content of the modules *Algebra review* and *Coordinate geometry*.
- Basic knowledge of the modules *Trigonometric functions and circular measure* and *Exponential and logarithmic functions*.

Motivation

The expression $y = x^2$, which links the two pronumerals x and y, can be thought of in several different ways.

In the early years of school, we seek pairs of values, such as $(x, y) = (3, 9)$, which **satisfy** the equation. These ordered pairs determine a **graph**. In this case, the graph is the standard parabola.



A **formula** is an equation relating different quantities using algebra. So $y = x^2$ is also a formula. In fact, $y = x^2$ is an example of a **function**, in the sense that each value of x uniquely determines a value of y.

In this module, we will study the concept of a function. The formula $A = \pi r^2$ gives A as a function of r. The formula $V = \pi r^2 h$ expresses V as a function of the two variables r and h. In this module, we will only consider functions of one variable, such as the polynomial

$$y = (x-1)(x-2)(x-3)(x-4).$$

A clear understanding of the concept of a function and a familiarity with function notation are important for the study of calculus. The use of functions and function notation in calculus can be seen in the module *Introduction to differential calculus*.

Content

Set theory

It is not possible to discuss functions sensibly without using the language and ideas of elementary set theory. In particular, we will use the following ideas from the module *Sets and Venn diagrams* (Years 7–8):

- sets and their elements
- equal sets
- listing the members of a set; for example, $A = \{2, 4, 6, 8\}$
- set membership; for example, $4 \in A$ and $5 \notin A$
- finite and infinite sets, and the number of elements in a finite set
- subsets
- unions and intersections.

Sets must be well defined, and if sets are defined using mathematical notation there are rarely any problems. A set is **well defined** if it is possible to determine whether or not a given object belongs to that set or not. The set of all words in the English language is not well defined.

Exercise 1

Think of several reasons why 'the set of all English words' is not well defined.

Further notation

Set-builder notation is useful for describing subsets of the real numbers. For example, consider the set defined by

$$S = \{x \in \mathbb{R} \mid -5 \le x \le 5\}$$

or, equivalently,

$$S = \{x \in \mathbb{R} : -5 \le x \le 5\}.$$

This can be read as 'S is the set of all x belonging to $\mathbb{R}$ such that $-5 \le x \le 5$'. In this construction:

- S is the name of the set
- $\{\quad\}$ holds the definition together
- $x \in \mathbb{R}$ says that x is a real number
- both $\mid$ and $:$ mean 'with the property that'
- $-5 \le x \le 5$ limits the allowed values of x.

So S is the following interval:

If we define the set $T = \{x \in \mathbb{R} : |x| \le 5\}$, then $S = T$.

Union and intersection are familiar operations on sets. Another useful operation on sets is **set difference**. For two sets A and B, we define

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

That is, $A \setminus B$ is the set of all elements of A that are not in B. We can read $A \setminus B$ as 'A take away B' or 'A minus B'. For example, the set of all non-zero real numbers can be written as $\{x \in \mathbb{R} \mid x \neq 0\}$ or, more simply, as $\mathbb{R} \setminus \{0\}$.

The **Cartesian product** of two sets A and B is defined by

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

That is, $A \times B$ is the set of all ordered pairs (a, b) with a in A and b in B. So, for example,

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

We write $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$, which explains the use of the notation $\mathbb{R}^2$ to describe the coordinate plane. This was introduced in the module *Coordinate geometry*.

Note. In this module and many others:

- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{Q}$ denotes the set of rational numbers
- $\mathbb{Z}$ denotes the set of integers
- $\mathbb{C}$ denotes the set of complex numbers.

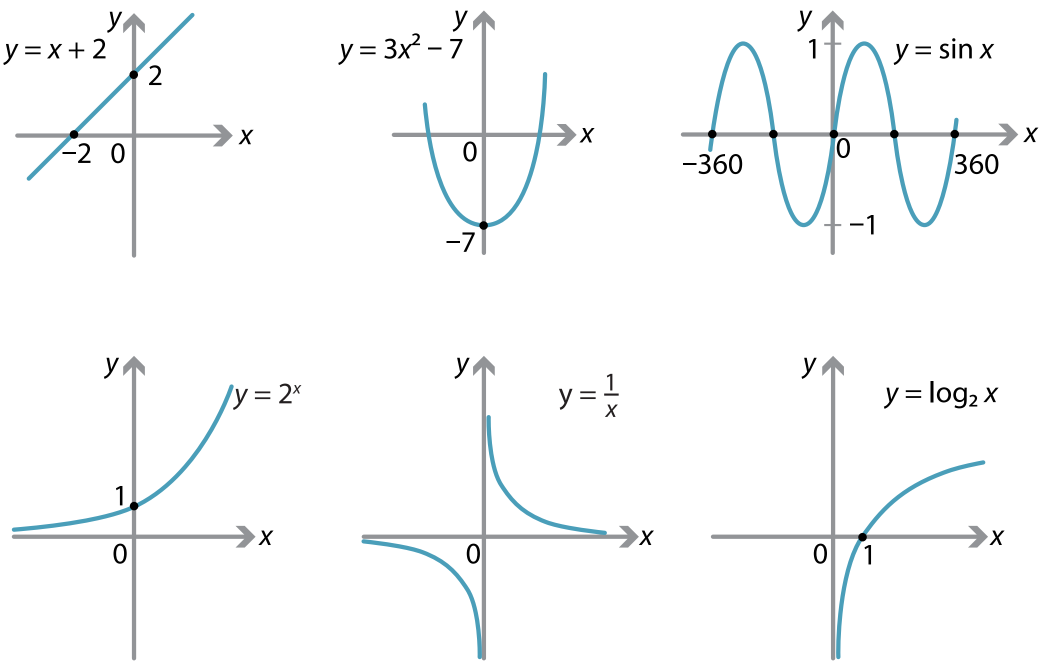
The concept of a function

When a quantity y is *uniquely determined* by some other quantity x as a result of some rule or formula, then we say that y **is a function of** x. (In other words, for each value of x, there is *at most one* corresponding value of y.)

We begin with six examples in which both x and y are real numbers:

- $y = x + 2$
- $y = 3x^2 - 7$
- $y = \sin x$
- $y = 2^x$
- $y = \dfrac{1}{x}$
- $y = \log_2 x$.

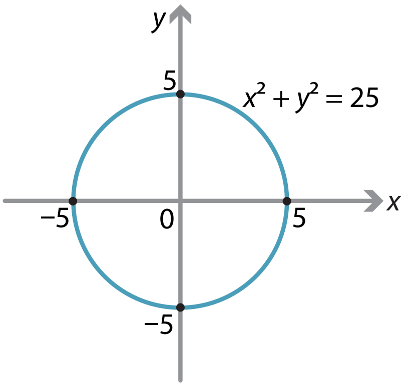
We draw their graphs in the usual way, with the x-axis horizontal and the y-axis vertical.

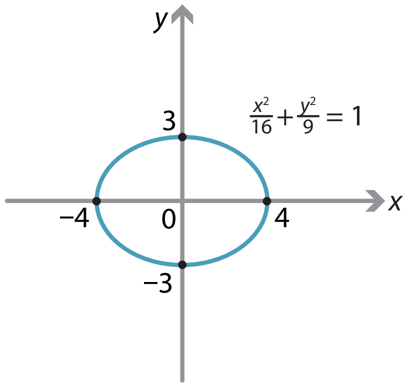


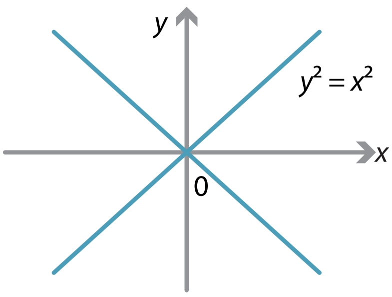
The first four functions are similar in that their formulas 'work' for all real numbers x. For $y = \dfrac{1}{x}$, we clearly need $x \neq 0$, and for $y = \log_2 x$, we need $x > 0$. We will discuss this further in the section *Domains and ranges*.

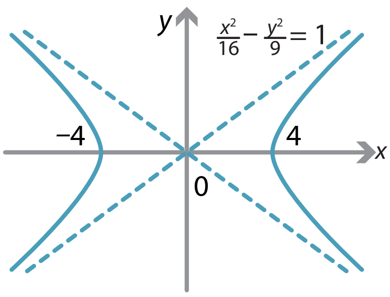
What is a relation?

There are many naturally occurring formulas whose graphs are not the graphs of functions. For example:









The first graph is a circle, the second is an ellipse, the third is two straight lines, and the fourth is a hyperbola. In each example, there are values of x for which there are two values of y. So these are not graphs of functions.

It turns out that the most useful concept to help describe and understand this issue is very general.

Definition

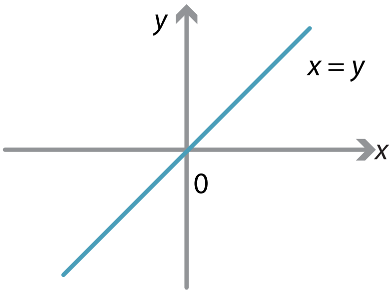
A **relation** on the real numbers is any subset of $\mathbb{R} \times \mathbb{R}$. That is, a relation on the reals is a set of ordered pairs of real numbers.

Thus the four graphs above and the graphs of the six example functions are all relations on the real numbers. Indeed, the graph of any function is a relation. Formally speaking, a **function** is a relation such that, for each x, there is at most one ordered pair (x, y).

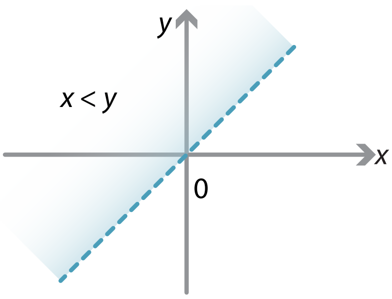
Example

The line $y = x$ divides the number plane into three relations.

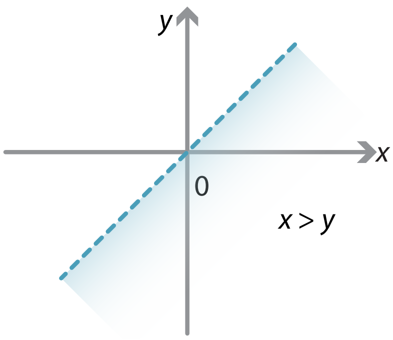
- The relation $R_1 = \{(x, y) \mid x = y\}$ is the line itself. Note that, for each x, there is only one value of y.



- The relation $R_2 = \{(x, y) \mid x < y\}$ consists of all points strictly above the line $y = x$. Note that, for each x, there are infinitely many values of y.



- The relation $R_3 = \{(x, y) \mid x > y\}$ consists of all points strictly below the line $y = x$.



We can generalise the previous example to any line in the plane, as follows.

Example

The equation of a line l in the plane is given by $ax + by + c = 0$. This line determines, in a natural way, three relations on the reals:

$$R_1 = \{(x, y) \mid ax + by + c = 0\}$$

$$R_2 = \{(x, y) \mid ax + by + c < 0\}$$

$$R_3 = \{(x, y) \mid ax + by + c > 0\}.$$

Example

Consider the circle $x^2 + y^2 = r^2$, for some $r > 0$. There are three relations closely connected with this circle.

- The circle itself:

$$R_1 = \{(x, y) \mid x^2 + y^2 = r^2\}.$$

- The interior of the circle:

$$R_2 = \{(x, y) \mid x^2 + y^2 < r^2\}.$$

- The exterior of the circle:

$$R_3 = \{(x, y) \mid x^2 + y^2 > r^2\}.$$

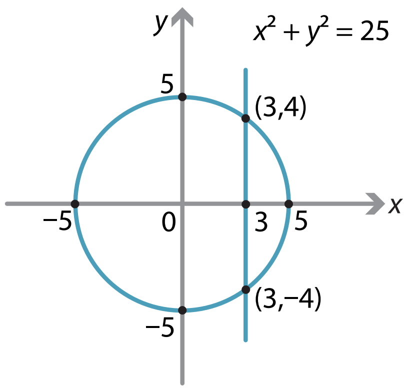
Graphs and the vertical-line test

We have seen the graphs of several naturally described functions. A sensible question to ask, for a given graph in $\mathbb{R}^2$, is whether it is the graph of a function.

The **vertical-line test** gives a simple geometric test for answering this question:

> If we can draw a vertical line $x = c$ that cuts the graph more than once, then the graph is not the graph of a function.

Returning to the graph of $x^2 + y^2 = 25$, we see that the vertical line $x = 3$ meets the graph at both $(3,4)$ and $(3,-4)$.



Hence, the graph of $x^2 + y^2 = 25$ is not the graph of a function. The line $x = 6$ does not meet the graph at all, but this does not matter. In general:

- for $-5 < c < 5$, the line $x = c$ meets the graph twice
- for $c = -5$ and for $c = 5$, the line $x = c$ meets the graph once
- for $c < -5$ and for $c > 5$, the line $x = c$ does not meet the graph.

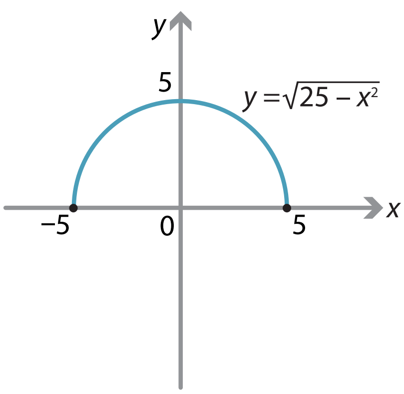
Relations which determine functions

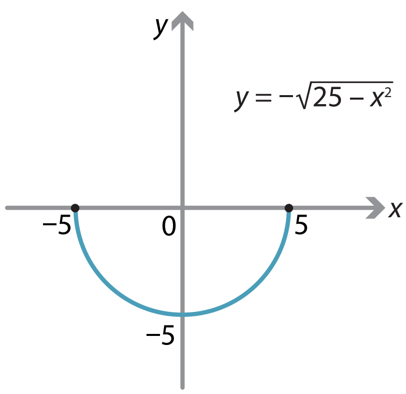
There is a natural way in which we can use the relation $x^2 + y^2 = 25$ to construct two functions. Solving $x^2 + y^2 = 25$ for y gives

$$y^2 = 25 - x^2$$

$$y = \sqrt{25 - x^2} \quad \text{or} \quad y = -\sqrt{25 - x^2}.$$

The graph of the first of these functions is the 'top half' of the circle, and the graph of the second is the 'bottom half' of the circle.





Example

Consider the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Find two functions whose graphs together include all points on the ellipse.

Solution

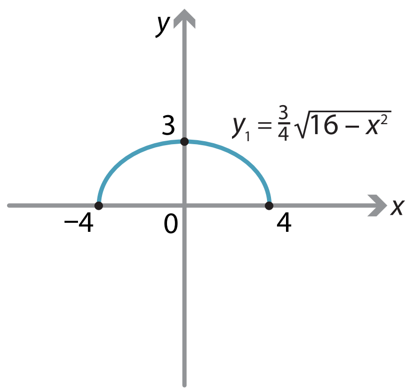
Solve for y:

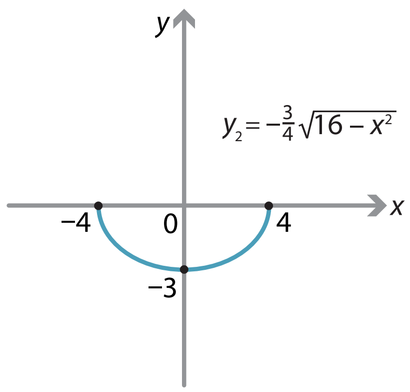
$$\begin{aligned} \frac{x^2}{16} + \frac{y^2}{9} &= 1 \\ \frac{y^2}{9} &= 1 - \frac{x^2}{16} \\ y^2 &= \frac{9}{16}\left(16 - x^2\right) \\ y &= \pm\frac{3}{4}\sqrt{16 - x^2}. \end{aligned}$$

Hence,

$$y_1 = \frac{3}{4}\sqrt{16 - x^2} \quad \text{and} \quad y_2 = -\frac{3}{4}\sqrt{16 - x^2}$$

are two such functions.





Note that the two graphs overlap in the points $(4, 0)$ and $(-4, 0)$. This is not an issue.

Domains and ranges

Recall our six example functions:

- $y = x + 2$
- $y = 3x^2 - 7$
- $y = \sin x$
- $y = 2^x$
- $y = \dfrac{1}{x}$
- $y = \log_2 x$.

For the first four functions, we can take x to be any real number. That is, we can substitute any x-value into the formula to obtain a unique y-value. We therefore say that the natural domain of the functions $y = x + 2$, $y = 3x^2 - 7$, $y = \sin x$ and $y = 2^x$ is the set of all real numbers, denoted by $\mathbb{R}$. On the other hand, for the function $y = \dfrac{1}{x}$, we cannot substitute $x = 0$; the set of allowable values of x is all non-zero reals.

Definition

The set of allowable values of x is called the **natural domain** of the function.

The natural domain is sometimes called the maximum domain; it is often simply called the **domain** of the function.

For example:

- The function $y = \dfrac{1}{x}$ has domain $\{x \in \mathbb{R} \mid x \neq 0\}$, which is also written as $\mathbb{R} \setminus \{0\}$.
- The function $y = \log_2 x$ has domain $\{x \in \mathbb{R} \mid x > 0\}$, which is also written as $\mathbb{R}^+$.

In a similar way, we can ask: What are all possible values of y, as x varies over the domain of the function? This set of values is called the **range** of the function.

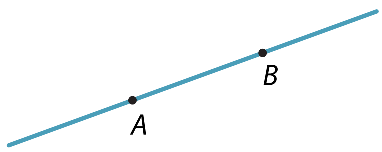
The domains and ranges for our six standard examples are given in the following table.

Example functions

Function	Domain	Range
$y = x + 2$	$\mathbb{R}$	$\mathbb{R}$
$y = 3x^2 - 7$	$\mathbb{R}$	$\{y : y \geq -7\}$
$y = \sin x$	$\mathbb{R}$	$\{y : -1 \leq y \leq 1\}$
$y = 2^x$	$\mathbb{R}$	$\{y : y > 0\}$
$y = \frac{1}{x}$	$\{x : x \neq 0\}$	$\{y : y \neq 0\}$
$y = \log_2 x$	$\{x : x > 0\}$	$\mathbb{R}$

Interval notation

In geometry, an interval is the set of all points between two points A and B on a line.



Sometimes the interval includes A and B, in which case it is called **closed**. If it contains neither A nor B, we call it an **open** interval.

In much of mathematics — particularly in calculus — we consider **real functions**, where both the domain of the function and the range of the function are subsets of the reals. Consequently, domains and ranges are very often intervals or unions of disjoint intervals.

Consider the sets A and B given by

$$A = \{x : 3 \le x \le 5\} \qquad \text{and} \qquad B = \{t : 3 \le t \le 5\}.$$

Clearly $A = B$, and the variables x and t are irrelevant! For this reason (and others), we use the notation $[3,5]$ to represent the set of all real numbers between 3 and 5, including 3 and 5. So

$$A = B = [3,5].$$

We call $[3,5]$ a **closed interval**, since it includes the endpoints. We use parentheses to denote an **open interval**, not including the endpoints, so

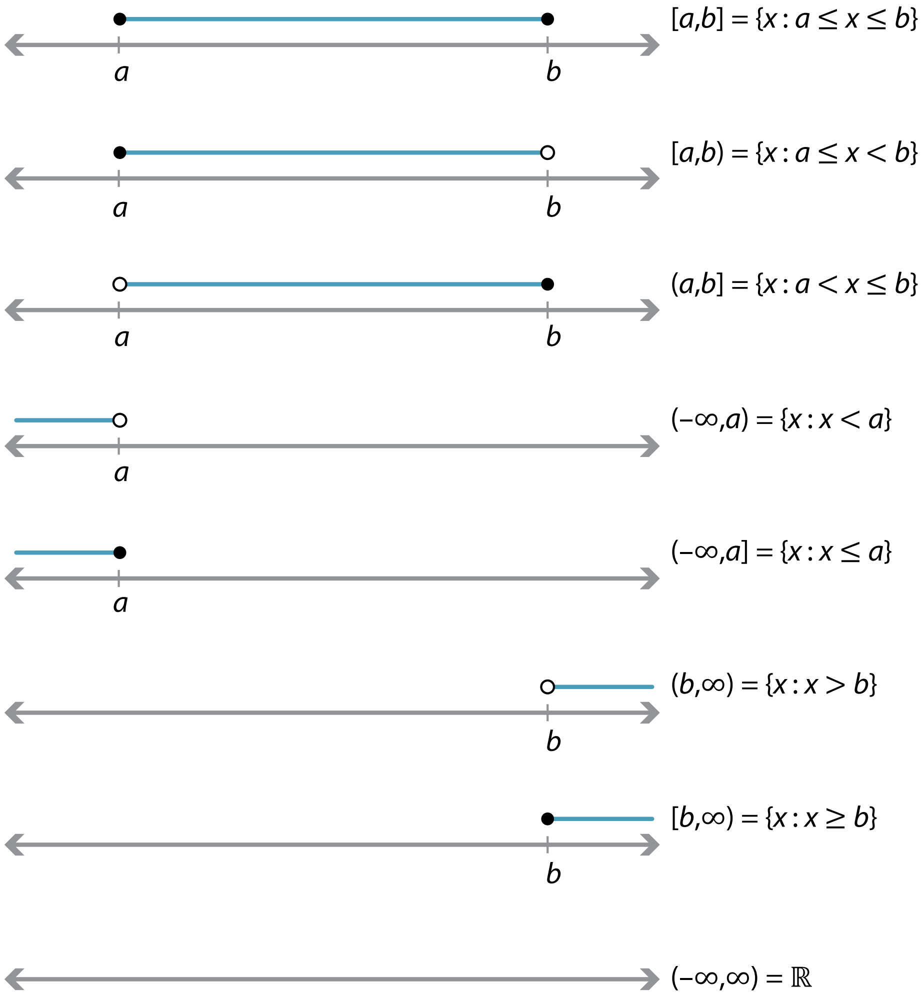
$$(3,5) = \{x : 3 < x < 5\}.$$

There are also **half-open** intervals $(3,5]$ and $[3,5)$. We use the symbols ∞ and $-\infty$ to define unbounded intervals:

$$\begin{aligned} (3,\infty) &= \{x : x > 3\} \qquad &&\text{(open interval)} \\ (-\infty,3] &= \{x : x \le 3\} \qquad &&\text{(closed interval).} \end{aligned}$$

Note that $(-\infty,3] \cup (3,\infty) = \mathbb{R}$.

To summarise, let $a, b \in \mathbb{R}$ with $a < b$. In the following diagrams, a solid dot means that the point is included and an open dot that the point is not included.



Of course, the intervals $[a,\infty)$, (a,∞), $(-\infty, b]$ and $(-\infty, b)$ could also have been included. Notice that

$$(-\infty, a) \cup [a, b] \cup (b,\infty) = \mathbb{R},$$

as a disjoint union. Similarly, we have

$$(-\infty, a] \cup (a, b) \cup [b,\infty) = \mathbb{R}.$$

We will often use interval notation to describe domains. For example, the domain of $y = \log_2 x$ is $(0,\infty)$, and the domain of $y = \sqrt{x}$ is $[0,\infty)$.

Sometimes it is more convenient not to use intervals. For example, the domain of the function

$$y = \frac{x-1}{(x-2)(x-3)}, \quad \text{for } x \notin \{2,3\},$$

is written most simply as $\mathbb{R} \setminus \{2,3\}$. (A quotient of two polynomials is called a **rational function**.)

Function notation

A **function** is a rule for transforming an object into another object. The object you start with is called the **input**, and comes from some set called the **domain**. What you end up with is called the **output**, and it comes from some set called the **codomain**. There is a standard and very convenient notation for functions.

For example, we write the function $y = x^2$ as

$$f(x) = x^2.$$

This is read as 'f of x is equal to x^2'. To calculate the value of the function for some input a, we simply substitute a for x in the formula for f. For example, for this function we have

$$f(3) = 3^2 = 9, \quad f(0) = 0, \quad f(-2) = 4, \quad f(a) = a^2, \quad f(x+2) = x^2 + 4x + 4.$$

We are now distinguishing between the *function* $f(x)$ and its *graph* $y = f(x)$. A function can be given by a rule such as

$$f(x) = \frac{x^3 \sin x}{(x-2)^2}, \quad \text{for } x \neq 2.$$

We can easily calculate values of this function, even though drawing its graph may be quite difficult. This new way of writing functions is called **function notation**, and was introduced to mathematics by Leonhard Euler in 1735. It is now completely standard.

Exercise 2

For the function $f(x) = \dfrac{x^2}{2} + x$, find

a $f(3)$

b $f(a+h)$.

In general, a function should be specified by giving both the rule and the domain. It is not essential to draw the graph of the function, but very often the sensible first step in solving an algebraic problem or a problem using calculus is to draw the graph of the function.

Non-standard functions

Most functions encountered in secondary school mathematics are easily graphed.

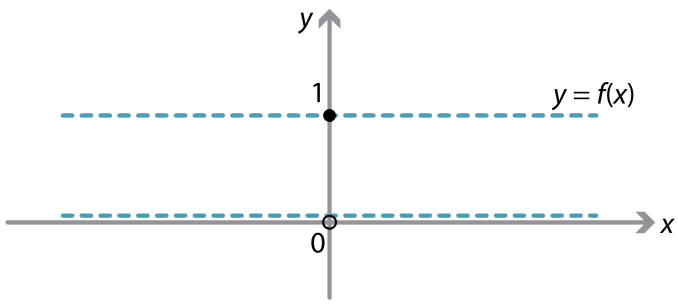
In the late 19th century, when these ideas were being seriously considered for the first time, examples such as the following were introduced:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Yes, this is a function! For example, we have

$$f(0) = f(3) = 1 \quad \text{and} \quad f(\sqrt{2}) = f(\pi) = 0.$$

It is not possible to draw the graph of the function, except perhaps as follows.



At first, this type of example was dismissed by many mathematicians. But such examples turned out to be very important in finding the correct definitions of *continuity* and *differentiability* in advanced calculus (known as analysis).

Finding domains and ranges

The natural domain of a function is the set of all allowable input values. We will call it the **domain** of the function f, denoted by $\mathrm{domain}(f)$. The **range** of the function f is the set of all possible output values:

$$\mathrm{range}(f) = \{ f(x) : x \in \mathrm{domain}(f) \}.$$

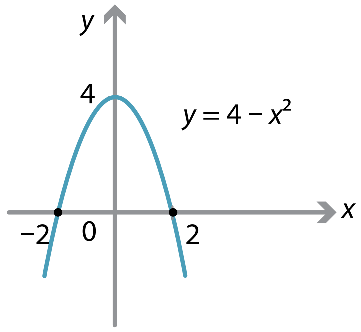
For a particular function $f(x)$, determining $\mathrm{domain}(f)$ and $\mathrm{range}(f)$ is sometimes quite complicated.

Example

What is the domain and range of the function $f(x) = 4 - x^2$?

Solution

Here a graph of the function helps.



Since $f(x)$ is defined for all real numbers, we have $\text{domain}(f) = \mathbb{R}$.

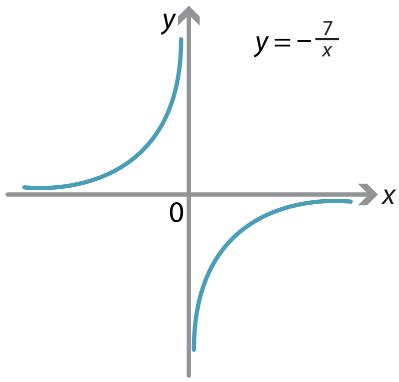
We can see from the graph that $\text{range}(f) = \{y : y \le 4\} = (-\infty, 4]$.

Example

What is the domain and range of $f(x) = -\dfrac{7}{x}$?

Solution

The graph of $y = f(x)$ is a rectangular hyperbola.



From the graph, we can see that

$$\text{domain}(f) = \{x \in \mathbb{R} \mid x \neq 0\} = \mathbb{R} \setminus \{0\}$$

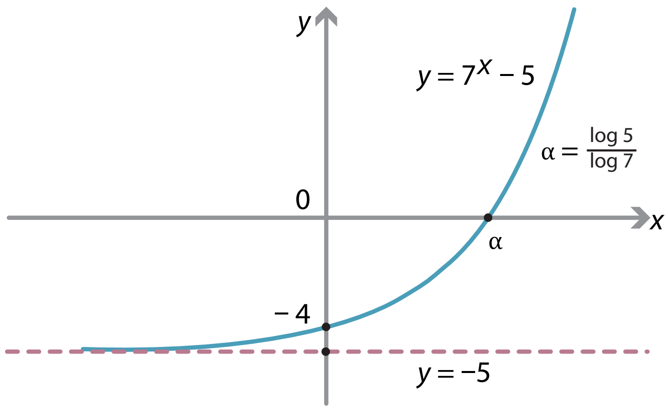
$$\text{range}(f) = \{y \in \mathbb{R} \mid y \neq 0\} = \mathbb{R} \setminus \{0\}.$$

Example

What is the domain and range of $f(x) = 7^x - 5$?

Solution

This is a vertical translation of an exponential function.



Hence, domain$(f) = \mathbb{R}$ and range$(f) = (-5, \infty)$.

Example

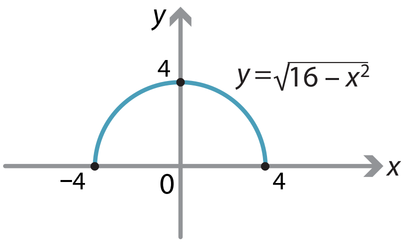
What is the domain and range of $f(x) = \sqrt{16 - x^2}$?

Solution

We have

$$\begin{aligned} y = \sqrt{16 - x^2} &\iff y^2 = 16 - x^2 \text{ and } y \geq 0 \\ &\iff x^2 + y^2 = 16 \text{ and } y \geq 0. \end{aligned}$$

So the graph of $f(x)$ is the top half of the circle with centre the origin and radius 4.



Hence, domain$(f) = [-4, 4]$ and range$(f) = [0, 4]$.

In the next example, we find the domain and range without first drawing the graph.

Example

Find the domain and range of $f(x) = \sqrt{1-2x}$.

Solution

We first find the domain of $f(x)$:

$$\begin{aligned} f(x) \text{ is defined } &\iff 1-2x \geq 0 \\ &\iff x \leq \tfrac{1}{2}. \end{aligned}$$

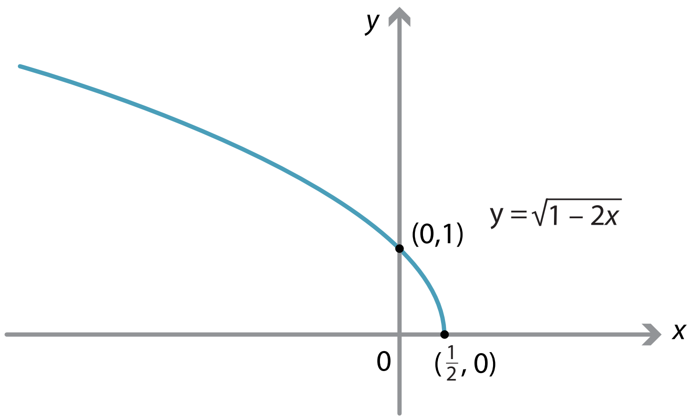
So $\text{domain}(f) = (-\infty, \frac{1}{2}]$.

Every non-negative real number can be obtained as a value of $f(x) = \sqrt{1-2x}$. Thus $\text{range}(f) = [0, \infty)$.

We can sketch the graph of this function by noticing that

$$\begin{aligned} y = \sqrt{1-2x} &\iff y^2 = 1-2x \text{ and } y \geq 0 \\ &\iff x = \tfrac{1}{2}(1-y^2) \text{ and } y \geq 0. \end{aligned}$$

So this is 'half' of a rotated and translated parabola.



Exercise 3

Find the domain and range of

a $f(x) = \sqrt{(x-1)(x-2)}$

b $g(x) = \sqrt{1 - 2\sin 2x}$.

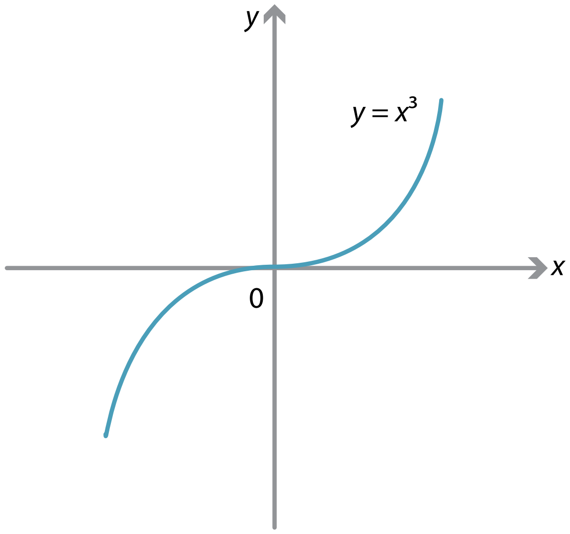
Example

Find the domain and range of $f(x) = x^{\frac{1}{3}}$. Sketch the graph of $y = f(x)$.

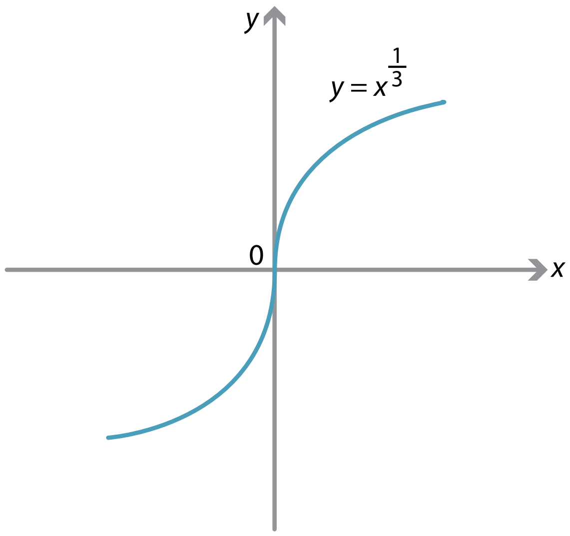
Solution

Clearly domain$(f) = \mathbb{R}$, since all real numbers have a cube root. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. Since f is continuous, it follows that range$(f) = \mathbb{R}$.

We have $y = x^{\frac{1}{3}}$ if and only if $x = y^3$. The graph of $y = x^3$ is as follows.



The graph of $x = y^3$, or $y = x^{\frac{1}{3}}$, is obtained by reflecting in the line $y = x$.



Exercise 4

What is the domain and range of $f(x) = 3\tan 2x$? Sketch the graph of $y = 3\tan 2x$.

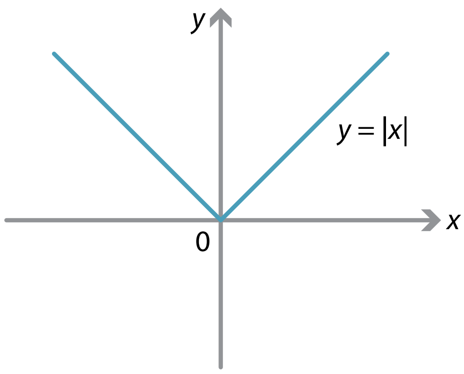
Example

Find the domain and range of $f(x) = |x|$, where as usual

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Solution

By drawing the graph of $y = |x|$, we see that $\text{domain}(f) = \mathbb{R}$ and $\text{range}(f) = [0, \infty)$.



Exercise 5

Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. So, for example, $\lfloor \pi \rfloor = 3$.

a Let $f(x) = \lfloor x \rfloor$. Sketch the graph of $y = f(x)$. Find $\text{domain}(f)$ and $\text{range}(f)$.

b Let $g(x) = x - \lfloor x \rfloor$. Sketch the graph of $y = g(x)$. Find $\text{domain}(g)$ and $\text{range}(g)$.

(In some books, the notation for $\lfloor x \rfloor$ is $[x]$.)

Links forward

Functions between sets

In the *Content* sections of this module, we have dealt exclusively with real functions. In the 20th century, the concept of a function was generalised to functions from any set to any other set. This has turned out to be a very powerful idea.

Definition

Let A and B be any two sets. A **function** f **from** A **to** B is a rule which maps each element x in A to an element $f(x)$ in B. A common notation is

$$f\colon A \to B$$

$$f\colon x \mapsto f(x)$$

Notes.

1 A function from A to B is often called a **map** from A to B.

2 The set A is called the domain of the function, and the set B is called the codomain.

3 The range of a function $f\colon A \to B$ is the set $\{\, y \in B : y = f(x) \text{ for some } x \in A\}$. Another notation for the range is $f(A) = \{\, f(x) : x \in A\}$.

4 There is a difficulty with domains. For example, consider the real function

$$f(x) = \frac{1}{x-2}.$$

One approach is define $A = \{x \in \mathbb{R} : x \neq 2\}$, and then define

$$f\colon A \to \mathbb{R}, \quad f(x) = \frac{1}{x-2}.$$

A difficulty with this approach is that the precise domain must be known for the function to be properly specified.

In the next example, we use $\mathbb{N}$ to denote the set of all positive integers (also called the set of natural numbers). That is, $\mathbb{N} = \{1, 2, 3, \dots\}$.

Example

An infinite sequence $a_1, a_2, a_3, a_4, \dots$ of real numbers can be thought of as a function $S\colon \mathbb{N} \to \mathbb{R}$ given by $S(n) = a_n$. For example:

- An **arithmetic sequence** is of the form

  $$a,\ a+d,\ a+2d,\ a+3d,\ \dots$$

  for some $a, d \in \mathbb{R}$. The associated function $S\colon \mathbb{N} \to \mathbb{R}$ is given by $S(n) = a + (n-1)d$.

- A **geometric sequence** is of the form

  $$a,\ ar,\ ar^2,\ ar^3,\ \dots$$

  for some $a, r \in \mathbb{R}$. The associated function $S\colon \mathbb{N} \to \mathbb{R}$ is given by $S(n) = ar^{n-1}$.

Exercise 6

Classify each of the following sequences as arithmetic, geometric or neither.

a $2, 5, 8, 11, 14, \dots$

b $3, -6, 12, -24, 48, \dots$

c $1, 0.1, 0.01, 0.001, 0.0001, \dots$

d $1, 1, 2, 3, 5, 8, 13, \dots$

e The sequence of decimal approximations to π:

$$3, 3.1, 3.14, 3.142, 3.1416, 3.14159, \dots.$$

Example

Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$. The range of f is $\mathbb{Z}$. Hence f is also a function from $\mathbb{R}$ to $\mathbb{Z}$.

Exercise 7

Determine the domain, codomain and range of $f: \mathbb{N} \to \mathbb{N}$ given by $f(n) = 2n + 1$.

Functions between finite sets

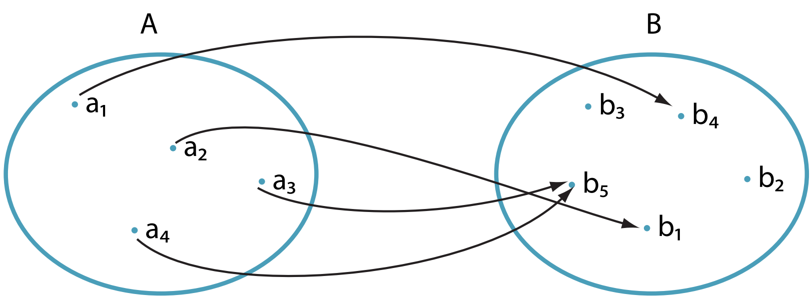
If A and B are finite sets, then a function from A to B can be specified by listing all elements in A and their images in B. For example, if

$$A = \{a_1, a_2, a_3, a_4\} \qquad \text{and} \qquad B = \{b_1, b_2, b_3, b_4, b_5\},$$

then we can define a function $f: A \to B$ by

$$f(a_1) = b_4, \quad f(a_2) = b_1, \quad f(a_3) = b_5 \quad \text{and} \quad f(a_4) = b_5.$$

This can be represented by an **arrow diagram**.



For this example, we have domain$(f) = A$, codomain$(f) = B$ and range$(f) = \{b_1, b_4, b_5\}$.

Exercise 8

Suppose $|A| = m$ and $|B| = n$. How many functions are there from A to B?

More examples of functions

A function defined on a language

The study of formal languages is an important part of theoretical computer science.

Let Σ be a finite set of symbols. In computer science, this is often called an **alphabet**. For example, we could take $\Sigma = \{a, b, c, \dots, z\}$ or $\Sigma = \{a, b\}$.

A **word** over Σ is a finite string of characters from Σ. For example, if $\Sigma = \{a, b\}$, then the following are examples of words over Σ:

$$a, \qquad b, \qquad ab, \qquad bbb, \qquad aababbab.$$

The **empty word**, which has no characters, is denoted by ε. Let Σ^* be the set of all words over Σ.

The **length** of a word is the number of characters in the string. Thus, for example,

$$l(\varepsilon) = 0, \qquad l(a) = l(b) = 1, \qquad l(abab) = 4.$$

This gives us a natural function $l\colon \Sigma^* \to \mathbb{N} \cup \{0\}$.

The function l induces a natural partition on the set of all words over Σ. Define Σ^i to be all words of length i, that is,

$$\Sigma^i = \{\, w \in \Sigma^* : l(w) = i \,\}.$$

Then we can write Σ^* as the disjoint union

$$\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma^i.$$

Binary operations as functions

There are four basic binary operations involving the real numbers:

$$3 + 2 = 5, \qquad 3 - 2 = 1, \qquad 3 \times 2 = 6, \qquad 3 \div 2 = \frac{3}{2}.$$

All of these can be thought of as functions. For example,

$$+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (a, b) \mapsto a + b.$$

We can also consider the restrictions $+\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $+\colon \mathbb{Q} \times \mathbb{Q}$. This idea is a starting point for abstract algebra.

Probability distributions as functions

Consider a finite sample space $\mathscr{E} = \{a_1, a_2, a_3, \dots, a_n\}$. Let $\mathscr{P}(\mathscr{E})$ denote the set of all subsets of $\mathscr{E}$, that is, the set of all events. (The notation $\mathscr{P}$ is used because 'the set of all subsets of S' is called the **power set** of S.) Then we can assign probabilities to events by a function

$$\Pr\colon \mathscr{P}(\mathscr{E}) \to [0,1]$$

that satisfies $\Pr(\mathscr{E}) = 1$ and $\Pr(A) = \sum_{a \in A} \Pr(\{a\})$, for all $A \subseteq \mathscr{E}$. See the module *Probability*.

History

Euler wrote six books on calculus between 1748 and 1770. All six were written in Latin, the lingua franca of scholars of the 18th century. There were two volumes entitled *Introduction to analysis of the infinite*, one volume on *Methods of differential calculus*, and three volumes on *Methods of the integral calculus*. Euler made *functions* the central idea in his books. This was a radical move, since both Newton and Leibniz in their development of the calculus had dealt with *curves*. The idea of a function was not fully developed in these books, since Euler restricted a function to mean a formula. However, he did consider functions of a complex variable as well as functions of a real variable. He also considered divergent series as well as convergent series.

Euler divided the set of functions into two basic classes: algebraic and transcendental. Algebraic functions are formed from variables and constants by addition, subtraction, multiplication, division, raising to an integer power, and extraction of roots. For Euler, the transcendental functions were limited to trigonometric, exponential and logarithmic functions.

Euler made extensive use of infinite series, infinite products and infinite continued fractions. As an example of the depth of his thinking, he was the first mathematician to seriously investigate the **Riemann zeta function**, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For example, Euler showed that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In the 19th century, the centrality of the complex numbers to mathematics, and in particular to both algebra and analysis (calculus), was realised. Functions of a complex variable were introduced, and it was shown (essentially) that any function given by a formula such as z^2 or $\sin z$, where $z = x + iy$, could be both integrated and differentiated. This material is typically taught in second-year university courses. (Reference: R. V. Churchill and J. W. Brown, *Complex variables and applications*, 5th edition, McGraw-Hill, 1990.)

In the 20th century, the idea of a function was generalised considerably further, leading to two branches of modern analysis called *harmonic analysis* and *functional analysis*.

Answers to exercises

Exercise 1

- Are 'diagonalise' and 'diagonalize' two words or the same word?
- Is the phrase 'deja vu' two English words? What about 'café'?
- Can we restrict English words to the words defined in the *Oxford English Dictionary*? Is the word 'gunna' defined? Was 'internet' in the *Oxford English Dictionary* in 1970?

Exercise 2

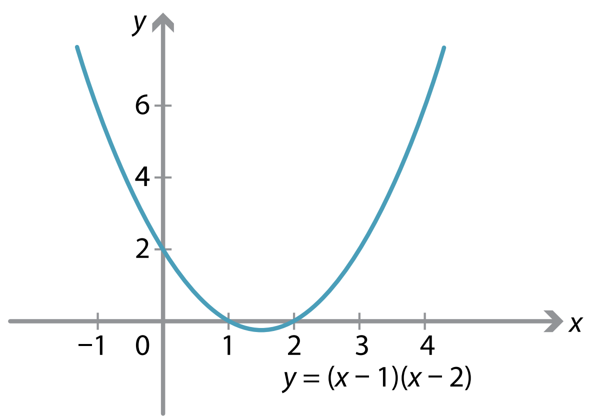
Let $f(x) = \dfrac{x^2}{2} + x$.

a $f(3) = \dfrac{3^2}{2} + 3 = \dfrac{15}{2}$.

b $f(a+h) = \dfrac{(a+h)^2}{2} + a + h$.

Exercise 3

a Let $f(x) = \sqrt{(x-1)(x-2)}$. Then $f(x)$ is defined if and only if $(x-1)(x-2) \geq 0$. So we sketch the graph of $y = (x-1)(x-2)$.

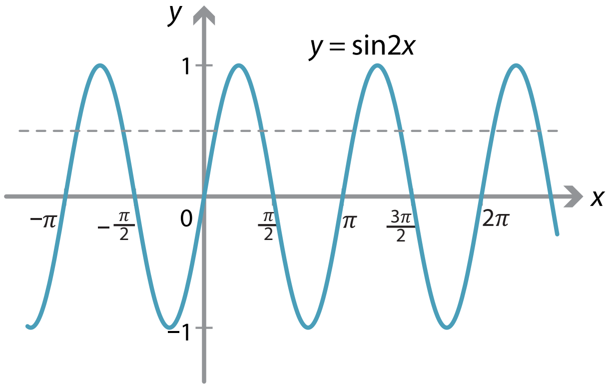


Hence, $\text{domain}(f) = (-\infty, 1] \cup [2, \infty)$ and $\text{range}(f) = [0, \infty)$.

b Let $g(x) = \sqrt{1-2\sin 2x}$. To find the domain of $g(x)$, we first note that

$$\begin{aligned} g(x) \text{ is defined} &\iff 1-2\sin 2x \geq 0 \\ &\iff 2\sin 2x \leq 1 \\ &\iff \sin 2x \leq \tfrac{1}{2}. \end{aligned}$$

So we sketch the graph of $y = \sin 2x$.



Now solve $\sin 2x = \frac{1}{2}$ for x:

$$2x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2k\pi, \quad \text{for some } k \in \mathbb{Z}$$

$$x = \frac{\pi}{12} + k\pi \quad \text{or} \quad x = \frac{5\pi}{12} + k\pi, \quad \text{for some } k \in \mathbb{Z}.$$

It follows that $\sin 2x \leq \frac{1}{2}$ if and only if x belongs to the interval $\left[\frac{5\pi}{12}, \frac{13\pi}{12}\right]$ or one of its translates by a multiple of π. Hence,

$$\text{domain}(g) = \bigcup_{k\in\mathbb{Z}} \left[\frac{5\pi}{12} + k\pi, \frac{13\pi}{12} + k\pi\right].$$

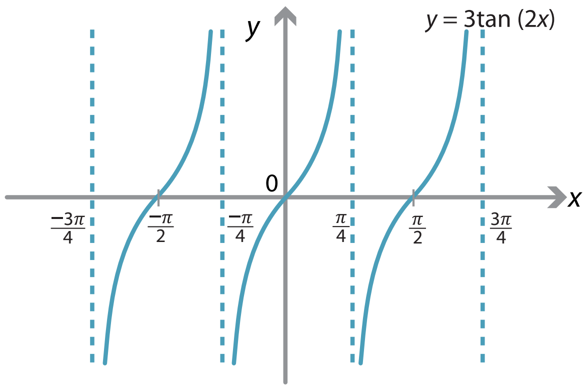
The maximum value of $g(x)$ is $\sqrt{1-2\times(-1)} = \sqrt{3}$. Hence, $\text{range}(g) = [0, \sqrt{3}]$.

Exercise 4

Let $f(x) = 3\tan 2x$. The domain of $\tan x$ is all real numbers except odd multiples of $\frac{\pi}{2}$, and the range of $\tan x$ is all reals. Hence,

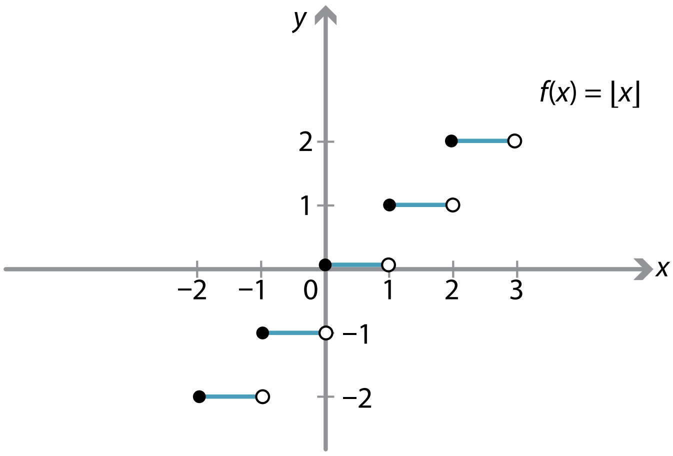
$$\text{domain}(f) = \mathbb{R} \setminus \{\dots, -\tfrac{3\pi}{4}, -\tfrac{\pi}{4}, \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \dots\}$$

and $\text{range}(f) = \mathbb{R}$.



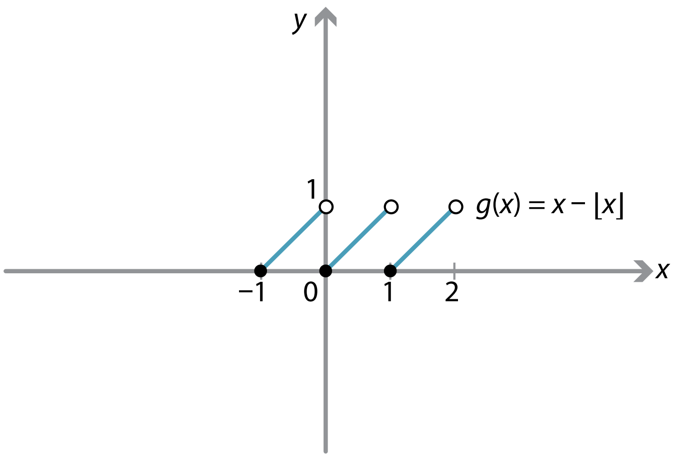
Exercise 5

a



domain$(f) = \mathbb{R}$ and range$(f) = \mathbb{Z}$.

b



domain$(g) = \mathbb{R}$ and range$(g) = [0, 1)$.

Exercise 6

a Arithmetic sequence with first term $a = 2$ and common difference $d = 3$.

b Geometric sequence with first term $a = 3$ and common ratio $r = -2$.

c Geometric sequence with first term $a = 1$ and common ratio $r = 0.1$.

d The Fibonacci sequence; it is neither geometric nor arithmetic.

e This sequence is neither geometric nor arithmetic.

Exercise 7

$\text{domain}(f) = \text{codomain}(f) = \mathbb{N}$ and $\text{range}(f) = \{3, 5, 7, 9, 11, \dots\}$.

Exercise 8

Assume $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. If we are defining a function from A to B, then there are n choices for where to map a_1, and then there are n choices for where to map a_2, and so on. Thus the total number of functions is n^m.

0 1 2 3 4 5 6 7 8 9 10 11 12