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A guide for teachers – Years 11 and 12

Calculus: Module 14

Exponential and logarithmic functions

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Exponential and logarithmic functions – A guide for teachers (Years 11–12)

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Exponential and logarithmic functions

Assumed knowledge

The content of the modules:

- *Functions II*
- *Introduction to differential calculus*
- *Applications of differentiation*
- *Integration*.

Furthermore, knowledge of the index laws and logarithm laws is assumed. These are covered in the TIMES module *Indices and logarithms* (Years 9–10) and briefly revised at the beginning of this module.

Motivation

> The greatest shortcoming of the human race is our inability to understand the exponential function.
>
> — Albert A. Bartlett

Our world involves phenomena and objects on many different scales.

Repeated multiplication by 10 can rapidly transform a microscopically small number to an astronomically large one. Multiplying by 10 a few times takes us immediately from the scale of atoms and molecules to the scale of microbiology, insects, humans, cities, continents, planets and beyond — from scales that are imperceptibly small to scales that are almost unfathomably vast. There are only 17 orders of magnitude between the size of a single human cell and the size of our solar system.[1] Understanding the functions

[1] A typical human cell is between 10^{-5} and 10^{-4} metres. The radius of the solar system, out to the orbit of Neptune, is roughly 4.5×10^{12} metres.

involved in such repeated multiplication — namely, exponential functions such as 10^x — is a useful step towards a grasp of these enormities.

Exponential functions, with all their properties of sudden growth and decay, arise in many natural phenomena, from the growth of living cells to the expansion of animal populations, to economic development, to radioactive decay. The quote from Professor Bartlett at the start of this section was made in the context of human population growth. The inverses of exponential functions — namely, logarithmic functions — occur prominently in fields as diverse as acoustics and seismology.

To understand these natural processes of growth and decay, it is important, then, to understand the properties of exponential and logarithmic functions.

In this module, we consider exponential and logarithmic functions from a pure mathematical perspective. We will introduce the function $y = e^x$, which is a solution of the differential equation $\frac{dy}{dx} = y$. It is a function whose derivative is itself. In the module *Growth and decay*, we will consider further applications and examples.

The module *Indices and logarithms* (Years 9–10) covered many properties of exponential and logarithmic functions, including the index and logarithm laws. Now, having more knowledge, we can build upon what we have learned, and investigate exponential and logarithmic functions in terms of their rates of change, antiderivatives, graphs and more.

In particular, we can ask questions like: How fast does an exponential function grow? It grows rapidly! But, with calculus, we can give a more precise answer.

A brief refresher

To jog your memory, we recall some basic definitions and rules for manipulating exponentials and logarithms. For further details, we refer to the module *Indices and logarithms* (Years 9–10).

Logarithms and exponentials are inverse operations. In particular, for $a > 1$,

$$x = a^y \iff y = \log_a x.$$

The following index laws hold for any bases $a, b > 0$ and any real numbers m and n:

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn} \qquad (ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Some simple consequences of the index laws are, for $a > 0$ and a positive integer n:

$$a^0 = 1 \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{-1} = \frac{1}{a}.$$

The following logarithm laws hold for any base $a > 1$, any positive x and y, and any real number n:

$$\log_a 1 = 0 \qquad \log_a a = 1$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a \frac{1}{x} = -\log_a x \qquad \log_a(x^n) = n \log_a x.$$

Also, recall the change of base formula:

$$\log_b x = \frac{\log_a x}{\log_a b},$$

for any $a, b > 1$ and any positive x.

Two approaches

In this module, we will introduce two new functions e^x and $\log_e x$. We will do this in two different ways.

The first approach develops the topic in an investigatory fashion, starting from the question: 'What is the derivative of 2^x?' However, as we proceed, we will point out some shortcomings of this approach.

Alternatively, we can begin from a definition of $\log_e x$ as an integral, and then define e^x as its inverse. The story is then told in a completely different order.

The first approach is probably easier for most students to understand, but the second approach is more concise and rigorous.

In general, when telling a mathematical story, there are various goals such as elegance, rigour, practicality, generality and understandability. Sometimes these goals conflict, and we have to compromise. Sometimes developing a subject in the most logically concise way does not make for easy reading. As with any other subject, learning mathematics from multiple perspectives leads to a deeper and more critical understanding.

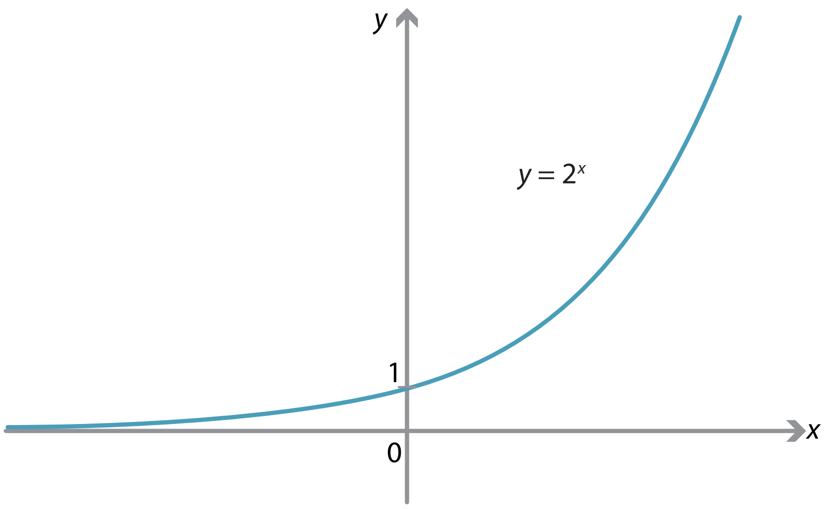
Content

How fast does an exponential function grow?

We will attempt to find the derivatives of exponential functions, beginning with 2^x. This is quite a long story, eventually leading us to introduce the number e, the exponential function e^x, and the natural logarithm. But we will then be able to differentiate functions of the form a^x in general.

The derivative of 2^x

We begin by attempting to find the derivative of $f(x) = 2^x$, which is graphed as follows.



Graph of $f(x) = 2^x$.

Examining this graph, we can immediately say something about the derivative $f'(x)$.

The graph of $y = 2^x$ is always sloping upwards and convex down. For large negative x, it is very flat but sloping upwards. As x increases, the graph slopes increasingly upwards; as x increases past 0, the gradient rapidly increases and the graph becomes close to vertical.

Therefore, we expect $f'(x)$ to be:

- always positive
- increasing
- approaching 0 as $x \to -\infty$
- rapidly increasing for x positive
- approaching ∞ as $x \to \infty$.

In other words, we expect $f'(x)$ to behave just like ... an exponential function. (We will see eventually that $f'(x) = \log_e 2 \cdot 2^x$.)

Let's first attempt to compute $f'(0)$ from first principles:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{2^h - 1}{h}. \end{aligned}$$

This is not an easy limit to compute exactly, but we can approximate it by substituting values of h.

Approximating $f'(0)$ for the function $f(x) = 2^x$

h	$\dfrac{2^h - 1}{h}$ (to 6 decimal places)
0.001	0.693387
0.0001	0.693171
0.00001	0.693150
0.000001	0.693147
0.0000001	0.693147

From the table above, it appears that

$$f'(0) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693147.$$

Let us press on and attempt to compute $f'(x)$ for x in general:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x(2^h - 1)}{h} \\ &= 2^x \cdot \lim_{h\to 0} \frac{2^h - 1}{h}. \end{aligned}$$

In the last step, we are able to take the 2^x through the limit sign, since it is independent of h. Note that the final limit is exactly the expression we found for $f'(0)$. So we can now express the derivative as

$$\begin{aligned} f'(x) &= f'(0) \cdot 2^x \\ &\approx 0.693147 \cdot 2^x. \end{aligned}$$

We have found that the derivative of 2^x is a constant times itself, confirming our initial expectations. (We will see later that this constant is $\log_e 2$.)

The derivative of a^x

There is nothing special about the number 2 above. If we take any number $a > 1$ and consider $f(x) = a^x$, the graph would have a similar shape to that of 2^x, and we could carry out similar computations for $f'(0)$ and $f'(x)$:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{a^h - 1}{h}, \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}. \end{aligned}$$

We have found that, for any $a > 1$, the function $f(x) = a^x$ has a derivative which is a constant multiple of itself:

$$\begin{aligned} f'(x) &= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= f'(0) \cdot a^x. \end{aligned}$$

However, we need to better understand the limit involved. Clearly, if we can choose the value of a so that this limit is 1, then $f'(x) = f(x)$ and so f is its own derivative.

The number e

Let's ponder further this limit for $f'(0)$, the derivative at 0 of $f(x) = a^x$:

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

We have already found that, when $a = 2$, the limit is approximately 0.693147. We can do the same calculations for other values of a, and find the approximate limit. We obtain the following table.

Approximating $f'(0)$ for functions $f(x) = a^x$

a	$\lim\limits_{h\to 0} \dfrac{a^h - 1}{h}$ (to 6 decimal places)
1	0
2	0.693147
3	1.098612
4	1.386294
5	1.609438

It appears that, when a increases, the limit for $f'(0)$ also increases. This is not too difficult to prove.

Exercise 1

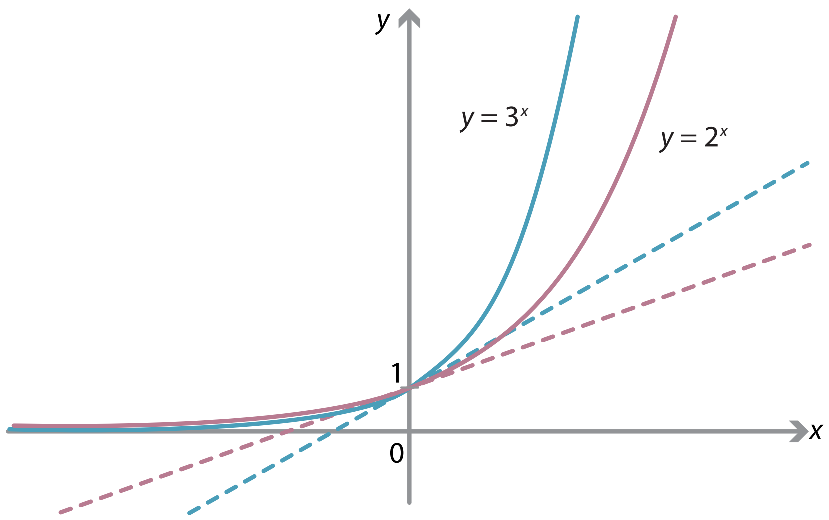
Show that, if $1 \le a < b$ and $h > 0$, then

$$\frac{a^h - 1}{h} < \frac{b^h - 1}{h},$$

and hence explain why

$$\lim_{h\to 0} \frac{a^h - 1}{h} \le \lim_{h\to 0} \frac{b^h - 1}{h}.$$

Geometrically, this means that, as a increases, the graph of $y = a^x$ becomes more sharply vertical, and the gradient of the graph at $x = 0$ increases.



Graphs of $y = 2^x$ and $y = 3^x$ with tangents shown at $x = 0$.

When $a = 2$, we have a gradient at $x = 0$ of 0.69315. When $a = 3$, we have a gradient of 1.0986. So we expect that there is a single value of a, between 2 and 3, for which the gradient at $x = 0$ is 1. It turns out that there is such a number,[2] which we shall call e. The number e is approximately 2.718281828.

Thus, the function $f(x) = e^x$ has $f'(0) = 1$ and, since $f'(x) = f'(0) \cdot e^x$, we have $f'(x) = e^x$. The function e^x is *its own derivative*. Equivalently, in Leibniz notation, $y = e^x$ satisfies

$$\frac{dy}{dx} = e^x \qquad \text{or, equivalently,} \qquad \frac{dy}{dx} = y.$$

The function $f(x) = e^x$ is often called *the* exponential function, and sometimes written as $\exp x$.

Note. This approach may appear to be a sleight of hand. We didn't really 'prove' that the derivative of e^x is itself, we just defined e to make it true. But the key point is that there is a number e that makes the function e^x its own derivative. We have given an argument (although not a rigorous proof) as to why there is such a number.

Summary

- We considered the derivative of $f(x) = 2^x$ and found that

  $$\begin{aligned} f'(x) &= 2^x \cdot \lim_{h\to 0} \frac{2^h - 1}{h} \\ &\approx 0.693147 \cdot 2^x, \end{aligned}$$

  so $f'(x)$ is a constant multiple of $f(x)$.
- We considered the derivative of the general function $f(x) = a^x$, where $a > 1$, and found that

  $$f'(x) = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h},$$

  so $f'(x)$ is a constant multiple of $f(x)$.
- We defined the number e so that the function $f(x) = e^x$ is its own derivative, that is, $f'(x) = f(x)$.

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[2] However, note we have not shown this. We have not shown that the limit $\lim_{h\to 0} \frac{a^h-1}{h}$ is a continuous function of a; otherwise, this limit might never take the value 1. And we have not shown that the limit is a strictly increasing function of a; if not, there might be multiple values of a for which the limit is 1. The alternative treatment later in this module avoids these issues.

Example

Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{x^2}$. What is $f'(x)$?

Solution

Use the chain rule. Let $g(x) = e^x$ and $h(x) = x^2$, so that $f(x) = g(h(x))$. Then

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= 2x\, e^{x^2}. \end{aligned}$$

Exercise 2

Find the derivatives of the following functions:

a $f(x) = x^2 e^x$ **b** $f(x) = e^{e^x}$.

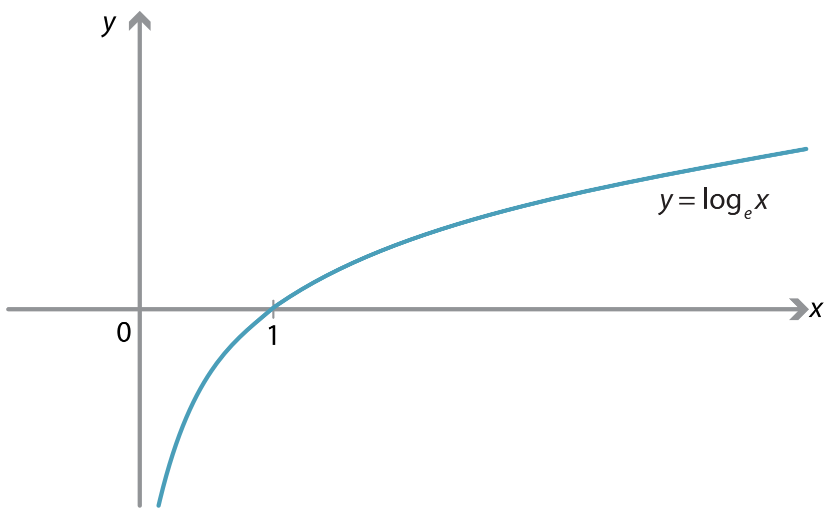
We will next introduce the natural logarithm. We will then be able to better express derivatives of exponential functions.

The natural logarithm

The logarithm to the base e is an important function. It is also known as the **natural logarithm**. It is defined for all $x > 0$:

$$y = \log_e x \iff x = e^y.$$

The alternative notation $\ln x$ (pronounced 'ell-en' x) is often used instead of $\log_e x$.



Graph of $y = \log_e x$ or, equivalently, $x = e^y$.

Since we know how to differentiate the exponential, we can now also differentiate the natural logarithm. If $y = \log_e x$, then $x = e^y$, so

$$\frac{dx}{dy} = e^y = x.$$

Now, by the chain rule, $1 = \dfrac{dy}{dy} = \dfrac{dy}{dx}\dfrac{dx}{dy}$, and so

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x}.$$

Therefore, we conclude that the derivative of $f(x) = \log_e x$ is $f'(x) = \dfrac{1}{x}$.

Exercise 3

Give an alternative proof that the derivative of $\log_e x$ is $\dfrac{1}{x}$, by differentiating both sides of the equation

$$x = e^{\log_e x}.$$

Knowing the derivative of $\log_e x$ allows us to differentiate many related functions.

Example

Find the derivative of $f(x) = \log_e(2x+5)$.

Solution

The chain rule gives

$$f'(x) = \frac{1}{2x+5} \cdot 2 = \frac{2}{2x+5}.$$

In general, for any real constants a and b with $a \neq 0$, we can consider the function $f(x) = \log_e(ax+b)$. Its derivative is, again using the chain rule,

$$f'(x) = \frac{a}{ax+b}.$$

A warning about domains. Any logarithm function $\log_a x$, with base $a > 1$, is defined only for $x > 0$; its domain is $(0,\infty)$. Yet the function $\dfrac{1}{x}$ is defined for all $x \neq 0$, including all negative x. Strictly speaking, the derivative of $\log_e x$ is the function $\dfrac{1}{x}$, *restricted* to the domain $(0,\infty)$.

In the previous example, $\log_e(2x+5)$ is only defined when $2x+5 > 0$, that is, $x > -\frac{5}{2}$; so the functions f and f' both have domain $(-\frac{5}{2},\infty)$.

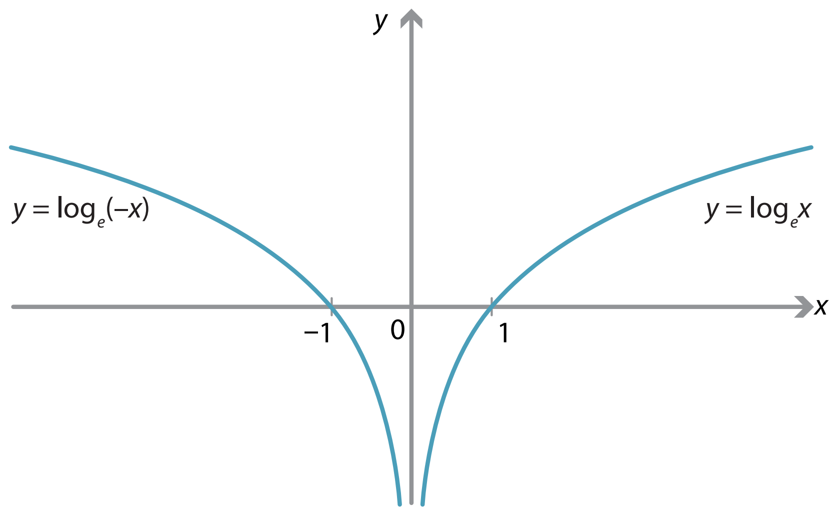
Exercise 4

Consider the function $f\colon (-\infty, 0) \to \mathbb{R}$ defined by $f(x) = \log_e(-x)$. Show that $f'(x) = \dfrac{1}{x}$, restricted to the domain $(-\infty, 0)$.

Using the preceding exercise, we can construct a function which is defined for *all* $x \neq 0$, and whose derivative is always $\dfrac{1}{x}$. We define $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ as

$$f(x) = \log_e |x| = \begin{cases} \log_e x & \text{if } x > 0 \\ \log_e(-x) & \text{if } x < 0. \end{cases}$$

As both $\log_e x$ and $\log_e(-x)$ have derivative $\dfrac{1}{x}$, we conclude that $f'(x) = \dfrac{1}{x}$ for all $x \neq 0$.



Graph of $y = \log_e |x|$, which is the union of the graphs $y = \log_e x$, for $x > 0$, and $y = \log_e(-x)$, for $x < 0$.

Exercise 5

What is the domain of the function $f(x) = \log_2(3 - 7x)$?

The derivative of 2^x, revisited

We now obtain a simple answer to our original question.

Example

What is the derivative of $f(x) = 2^x$?

Solution

Since $2 = e^{\log_e 2}$, we have $f(x) = e^{\log_e 2 \cdot x}$. Using the chain rule, we obtain

$$\begin{aligned} f'(x) &= \log_e 2 \cdot e^{\log_e 2 \cdot x} \\ &= \log_e 2 \cdot 2^x. \end{aligned}$$

Previously we found that $f'(x) \approx 0.693147 \cdot 2^x$. We now see that the constant is $\log_e 2$.

Exercise 6

Find the derivatives of the following functions:

a $f(x) = 2^{x^2}$ **b** $f(x) = 3^{4x^2+2x-7}$.

Exercise 7

Use what we've done so far to explain why

$$\lim_{h \to 0} \frac{2^h - 1}{h} = \log_e 2.$$

Derivatives of general exponential functions

Since we can now differentiate e^x, using our knowledge of differentiation we can also differentiate other functions.

In particular, we can now differentiate functions of the form $f(x) = e^{kx}$, where k is a real constant. From the chain rule, we obtain

$$f'(x) = k\,e^{kx}.$$

We saw in the previous section, when differentiating 2^x, that it can be written as $e^{\log_e 2 \cdot x}$, which is of the form e^{kx}. The same technique can be used to differentiate any function a^x, where a is a positive real number. A function a^x is just a function of the form e^{kx} in disguise.

Writing a as $e^{\log_e a}$, we can rewrite $f(x) = a^x$, using the index laws, as

$$f(x) = a^x = \left(e^{\log_e a}\right)^x = e^{x \log_e a}.$$

The function is then in the form e^{kx} (with $k = \log_e a$) and differentiating gives

$$f'(x) = \log_e a \cdot a^x.$$

So the derivative of a^x is a constant times itself, and that constant is $\log_e a$.

Exercise 8

Use what we've done so far to explain why, for any $a > 0$,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \log_e a.$$

Exercise 9

Let $f(x) = x^x$, for $x > 0$. Differentiate $f(x)$ and find its stationary points.

Derivatives of general logarithmic functions

Consider a logarithmic function $f(x) = \log_a x$, where $a > 1$ is a constant. By using the change of base rule, we can write $f(x)$ in terms of the natural logarithm, and then differentiate it:

$$f(x) = \log_a x = \frac{\log_e x}{\log_e a}$$

and so, since $\log_e a$ is a constant, the derivative is

$$f'(x) = \frac{1}{x \log_e a}.$$

Example

Differentiate $f(x) = \log_2(3 - 7x)$.

Solution

Since

$$f(x) = \frac{\log_e(3 - 7x)}{\log_e 2},$$

noting that $\log_e 2$ is just a constant and using the chain rule, we have

$$\begin{aligned} f'(x) &= \frac{1}{\log_e 2} \cdot \frac{1}{3 - 7x} \cdot (-7) \\ &= \frac{-7}{(3 - 7x) \log_e 2}. \end{aligned}$$

Exercise 10

Consider $f(x) = \log_a(xy)$ and $g(x) = \log_a x + \log_a y$, where $a > 1$ and x, y are positive. As our notation suggests, think of x as a variable and y as a constant.

a Show that $f'(x) = g'(x)$.

b Show that $f(1) = g(1)$.

c Conclude that $f(x) = g(x)$ for all positive x.

This gives an alternative proof of one of the logarithm laws.

Antiderivatives of exponential and logarithmic functions

We've seen various derivatives so far, including

$$\frac{d}{dx} e^x = e^x \qquad \text{and, more generally,} \qquad \frac{d}{dx} e^{kx} = k\,e^{kx},$$

where k is any non-zero real constant. Also, we've seen

$$\frac{d}{dx} \log_e x = \frac{1}{x} \qquad \text{and, more generally,} \qquad \frac{d}{dx} \log_e(ax+b) = \frac{a}{ax+b},$$

for $ax + b > 0$, where a, b are real constants with $a \neq 0$. From this we can deduce several antiderivatives.

The basic indefinite integrals are

$$\int e^x\,dx = e^x + c \qquad \text{and} \qquad \int \frac{1}{x}\,dx = \log_e x + c$$

and, more generally,

$$\int e^{kx}\,dx = \frac{1}{k} e^{kx} + c \qquad \text{and} \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a} \log_e(ax+b) + c,$$

where c as usual is a constant of integration.

We can use these antiderivatives to evaluate definite integrals.

Example

Find

$$\int_e^{e^2} \frac{1}{x}\,dx.$$

Solution

$$\int_e^{e^2} \frac{1}{x}\,dx = [\log_e x]_e^{e^2}$$

$$= \log_e(e^2) - \log_e e = 2 - 1 = 1$$

Exercise 11

Prove that, for any $x > 0$,

$$\int_1^x \frac{1}{t}\,dt = \log_e x.$$

Exercise 12

Prove that, for any $x > 0$ and $n > 0$,

$$\int_{x^n}^{x^{n+1}} \frac{1}{t}\,dt = \log_e x.$$

Exercise 13

Differentiate $f(x) = x\log_e x - x$. Hence find the indefinite integral

$$\int \log_e x\,dx.$$

Warning! As we mentioned previously, $\log_e x$ is only defined for $x > 0$, while $\frac{1}{x}$ is defined for all $x \neq 0$. So the equation

$$\int \frac{1}{x}\,dx = \log_e x + c$$

is valid only for $x > 0$ and, more generally, the equation

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$$

is valid only when $ax + b > 0$. Although it is more complicated, it is sometimes necessary to consider the function $\log_e|x|$, which is defined for all $x \neq 0$ and which also has derivative $\frac{1}{x}$. The equation

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

is valid for all $x \neq 0$, and the equation

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$$

is valid for all x such that $ax + b \neq 0$.

Graphing exponential functions

Having seen the general shape of the graphs $y = a^x$, we can graph related functions, including those obtained by transformations such as dilations, reflections or translations.

Let us first revise these transformations; for more details, we refer to the module *Functions II*.

- A **dilation in the x-direction** from the y-axis with factor k maps

  $$(x, y) \mapsto (kx, y).$$

  The plane is stretched out horizontally from the y-axis by a factor of k.
- A **dilation in the y-direction** from the x-axis with factor k maps

  $$(x, y) \mapsto (x, ky).$$

  The plane is stretched vertically from the x-axis by a factor of k.
- The **reflection in the y-axis** maps

  $$(x, y) \mapsto (-x, y).$$

  The plane is reflected horizontally in the vertical axis: everything in the plane on one side of the y-axis goes to its mirror image on the other side of the y-axis.
- The **reflection in the x-axis** maps

  $$(x, y) \mapsto (x, -y).$$

  The plane is reflected vertically in the horizontal axis.
- A **translation in the x-direction** by a units maps

  $$(x, y) \mapsto (x + a, y).$$

  If $a > 0$, the translation is to the right, and if $a < 0$, the translation is to the left.
- A **translation in the y-direction** by a units maps

  $$(x, y) \mapsto (x, y + a).$$

  If $a > 0$, the translation is upwards, and if $a < 0$, the translation is downwards.

Note that, as the x-axis is an asymptote for the graph $y = a^x$, any graph obtained by dilating, reflecting or translating such a graph will also have an asymptote.

Note also that, for any $a > 0$ with $a \neq 1$, the function $y = a^x$ has no critical points: the derivative $\dfrac{dy}{dx} = \log_e a \cdot a^x$ is never 0. A graph obtained from $y = a^x$ by dilations, reflections in the axes and translations also has no critical points.

Example

Sketch the graph of $y = 2^{x+1} - 3$.

Solution

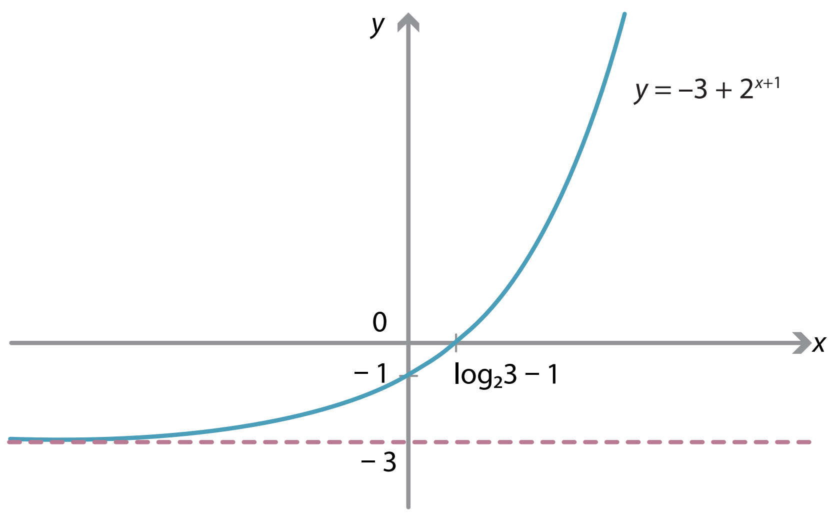
This equation can be written as $y = -3 + g(x+1)$ where $g(x) = 2^x$. Hence the graph is obtained from $y = 2^x$ by successively performing the following transformations:

- translation by 1 in the negative x-direction, which gives the graph of $y = 2^{x+1}$
- translation by 3 in the negative y-direction, which gives the graph of $y = 2^{x+1} - 3$.

These transformations shift the asymptote to $y = -3$.

Substituting $x = 0$ gives a y-intercept of $-3 + 2 = -1$. Substituting $y = 0$ gives $2^{x+1} = 3$, so that $x = \log_2 3 - 1$.

This is enough information to sketch the graph.



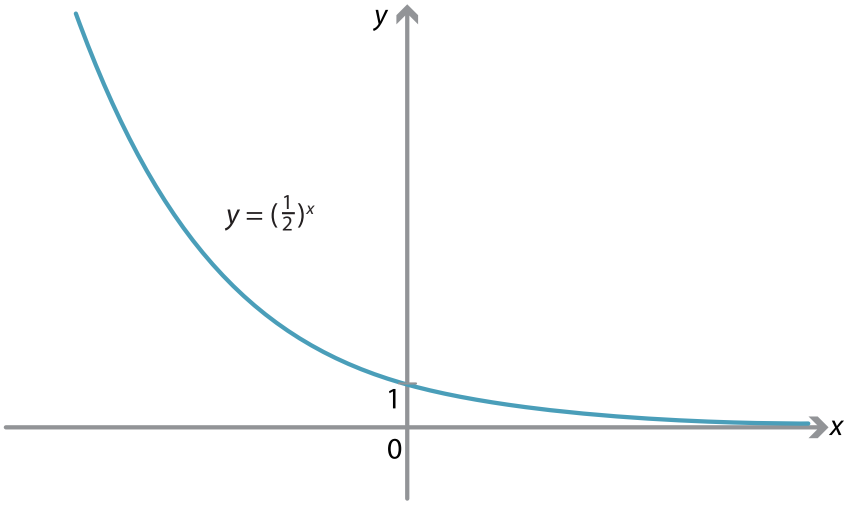
Note that, using the change of base rule, we can alternatively write the x-intercept of the graph as

$$\log_2 3 - 1 = \frac{\log_e 3}{\log_e 2} - 1.$$

Exercise 14

Sketch the graph of $y = 5 - 4 \cdot 3^{2x+1}$.

When $0 < a < 1$, the graph $y = f(x)$ of the function $f(x) = a^x$ has a similar shape as for the case $a > 1$, but now $f(x) \to 0$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.



Graph of $f(x) = \left(\frac{1}{2}\right)^x$.

Exercise 15

a Explain why the graph of $y = \left(\frac{1}{2}\right)^x$ is obtained from the graph of $y = 2^x$ by reflecting in the y-axis.

b Hence sketch the graph of $y = 3 - \left(\frac{1}{2}\right)^{x+1}$.

Exercise 16

Consider the graph $y = 3^x$. Explain why the following two transformations on the graph have the same effect:

- dilation in the y-direction from the x-axis with factor 9
- translation by 2 units to the left (that is, in the negative x-direction).

Graphing logarithmic functions

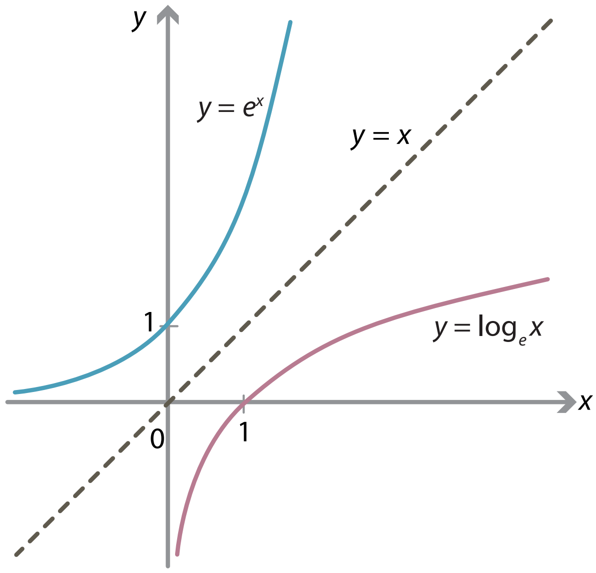
For any $a > 1$, the functions $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions, since

$$f(g(x)) = a^{\log_a x} = x$$

and

$$g(f(x)) = \log_a(a^x) = x.$$

The graphs of $y = f(x)$ and $y = g(x)$ are therefore obtained from each other by reflection in the line $y = x$.



The symmetry between the graphs of $y = e^x$ and $y = \log_e x$.

The graph of the logarithmic function $\log_a x$, for any $a > 1$, has the y-axis as a vertical asymptote, but has no critical points. Similarly, any other graph obtained from this graph by dilations, reflections in the axes and translations also has a vertical asymptote and no critical points.

Note. The graph of a logarithmic function does *not* have a horizontal asymptote. As x becomes large positive, the graph of $y = \log_a x$ becomes very flat: the derivative

$$\frac{dy}{dx} = \frac{1}{x \log_e a}$$

approaches 0. Thus, $\log_a x$ increases very slowly. However $\log_a x$ still increases all the way to infinity:

$$\log_a x \to \infty \text{ as } x \to \infty.$$

One way to see this is to look at the symmetry in the graph above. The graph $y = a^x$ is defined for all values of x, but when x is large the corresponding value of y is very large. By symmetry, $y = \log_a x$ takes all values of y, but when y is large the corresponding value of x is very large.

Exercise 17

Prove directly that $\log_a x$ can take an arbitrarily large value. Consider a large positive number N. Does $\log_a x$ ever take the value N?

Exercise 18

Show that there is a point on the graph of $y = \log_e x$ such that, to go one unit upwards along the graph, you have to go a million units to the right.

That is, find x and y such that

$$y = \log_e x \qquad \text{and} \qquad y + 1 = \log_e(x + 10^6).$$

Example

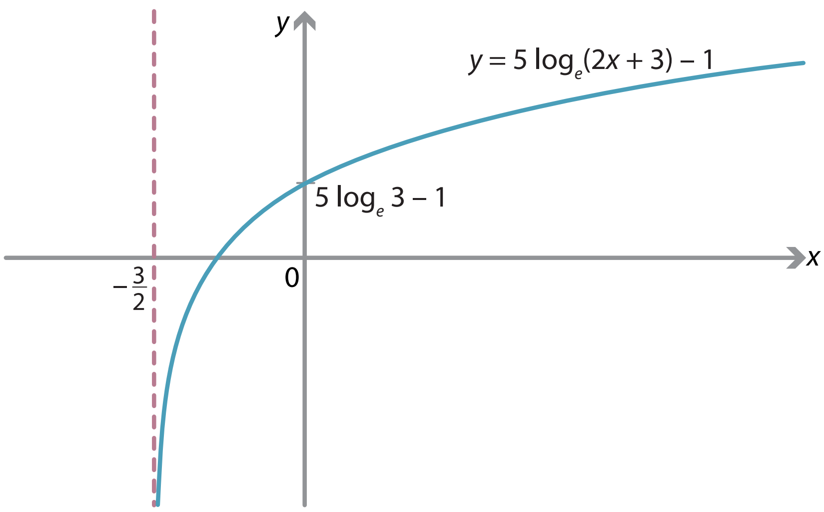
Draw the graph of $y = 5\log_e(2x + 3) - 1$.

Solution

The equation can be written as $y = 5f\big(2(x + \frac{3}{2})\big) - 1$, where $f(x) = \log_e x$. Therefore, the graph is obtained from the graph of $y = \log_e x$ by successively applying the following transformations:

- dilation by a factor of $\frac{1}{2}$ in the x-direction from the y-axis, which gives the graph of $y = \log_e(2x)$
- dilation by a factor of 5 in the y-direction from the x-axis, which gives $y = 5\log_e(2x)$
- translation by $\frac{3}{2}$ in the negative x-direction, giving $y = 5\log_e\big(2(x + \frac{3}{2})\big) = 5\log_e(2x + 3)$
- translation by 1 in the negative y-direction, which gives $y = 5\log_e(2x + 3) - 1$.

These transformations produce a vertical asymptote at $x = -\frac{3}{2}$. There are no critical points.



Exercise 19

Consider the graph $y = \log_3 x$. Explain why the following two transformations of the graph have the same effect:

- dilation in the x-direction from the y-axis with factor 9
- translation by 2 units down (that is, in the negative y-direction).

Relate your answer to exercise 16.

A rigorous approach to logarithms and exponentials

What does an exponential mean anyway?

Throughout this module, we've assumed that functions like $f(x) = 2^x$ are defined for all real numbers x. But are they really?

There's no problem defining 2^x when x is a positive integer; this just means repeated multiplication and is certainly well defined.

There's also no problem when x is a negative integer, using the index law

$$2^{-n} = \frac{1}{2^n}.$$

For example, $2^{-3} = \frac{1}{8}$.

Nor is there a problem when x is a *rational number*. Using the index laws again,

$$2^{\frac{a}{b}} = (2^a)^{\frac{1}{b}} = \sqrt[b]{2^a}.$$

For example, $2^{\frac{3}{2}} = \sqrt{2^3} = \sqrt{8} = 2\sqrt{2}$.

However, it's not so clear what to do when x is *irrational*. What does $2^{\sqrt{2}}$ mean? So far in the module, this issue has been quietly suppressed.

One approach we might use is *continuity*. We could take a sequence of rational numbers $r_1, r_2, r_3, \dots$ which approach $\sqrt{2}$, and consider $2^{r_1}, 2^{r_2}, 2^{r_3}, \dots$. If these numbers approach a limit, then we can call that limit $2^{\sqrt{2}}$.

In any case, it's not a trivial matter to define exponential functions like 2^x for irrational x.

One way to avoid all of the difficulties is to develop the entire story a different way, starting with the logarithmic function. As discussed in the *Motivation* section, this approach may appear less natural but is more rigorous and abstract.

The natural logarithm, rigorously

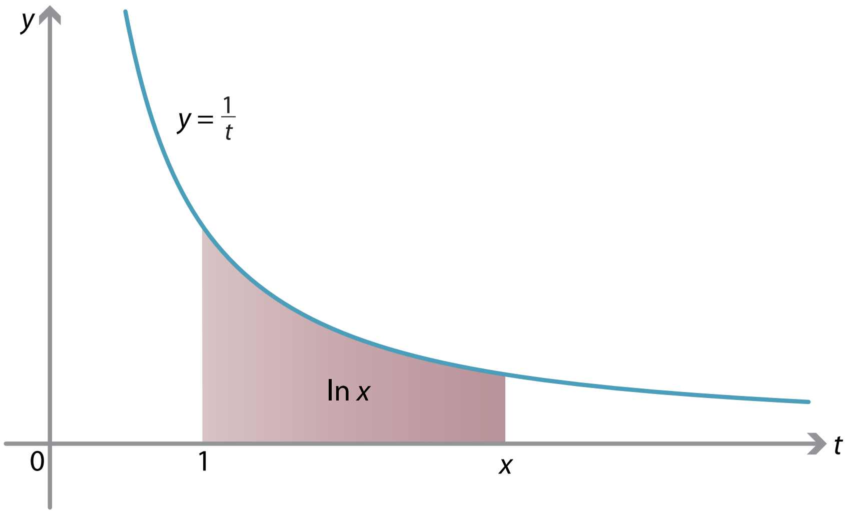
We begin by defining the *natural logarithm* as an *integral*.

Definition

For any real number $x > 0$,

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

This equation was exercise 11. It is now a *definition*.



Definition of the natural logarithm as an integral.

As an aside, note the standard fact that the integral of t^n is

$$\frac{1}{n+1}\, t^{n+1}.$$

This is true for any $n \neq -1$. The integral above is of t^n when $n = -1$, precisely the value of n for which this standard formula does not apply.

From this definition, we can see immediately that $\ln 1 = 0$. We can also see, using the fundamental theorem of calculus, that the derivative of $\ln x$ is $\frac{1}{x}$. (We refer to the module *Integration* for the details). So the function $\ln x$ is increasing, for all $x > 0$, as its gradient $\frac{1}{x}$ is positive. (This can also be seen from the diagram above, where $\ln x$ is shown as a signed area.) It now follows that $\ln x < 0$, for $x \in (0, 1)$, and $\ln x > 0$, for $x \in (1, \infty)$. It is clear that $\ln x$ is a continuous function, and it's not too difficult to show that $\ln x \to +\infty$ as $x \to \infty$, and that $\ln x \to -\infty$ as $x \to 0$.

Our newly defined function $\ln x$ and our previously defined function $\log_e x$ both have the same derivative $\frac{1}{x}$. Since $\ln 1 = 0 = \log_e 1$, it follows that $\ln x$ and $\log_e x$ are in fact the same function — that is, if you manage to overcome all the difficulties with our previous definition of $\log_e x$ and arrive at a well-defined function.

From the definition, it's not clear that the new function $\ln x$ behaves like a logarithm at all. However, we will now show directly that this new function obeys the logarithm laws. We use a similar method to exercise 10.

Take a positive number y, considered as a constant, and differentiate the two functions $f(x) = \ln(xy)$ and $g(x) = \ln x + \ln y$. We obtain

$$f'(x) = \frac{y}{xy} = \frac{1}{x} \qquad \text{and} \qquad g'(x) = \frac{1}{x},$$

so $f'(x) = g'(x)$, and therefore $f(x), g(x)$ differ by a constant. As $f(1) = \ln y = g(1)$, it follows that $f(x) = g(x)$. This shows that

$$\ln(xy) = \ln x + \ln y,$$

and so proves one of the logarithm laws.

Using a similar method, we can show that

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

and

$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

Next, consider the two functions $f(x) = \ln(x^n)$ and $g(x) = n \ln x$, where n is any rational number.[3] Differentiating these functions gives

$$f'(x) = \frac{1}{x^n} \cdot nx^{n-1} = \frac{n}{x} \qquad \text{and} \qquad g'(x) = \frac{n}{x}.$$

As f and g have the same derivative, they must differ by a constant. We further have $f(1) = 0 = g(1)$, so $f(x) = g(x)$ and we have proved the logarithm law

$$\ln(x^n) = n \ln x.$$

As our new function $\ln x$ obeys the familiar logarithm laws, we are justified in calling it a logarithm!

---

[3] Recall we said before that we only really know how to take rational powers.

Exponentials, rigorously

Having established our new version of the natural logarithm function, we now turn to exponentials.

The function $\ln x$ mapping $(0,\infty)$ to $\mathbb{R}$ is a continuous function, strictly increasing from $-\infty$ to $+\infty$, and hence has an inverse function. This inverse function has domain $\mathbb{R}$ and range $(0,\infty)$. We will, for the moment, call this inverse function $\exp x$. So, by definition,

$$\exp x = \ln^{-1}(x).$$

We then have

$$\exp(\ln x) = x \quad \text{for all positive } x,$$

$$\ln(\exp x) = x \quad \text{for all real } x.$$

We define the number e to be $\exp 1$. So $\exp 1 = e$ and $\ln e = 1$. It will turn out that $\exp x = e^x$; however this is not at all clear from the definition.

We can compute the derivative of $\exp x$, since it is the inverse of $\ln x$, and we know that the derivative of $\ln x$ is $\frac{1}{x}$. Let $y = \exp x$. Then $x = \ln y$ and we have

$$\frac{dx}{dy} = \frac{1}{y},$$

so

$$\frac{d}{dx}\exp x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = y = \exp x.$$

We can also show that exp satisfies the index laws. For instance, we have

$$\begin{aligned}\ln\big(\exp x \cdot \exp y\big) &= \ln(\exp x) + \ln(\exp y)\\ &= x + y\\ &= \ln\big(\exp(x+y)\big).\end{aligned}$$

Here we just used a logarithm law and the fact that ln and exp are inverses. Since the function ln is one-to-one and we have just shown that $\ln\big(\exp x \cdot \exp y\big) = \ln\big(\exp(x+y)\big)$, we conclude the index law

$$\exp x \cdot \exp y = \exp(x+y).$$

Using a similar method, we can show that exp also obeys the index law

$$\exp(x-y) = \frac{\exp x}{\exp y}.$$

For the remaining index law, take a rational number r; we will show $\exp(rx) = (\exp x)^r$. We observe

$$\begin{aligned} \ln\big((\exp x)^r\big) &= r\ln(\exp x) \\ &= rx \\ &= \ln\big(\exp(rx)\big), \end{aligned}$$

where we just used a logarithm law and the fact that ln and exp are inverses. We have $\ln\big((\exp x)^r\big) = \ln\big(\exp(rx)\big)$, and cancelling ln's establishes the index law

$$(\exp x)^r = \exp(rx),$$

for any rational number r. Since $\exp 1 = e$, we then have, for any rational number r,

$$\exp r = \exp(r\cdot 1) = (\exp 1)^r = e^r.$$

We can now use this to *define* irrational powers. We declare for any *real* number x, possibly irrational, that

$$e^x = \exp x.$$

We can go further and use this idea to define any real power a^x of any positive number a. When r is rational, we have from the index law above,

$$\exp(r\ln a) = \big(\exp(\ln a)\big)^r = a^r,$$

and so for any *real* number x we can *define* a^x as follows.

Definition

For any real number x and any $a > 0$,

$$a^x = \exp(x\ln a).$$

It's not difficult to show that a^x varies continuously with a and x.

Exercise 20

Let α be any real number and let $f(x) = x^\alpha$, for $x > 0$. Using the above definition for x^α, prove that $f'(x) = \alpha x^{\alpha-1}$.

In this way, we have rigorously defined the functions $\ln x$ and e^x, found their derivatives, established the index and logarithm laws, and rigorously defined irrational powers.

Links forward

A series for e^x

Consider the series

$$f(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

An infinite sum like this, with increasing powers of x, is like an infinite version of a polynomial and known as a **power series**. The denominators are all **factorials**, such as $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. We can write the series succinctly in summation notation, as

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(In the $n = 0$ term, we set $x^0 = 1$ and follow the convention that $0! = 1$.) Substituting $x = 0$ gives $f(0) = 1$. It turns out that, for any x, this series converges, and the function f is continuous and differentiable. And we can see that, if we differentiate the series term-by-term, we obtain ... the same series!

$$\begin{aligned} f'(x) &= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x). \end{aligned}$$

Hence the series $f(x)$ converges to e^x. In particular, substituting $x = 1$, we obtain an amazing formula for e:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Derangements and e

Suppose that a group of friends play a gift-giving game for the festive season. Let the number of friends be n. Each person writes their name down, and the names go into a hat. Each person in turn then takes a name out of the hat, until everyone has taken a name out and the hat is empty. You give a gift to the person whose name you pull out of the hat.

Of course, it won't do to give a gift to yourself, so we hope nobody pulls out their own name! What are the chances of this happening?

Obviously, the answer might depend on n. As it turns out, as n becomes large, the answer approaches $\frac{1}{e}$.

This question can be expressed as a problem about permutations. Number the people in the group from 1 to n. The function assigning each person to the person whose name they pull out of the hat is a one-to-one function

$$f\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\},$$

also known as a **permutation**. If person i pulls their own name out of the hat, then $f(i) = i$, and i is said to be a **fixed point** of the permutation.

A good permutation for the purposes of gift-giving is one with no fixed points. A permutation with no fixed points is called a **derangement**. Our question is really asking what fraction of permutations are derangements.

The number of permutations of the n friends is not difficult to calculate. The first person can pick n possible names out of the hat, the second person $n-1$ names, and so on. The number of permutations is $n!$.

We can count the number of derangements D_n as follows, using a technique known as the **principle of inclusion–exclusion**.

Starting from all $n!$ permutations, we exclude those which have a fixed point. There are n possible people who could be a fixed point, and once there is a fixed point, the remaining $n-1$ people can be permuted in $(n-1)!$ ways. So we subtract $n\cdot(n-1)!$.

But, in excluding those permutations, we have excluded some permutations more than once. Permutations with *two* fixed points will have been subtracted twice. So, for each pair of people, we add back on the number of permutations which have them both as fixed points. There are $\binom{n}{2}$ pairs of people, and the remaining $n-2$ people can be permuted in $(n-2)!$ ways. So we add back on $\binom{n}{2}(n-2)!$.

However, now permutations with *three* fixed points will have been added back on too many times, and have been counted when they shouldn't be. So, for every group of three people, we subtract the number of permutations which have them all as fixed points. There are $\binom{n}{3}$ triples of people, and the remaining $n-3$ people can be permuted in $(n-3)!$ ways, so we subtract off $\binom{n}{3}(n-3)!$.

Continuing in this way, we find that the number of derangements of the n people is

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \binom{n}{4}(n-4)! - \cdots + (-1)^n\binom{n}{n}0!.$$

(The sign of the last term depends on n, and following the usual convention we set $0! = 1$.)

Using the formula $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, we can simplify this expression to

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}.$$

Therefore, the fraction of permutations which are derangements is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}.$$

If we remember the formula for e^x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

then we see that, as $n \to \infty$,

$$\frac{D_n}{n!} \to e^{-1} = \frac{1}{e}.$$

So, with a large enough group of friends, the probability of good gift-giving is $\dfrac{1}{e}$.

Striking it lucky with e

You're digging for gold on the goldfields. It's a plentiful gold field, and every time you dig, you obtain an amount of gold between 0 and 1 gram. The amount of gold you find each time is random and uniformly distributed. How many times do you expect to have to dig before you obtain a total of 1 gram of gold?

You might think that the answer is twice! Actually, the expected number of digs is e.

To put the question in pure mathematical terms, the amount of gold (in grams) you obtain each time you dig is a random variable uniformly distributed in $[0,1]$. Let X_1 be the amount you obtain the first time you dig, let X_2 be the amount the second time you dig and, in general, let X_n be the amount of gold found the nth time you dig. So each X_n is a random variable uniformly distributed in $[0,1]$.

The number of times you have to dig to obtain 1 gram is then the least n such that $X_1 + X_2 + \cdots + X_n \geq 1$. Let this number be M. It is also a random variable:

$$M = \min\{n : X_1 + X_2 + \cdots + X_n \geq 1\}.$$

The number of times you expect to have to dig to obtain 1 gram of gold is the expected value $E(M)$ of M.

The probability that you obtain a full gram in your first dig is zero — you have to get a whole gram, and the chances of that are vanishingly small: $\Pr(M=1)=0$.

The probability you obtain a gram in two digs is $\frac{1}{2}$: you could get anywhere from 0 to 2 grams, and 1 is right in the middle.

In general, the probability that you obtain a gram on the nth dig, but not earlier, can be expressed as follows. It's the probability that you have less than a gram after $n-1$ digs, but are not still below a gram after n digs:

$$\begin{aligned}\Pr(M=n) &= \Pr(X_1+\cdots+X_{n-1}<1 \text{ and } X_1+\cdots+X_n \geq 1)\\ &= \Pr(X_1+\cdots+X_{n-1}<1)-\Pr(X_1+\cdots+X_n<1).\end{aligned}$$

As it turns out,[4] the probability that you have less than a gram after n digs is

$$\Pr(X_1+\cdots+X_n<1)=\frac{1}{n!}.$$

Proving this requires some work with integration and probability distribution functions.

From this, we obtain

$$\Pr(M=n)=\frac{1}{(n-1)!}-\frac{1}{n!}.$$

Thus $\Pr(M=1)=0$ and, for $n>1$, we can simplify to

$$\begin{aligned}\Pr(M=n) &= \frac{1}{(n-1)!}\left(1-\frac{1}{n}\right)\\ &= \frac{1}{(n-1)!}\cdot\frac{n-1}{n}\\ &= \frac{1}{n}\cdot\frac{1}{(n-2)!}.\end{aligned}$$

Knowing this probability, we see that the expected number of digs required is

$$\begin{aligned}E(M) &= 1\Pr(M=1)+2\Pr(M=2)+3\Pr(M=3)+4\Pr(M=4)+\cdots\\ &= 1\cdot 0+2\cdot\frac{1}{2}\cdot\frac{1}{0!}+3\cdot\frac{1}{3}\cdot\frac{1}{1!}+4\cdot\frac{1}{4}\cdot\frac{1}{2!}+\cdots\\ &= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots=e.\end{aligned}$$

We have yet to hear of any goldfields with such a distribution of gold.

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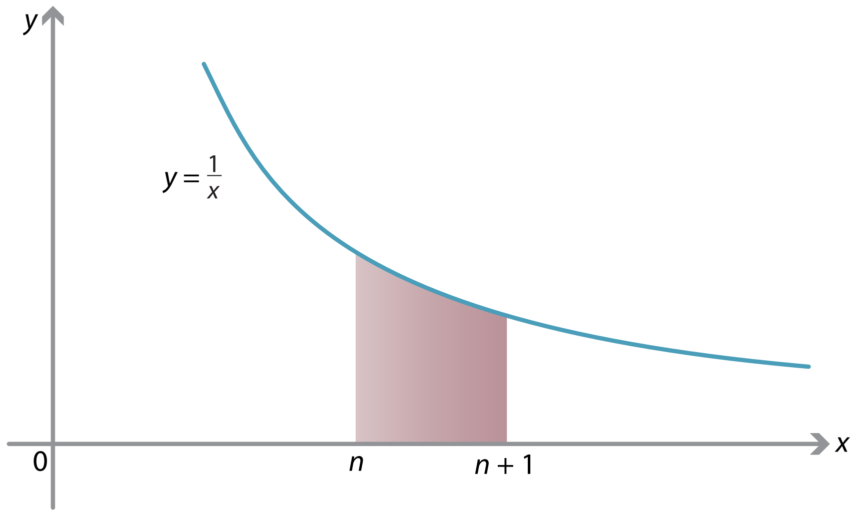
[4] Details may be found in K. G. Russell, 'Estimating the value of e by simulation', *The American Statistician*, Vol. 45, No. 1 (February 1991), pp. 66–68.

Sandwiching e

By considering the integral

$$\int_n^{n+1} \frac{1}{x}\,dx,$$

where n is positive, we will obtain another amazing formula for e.



On the one hand, we can compute the integral exactly:

$$\begin{aligned}\int_n^{n+1} \frac{1}{x}\,dx &= \big[\ln x\big]_n^{n+1} \\ &= \ln(n+1) - \ln n \\ &= \ln \frac{n+1}{n} \\ &= \ln\Big(1 + \frac{1}{n}\Big).\end{aligned}$$

On the other hand, on the interval $[n, n+1]$, we see that the function $\frac{1}{x}$ satisfies

$$\frac{1}{n+1} \le \frac{1}{x} \le \frac{1}{n}.$$

Using these inequalities to estimate the integral (see the module *Integration* for details), we have

$$\frac{1}{n+1} \le \int_n^{n+1} \frac{1}{x}\,dx = \ln\Big(1 + \frac{1}{n}\Big) \le \frac{1}{n}.$$

Multiplying through by n and using a logarithm law gives

$$\frac{n}{n+1} \leq n \ln\left(1+\frac{1}{n}\right) = \ln\left(1+\frac{1}{n}\right)^n \leq 1.$$

The expression $\ln(1+\frac{1}{n})^n$ is sandwiched between $\frac{n}{n+1}$ and 1. In fact, as $n \to \infty$, $\frac{n}{n+1} \to 1$ as well, and so we can deduce

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1.$$

It follows that the limit of $(1+\frac{1}{n})^n$, as $n \to \infty$, is a number whose natural logarithm is 1, that is, e. We have our new formula for e:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

Exercise 21

Using a similar technique, prove that for any real x,

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

History and applications

Bernoulli, compound interest and Euler

Although some related ideas appeared in earlier works, arguably the first 'discovery' of the number e was made by Jacob Bernoulli in 1683, in considering compound interest.

Suppose that a bank offers to take your money, and will give you 100% interest on it in a year's time. So if you give the bank \$1000 now, you will get \$2000 back in a year. (This bank pays well!)

Alternatively, the bank says, it will give you half that interest rate, but it will give you the interest twice as often: so it will give you 50% interest in half a year's time, and then a further 50% on that at the end of the year. Your \$1000 will become \$1500 after half a year, and then \$2250 at the end of the year. By calculating half the interest twice as often, you end up with significantly more at the end.

Now the bank says, it can give you a third of the interest rate, but three times as often. So, $33\frac{1}{3}$% interest is paid out three times throughout the year. Your \$1000 becomes \$1333.33, then \$1777.78, then \$2370.37 (to the nearest cent) at the end of the year. You end up with more again.

Continuing on in this fashion, the bank might calculate one fourth, one fifth, etc. the interest, four, five, etc. times as often. In general, it might offer you an interest rate of $\frac{100}{n}$%, calculated n times a year. You money is multiplied by

$$1+\frac{1}{n},$$

n times throughout the year. That is, your money is multiplied, overall, by

$$\left(1+\frac{1}{n}\right)^n.$$

We saw in the section *Sandwiching e* that, as n approaches infinity (one billionth the interest rate, one billion times a year!), this amount approaches none other than e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Thus, in the limit of *continuously compounded interest*, your \$1000 after a year becomes \$1000 × e, approximately \$2718.28.

In general, consider an interest rate of x (so $x = 1$ for our interest rate of 100% above). Again consider an interest rate of $\frac{x}{2}$, calculated twice as often; or an interest rate of $\frac{x}{3}$, calculated three times as often; and, in general, an interest rate of $\frac{x}{n}$, calculated n times as often. We obtain, in the limit, that our money is multiplied by

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

each year. As exercise 21 shows, this limit turns out to be e^x.

Although some of the above ideas were discussed by Bernoulli, the notation e was not used until much later. Its first known appearance is in a letter of Euler from 1731.

Today the notation e is often associated with Euler's name, and e is sometimes called Euler's number. (This can be a problem, since there is another mathematical constant $\gamma \approx 0.5772$ which often goes by the name of Euler's constant.) However, when e was first used it was not meant to refer to Euler — there is no reason to believe Euler used the letter in honour of his own name.

Appendix

Is e rational?

It turns out that e is irrational. We can give a proof of this fact.

Suppose to the contrary that e can be written as a fraction, so

$$e = \frac{a}{b},$$

where a and b are positive integers. Using the series for e, we then have

$$e = \frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

The key idea of the proof is to consider this series *up to the term* $\frac{1}{b!}$. So we think of the above sum as

$$\frac{a}{b} = \Big(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{b!}\Big) + \Big(\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \cdots\Big).$$

Now multiply through by $b!$. This gives

$$b!\,\frac{a}{b} = \Big(b! + \frac{b!}{1!} + \frac{b!}{2!} + \cdots + \frac{b!}{b!}\Big) + \Big(\frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots\Big),$$

and so

$$(b-1)!\,a = \Big(b! + \frac{b!}{1!} + \cdots + \frac{b!}{b!}\Big) + \Big(\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots\Big).$$

All the terms are positive. The left-hand side is an integer, and all the terms in the first bracket on the right-hand side are also integers, as the denominators cancel against $b!$. Therefore, the sum x of the terms in the second bracket must be a positive integer.

However, all the terms in the second bracket are rather small. The denominators $(b+1)$, $(b+1)(b+2)$, $(b+1)(b+2)(b+3)$ increase very quickly. So x is a small positive integer ... suspiciously small. Noting that $(b+1)(b+2)\cdots(b+k) > (b+1)^k$, for all $k \geq 2$, we can estimate x as

$$\begin{aligned} x &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \\ &< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots . \end{aligned}$$

This is a geometric series, and its sum is

$$\frac{\frac{1}{b+1}}{1 - \frac{1}{b+1}} = \frac{1}{b+1} \cdot \frac{b+1}{b} = \frac{1}{b}.$$

So x is a positive integer with $x < \dfrac{1}{b} \le 1$. There are not many positive integers less than one! This is a contradiction, and so our initial assumption must have been wrong. We conclude that e is irrational.

In fact, it can be shown that, like π, the number e is **transcendental**: it is not the root of any polynomial with rational coefficients.

Answers to exercises

Exercise 1

If $1 \le a < b$ and $h > 0$, then $a^h < b^h$. Subtracting 1 from both sides and dividing by h (which is positive) gives the first inequality. This inequality holds for all $h > 0$, hence taking a limit as $h \to 0$, we have the desired inequality. (Note that, in the limit, the strict inequality becomes non-strict.)

Exercise 2

a Using the product rule, $f'(x) = 2xe^x + x^2e^x = (2x + x^2)e^x$.

b Using the chain rule, $f'(x) = e^{e^x} \cdot e^x = e^{x+e^x}$.

Exercise 3

Differentiating both sides gives

$$\begin{aligned} 1 &= e^{\log_e x} \cdot \frac{d}{dx} \log_e x \\ &= x \cdot \frac{d}{dx} \log_e x. \end{aligned}$$

It follows that $\dfrac{d}{dx} \log_e x = \dfrac{1}{x}$.

Exercise 4

Using the chain rule,

$$f'(x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

As f has domain $(-\infty, 0)$ and f' is defined on this entire interval, the function f' also has domain $(-\infty, 0)$.

Exercise 5

The function $f(x)$ is defined when $3 - 7x > 0$, i.e., $x < \frac{3}{7}$. So the domain is $(-\infty, \frac{3}{7})$.

Exercise 6

a Writing $f(x) = e^{\log_e 2 \cdot x^2}$ and using the chain rule gives

$$\begin{aligned} f'(x) &= \log_e 2 \cdot 2x \cdot e^{\log_e 2 \cdot x^2} \\ &= \log_e 2 \cdot x \cdot 2^{x^2+1}. \end{aligned}$$

b Writing $f(x) = e^{\log_e 3 \cdot (4x^2+2x-7)}$ and using the chain rule gives

$$\begin{aligned} f'(x) &= \log_e 3 \cdot (8x+2) \cdot e^{\log_e 3 \cdot (4x^2+2x-7)} \\ &= \log_e 3 \cdot (8x+2) \cdot 3^{4x^2+2x-7}. \end{aligned}$$

Exercise 7

From first principles, if $f(x) = 2^x$, then

$$f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

We have now shown that $f'(x) = \log_e 2 \cdot 2^x$, so $f'(0) = \log_e 2$. The desired equality follows.

Exercise 8

From first principles, if $f(x) = a^x$, then

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Knowing now that $f'(x) = \log_e a \cdot a^x$, and so $f'(0) = \log_e a$, gives the desired equality.

Exercise 9

Writing

$$f(x) = x^x = \left(e^{\log_e x}\right)^x = e^{x \log_e x}$$

and using the chain and product rules, we obtain

$$\begin{aligned} f'(x) &= e^{x \log_e x} \cdot \frac{d}{dx}(x \log_e x) \\ &= x^x \left(x \cdot \frac{1}{x} + 1 \cdot \log_e x\right) \\ &= x^x (1 + \log_e x). \end{aligned}$$

The function $f(x)$ is only defined for $x > 0$, so $x^x \neq 0$. Thus the stationary points occur precisely when $\log_e x = -1$, that is, $x = e^{-1} = \frac{1}{e}$.

Exercise 10

a First, using the change of base rule, we have

$$f(x) = \frac{\log_e(xy)}{\log_e a} \qquad \text{and} \qquad g(x) = \frac{\log_e x}{\log_e a} + \frac{\log_e y}{\log_e a}.$$

So, using the chain rule (and thinking of y as a constant), we compute

$$f'(x) = \frac{1}{\log_e a} \cdot \frac{1}{xy} \cdot y = \frac{1}{x \log_e a}$$

and

$$g'(x) = \frac{1}{\log_e a} \cdot \frac{1}{x} = \frac{1}{x \log_e a}.$$

Hence $f'(x) = g'(x)$.

b Since $\log_a 1 = 0$, we compute $f(1) = \log_a y$ and $g(1) = \log_a y$.

c As the functions f and g have identical derivatives, they are equal up to a constant. Since $f(1) = g(1)$, it follows that $f(x)$ and $g(x)$ must be equal for all x.

Exercise 11

$$\begin{aligned}\int_1^x \frac{1}{t}\,dt &= \big[\log_e t\big]_1^x \\ &= \log_e x - \log_e 1 = \log_e x\end{aligned}$$

Exercise 12

$$\begin{aligned}\int_{x^n}^{x^{n+1}} \frac{1}{t}\,dt &= \big[\log_e t\big]_{x^n}^{x^{n+1}} \\ &= \log_e\big(x^{n+1}\big) - \log_e\big(x^n\big) = \log_e\Big(\frac{x^{n+1}}{x^n}\Big) = \log_e x\end{aligned}$$

Exercise 13

Using the product rule, we compute

$$f'(x) = 1 \cdot \log_e x + x \cdot \frac{1}{x} - 1 = \log_e x.$$

Thus $f(x)$ is an antiderivative of $\log_e x$, and we have

$$\int \log_e x\,dx = x \log_e x - x + c,$$

where c is a constant of integration.

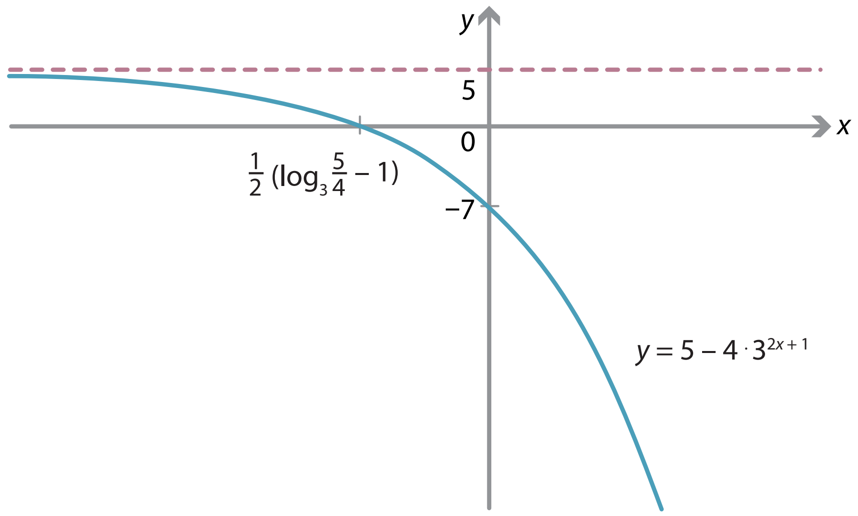
Exercise 14

Substituting $x = 0$ gives a y-intercept of $5 - 4 \cdot 3 = -7$. Substituting $y = 0$, we obtain $0 = 5 - 4 \cdot 3^{2x+1}$, and so $3^{2x+1} = \frac{5}{4}$. Hence the x-intercept is $\frac{1}{2}(\log_3 \frac{5}{4} - 1) \approx -0.4$.

The equation $y = 5 - 4 \cdot 3^{2x+1}$ can be written as $y = -4g\big(2(x + \frac{1}{2})\big) + 5$, where $g(x) = 3^x$. Hence the graph here is obtained from the graph of $y = 3^x$ by applying the following transformations:

- dilation by a factor of $\frac{1}{2}$ in the x-direction from the y-axis, giving $y = e^{2x}$
- dilation by a factor of 4 in the y-direction from the x-axis, giving $y = 4e^{2x}$
- reflection in the x-axis, giving $y = -4e^{2x}$
- translation by $\frac{1}{2}$ in the negative x-direction, giving $y = -4e^{2(x+\frac{1}{2})}$
- translation by 5 in the positive y-direction, giving $y = -4e^{2(x+\frac{1}{2})} + 5 = 5 - 4e^{2x+1}$.

Under these transformations, the asymptote moves to $y = 5$.



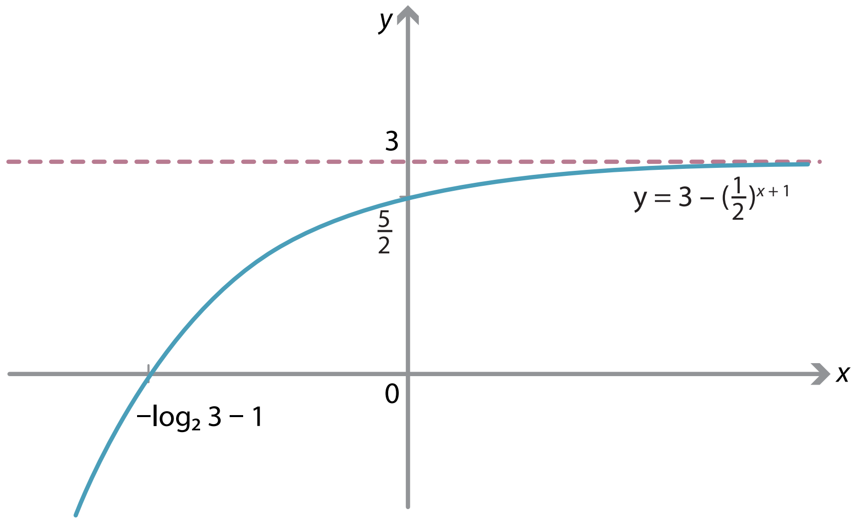
Exercise 15

a Reflecting the graph of $y = 2^x$ in the y-axis gives the graph of $y = 2^{-x}$. By the index laws, we see that $2^{-x} = \left(\frac{1}{2}\right)^x$.

b The equation $y = 3 - \left(\frac{1}{2}\right)^{x+1}$ can be written as $y = 3 - g\big(-(x+1)\big)$, where $g(x) = 2^x$. Hence the graph is obtained from that of $y = 2^x$ by the following transformations:

- reflection in the y-axis, giving $y = \left(\frac{1}{2}\right)^x$
- reflection in the x-axis, giving $y = -\left(\frac{1}{2}\right)^x$
- translation by 1 in the negative x-direction, giving $y = -\left(\frac{1}{2}\right)^{x+1}$
- translation by 3 in the positive y-direction, giving $y = 3 - \left(\frac{1}{2}\right)^{x+1}$.

Under these transformations, the asymptote moves to $y = 3$.



Exercise 16

Dilating the graph $y = 3^x$ in the y-direction from the x-axis with factor 9 gives the graph $y = 9 \cdot 3^x$. Translating the graph $y = 3^x$ by 2 units to the left gives the graph $y = 3^{x+2}$. As $3^{x+2} = 3^2 \cdot 3^x = 9 \cdot 3^x$, these two graphs are identical.

Exercise 17

To solve the equation $\log_a x = N$ is as simple as rewriting it as $x = a^N$. So, yes, $\log_a x$ does take the value N, for the (very large) value of $x = a^N$.

Exercise 18

Subtracting the first equation from the second gives $\log_e(x + 10^6) - \log_e x = 1$. Using a logarithm law, this simplifies to

$$\log_e\Big(\frac{x + 10^6}{x}\Big) = 1 \implies 1 + \frac{10^6}{x} = e.$$

Therefore $x = \dfrac{10^6}{e-1}$ and $y = \log_e x = \log_e\Big(\dfrac{10^6}{e-1}\Big)$.

Exercise 19

Dilating the graph $y = \log_3 x$ in the x-direction from the y-axis with factor 9 gives the graph $y = \log_3(\frac{x}{9})$. Translating the graph $y = \log_3 x$ down by 2 units gives the graph $y = \log_3 x - 2$. By the logarithm laws, we have

$$\log_3\Big(\frac{x}{9}\Big) = \log_3 x - \log_3 9 = \log_3 x - 2,$$

so these two graphs are identical.

The equation $y = \log_3 x$ is equivalent to $x = 3^y$; the function $f(x) = \log_3 x$ is the inverse of the function $f^{-1}(x) = 3^x$ considered in exercise 16. One graph is obtained from the other by reflection in the line $y = x$. The two transformations considered here are the mirror images (under reflection in $y = x$) of the two transformations considered in exercise 16.

Exercise 20

By our definition, $f(x) = x^\alpha = \exp(\alpha \ln x)$, and so from the chain rule we obtain

$$f'(x) = \frac{\alpha}{x} \exp(\alpha \ln x) = \frac{\alpha}{x} \cdot x^\alpha = \alpha x^{\alpha-1}.$$

Note. In the final step of the calculation above, we used an index law. We first need to check that the index laws hold for our new definition of powers. (This is easy to do, since we have already established the index laws for exp and the logarithm laws for ln.)

Exercise 21

We will give a proof assuming $x > 0$; the negative case is similar. Consider the integral

$$\int_n^{n+x} \frac{1}{t}\, dt.$$

We can compute this exactly as

$$\begin{aligned}\int_n^{n+x} \frac{1}{t}\, dt &= \big[\ln t\big]_n^{n+x} \\ &= \ln(n+x) - \ln n \\ &= \ln \frac{n+x}{n} \\ &= \ln\Big(1 + \frac{x}{n}\Big).\end{aligned}$$

On the other hand, we can bound $\dfrac{1}{t}$ on the interval $[n, n+x]$ by

$$\frac{1}{n+x} \le \frac{1}{t} \le \frac{1}{n}.$$

Estimating the integral then gives

$$\frac{x}{n+x} \le \ln\Big(1 + \frac{x}{n}\Big) \le \frac{x}{n}.$$

Multiplying through by n yields

$$\frac{xn}{n+x} \le \ln\Big(1 + \frac{x}{n}\Big)^n \le x.$$

As $n \to \infty$, $\dfrac{xn}{n+x} \to x$, so we obtain $\ln\Big(1 + \dfrac{x}{n}\Big)^n \to x$, hence $\Big(1 + \dfrac{x}{n}\Big)^n \to e^x$.

0 1 2 3 4 5 6 7 8 9 10 11 12