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A guide for teachers – Years 11 and 12

Calculus: Module 12

Applications of differentiation

Education Services Australia

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Applications of differentiation – A guide for teachers (Years 11–12)

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Applications of differentiation

Assumed knowledge

The content of the modules:

- *Functions I*
- *Polynomials*
- *Limits and continuity*
- *Introduction to differential calculus*
- *The calculus of trigonometric functions*
- *Exponential and logarithmic functions.*

Motivation

Calculus is used in pure and applied mathematics, the physical sciences, the biological and medical sciences, computer science, statistics, engineering, economics, and in many other areas.

Some simple examples of applications in biology and physics are given in the *History and applications* section of this module. Further applications to other fields are discussed in the modules *The calculus of trigonometric functions*, *Exponential and logarithmic functions*, *Growth and decay* and *Motion in a straight line*.

In this module, we consider three topics:

- graph sketching
- maxima and minima problems
- related rates.

We will mainly focus on nicely behaved functions which are differentiable at each point of their domains. Some of the examples are very straightforward, while others are more difficult and require technical skills to arrive at a solution.

Content

Graph sketching

Increasing and decreasing functions

Let f be some function defined on an interval.

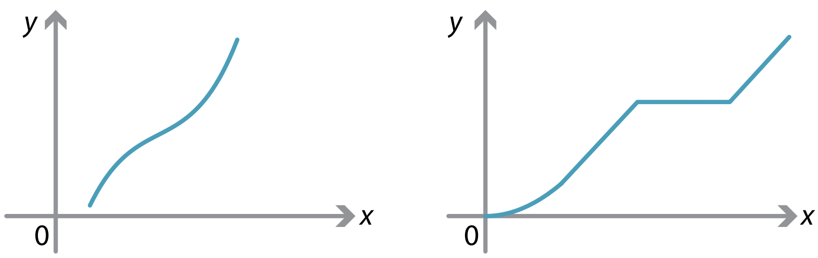
Definition

The function f is **increasing** over this interval if, for all points x_1 and x_2 in the interval,

$$x_1 \le x_2 \implies f(x_1) \le f(x_2).$$

This means that the value of the function at a larger number is greater than or equal to the value of the function at a smaller number.

The graph on the left shows a differentiable function. The graph on the right shows a piecewise-defined continuous function. Both these functions are increasing.



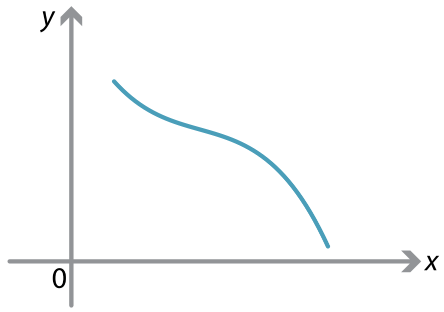
Examples of increasing functions.

Definition

The function f is **decreasing** over this interval if, for all points x_1 and x_2 in the interval,

$$x_1 \leq x_2 \implies f(x_1) \geq f(x_2).$$

The following graph shows an example of a decreasing function.



Example of a decreasing function.

Note that a function that is constant on the interval is both increasing and decreasing over this interval. If we want to exclude such cases, then we omit the equality component in our definition, and we add the word *strictly*:

- A function is **strictly increasing** if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- A function is **strictly decreasing** if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

We will use the following results. These results refer to intervals where the function is differentiable. Issues such as endpoints have to be treated separately.

- If $f'(x) > 0$ for all x in the interval, then the function f is strictly increasing.
- If $f'(x) < 0$ for all x in the interval, then the function f is strictly decreasing.
- If $f'(x) = 0$ for all x in the interval, then the function f is constant.

Stationary points

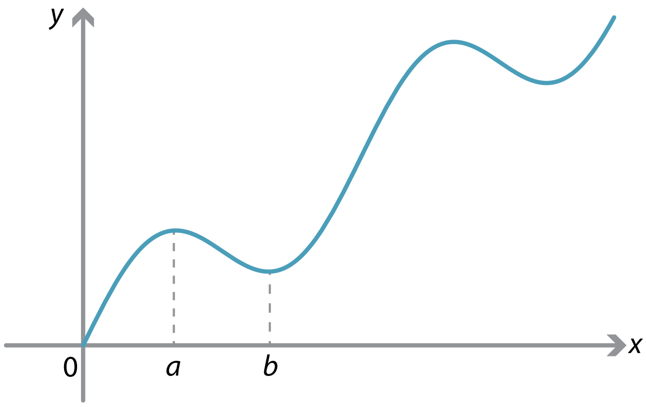
Definitions

Let f be a differentiable function.

- A **stationary point** of f is a number x such that $f'(x) = 0$.
- The point c is a **maximum point** of the function f if and only if $f(c) \geq f(x)$, for all x in the domain of f. The value $f(c)$ of the function at c is called the **maximum value** of the function.
- The point c is a **minimum point** of the function f if and only if $f(c) \leq f(x)$, for all x in the domain of f. The value $f(c)$ of the function at c is called the **minimum value** of the function.

Local maxima and minima

In the following diagram, the point a looks like a maximum provided we stay close to it, and the point b looks like a minimum provided we stay close to it.

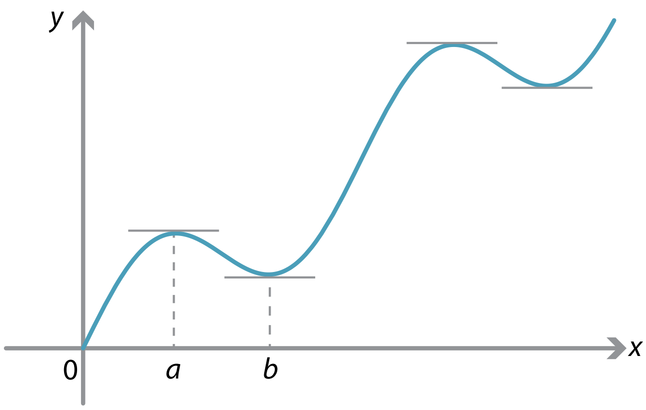


Definitions

- The point c is a **local maximum point** of the function f if there exists an interval (a,b) with $c \in (a,b)$ such that $f(c) \geq f(x)$, for all $x \in (a,b)$.
- The point c is a **local minimum point** of the function f if there exists an interval (a,b) with $c \in (a,b)$ such that $f(c) \leq f(x)$, for all $x \in (a,b)$.

These are sometimes called **relative maximum** and **relative minimum** points. Local maxima and minima are often referred to as **turning points**.

The following diagram shows the graph of $y = f(x)$, where f is a differentiable function. It appears from the diagram that the tangents to the graph at the points which are local maxima or minima are horizontal. That is, at a local maximum or minimum point c, we have $f'(c) = 0$, and hence each local maximum or minimum point is a stationary point.



The result appears graphically obvious, but we will present a formal proof in the case of a local maximum.

Theorem

Let f be a differentiable function. If c is a local maximum point, then $f'(c) = 0$.

Proof

Consider the interval $(c-\delta, c+\delta)$, with $\delta > 0$ chosen so that $f(c) \geq f(x)$ for all $x \in (c-\delta, c+\delta)$.

For all positive h such that $0 < h < \delta$, we have $f(c) \geq f(c+h)$ and therefore

$$\frac{f(c+h)-f(c)}{h} \leq 0.$$

Hence,

$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0. \qquad (1)$$

For all negative h such that $-\delta < h < 0$, we have $f(c) \geq f(c+h)$ and therefore

$$\frac{f(c+h)-f(c)}{h} \geq 0.$$

Hence,

$$f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0. \qquad (2)$$

From (1) and (2), it follows that $f'(c) = 0$. □

The first derivative test for local maxima and minima

The derivative of the function can be used to determine when a local maximum or local minimum occurs.

Theorem (First derivative test)

Let f be a differentiable function. Suppose that c is a stationary point, that is, $f'(c) = 0$.

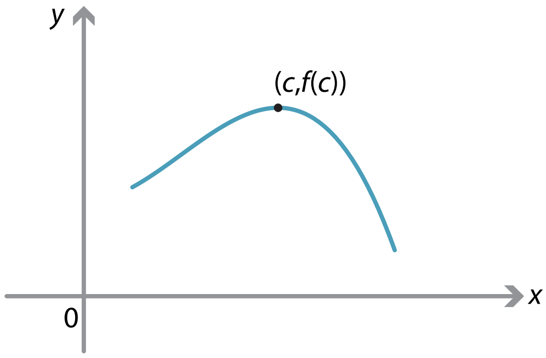
- **a** *If there exists $\delta > 0$ such that $f'(x) > 0$, for all $x \in (c-\delta, c)$, and $f'(x) < 0$, for all $x \in (c, c+\delta)$, then c is a local maximum point.*
- **b** *If there exists $\delta > 0$ such that $f'(x) < 0$, for all $x \in (c-\delta, c)$, and $f'(x) > 0$, for all $x \in (c, c+\delta)$, then c is a local minimum point.*

Proof

- **a** The function is increasing on the interval $(c-\delta, c)$, and decreasing on the interval $(c, c+\delta)$. Hence, $f(c) \geq f(x)$ for all $x \in (c-\delta, c+\delta)$.
- **b** The function is decreasing on the interval $(c-\delta, c)$, and increasing on the interval $(c, c+\delta)$. Hence, $f(c) \leq f(x)$ for all $x \in (c-\delta, c+\delta)$. □

In simple language, the first derivative test says:

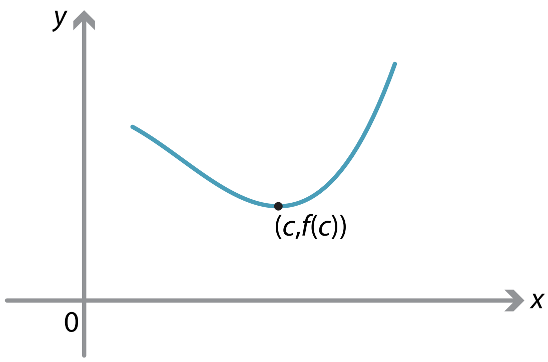
- If $f'(c) = 0$ with $f'(x) > 0$ immediately to the left of c and $f'(x) < 0$ immediately to the right of c, then c is a local maximum point.



We can also illustrate this with a gradient diagram.

Value of x		c	
Sign of $f'(x)$	$+$	0	$-$
Slope of graph $y = f(x)$	╱	—	╲

- If $f'(c) = 0$ with $f'(x) < 0$ immediately to the left of c and $f'(x) > 0$ immediately to the right of c, then c is a local minimum point.

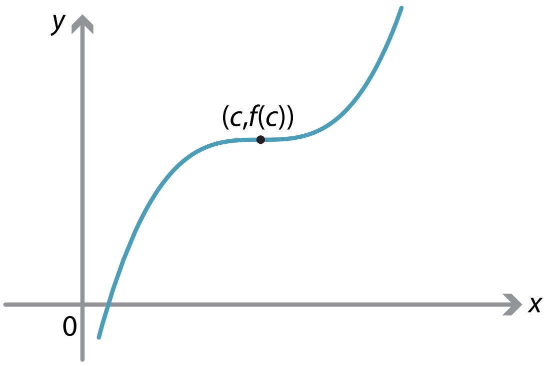


We can also illustrate this with a gradient diagram.

Value of x		c	
Sign of $f'(x)$	$-$	0	$+$
Slope of graph $y = f(x)$	╲	—	╱

There is another important type of stationary point:

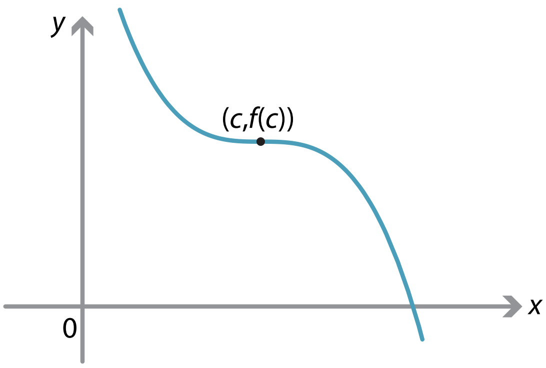
- If $f'(c) = 0$ with $f'(x) > 0$ on both sides of c, then c is a **stationary point of inflexion**.



Here is a gradient diagram for this situation.

Value of x		c	
Sign of $f'(x)$	$+$	0	$+$
Slope of graph $y = f(x)$	╱	—	╱

- If $f'(c) = 0$ with $f'(x) < 0$ on both sides of c, then c is a **stationary point of inflexion**.



Here is a gradient diagram for this situation.

Value of x		c	
Sign of $f'(x)$	$-$	0	$-$
Slope of graph $y = f(x)$	╲	—	╲

Example

Find the stationary points of $f(x) = 3x^4 + 16x^3 + 24x^2 + 3$, and determine their nature.

Solution

The derivative of f is

$$\begin{aligned} f'(x) &= 12x^3 + 48x^2 + 48x \\ &= 12x(x^2 + 4x + 4) \\ &= 12x(x+2)^2. \end{aligned}$$

So $f'(x) = 0$ implies $x = 0$ or $x = -2$.

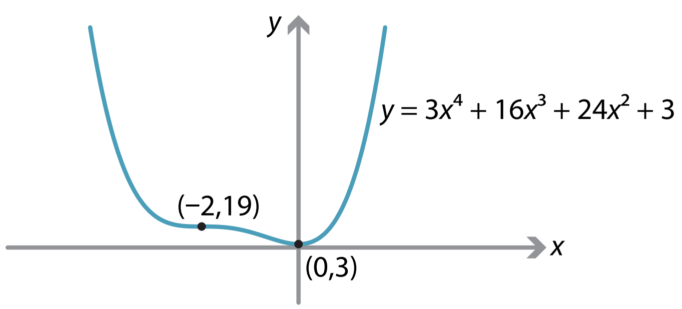
- If $x < -2$, then $f'(x) < 0$.
- If $-2 < x < 0$, then $f'(x) < 0$.
- If $x > 0$, then $f'(x) > 0$.

We can represent this in a gradient diagram.

Value of x		-2		0	
Sign of $f'(x)$	$-$	0	$-$	0	$+$
Slope of graph $y = f(x)$	$\diagdown$	—	$\diagdown$	—	$\diagup$

Hence, there are stationary points at $x = 0$ and $x = -2$: there is a local minimum at $x = 0$, and a stationary point of inflexion at $x = -2$.

The graph of $y = f(x)$ is shown in the following diagram, but not all the features of the graph have been carefully considered at this stage.



Exercise 1

Assume that the derivative of the function f is given by $f'(x) = (x-1)^2(x-3)$. Find the values of x which are stationary points of f, and state their nature.

Exercise 2

Find the stationary points of $f(x) = x^3 - 5x^2 + 3x + 2$, and determine their nature.

Use of the second derivative

The second derivative is introduced in the module *Introduction to differential calculus*. Using functional notation, the second derivative of the function f is written as f''. Using Leibniz notation, the second derivative is written as $\dfrac{d^2y}{dx^2}$, where y is a function of x.

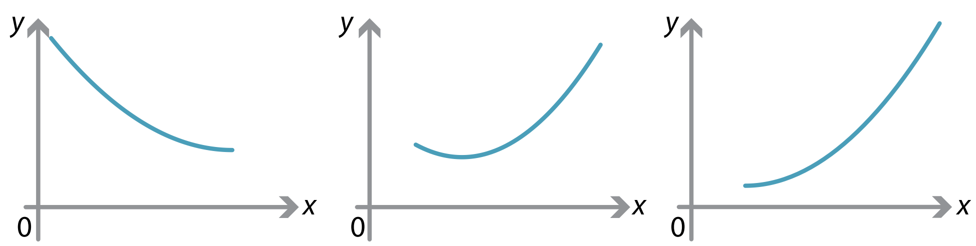
In the module *Motion in a straight line*, it is shown that the acceleration of a particle is the second derivative of its position with respect to time. That is, if the position of the particle at time t is denoted by $x(t)$, then the acceleration of the particle is $\ddot{x}(t)$.

Recall that, in kinematics, $\dot{x}$ means $\dfrac{dx}{dt}$ and $\ddot{x}$ means $\dfrac{d^2x}{dt^2}$.

Concave up and concave down

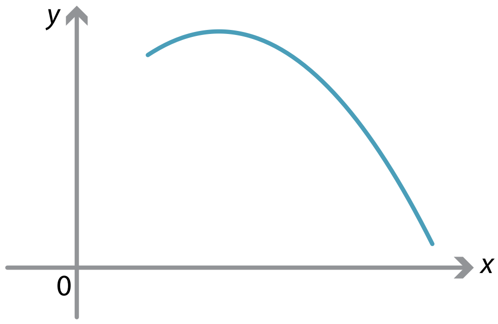
Let f be a function defined on the interval (a, b), and assume that f' and f'' exist at all points in (a, b). We consider the shape of the curve $y = f(x)$.

If $f''(x) > 0$, for all $x \in (a, b)$, then the slope of the curve is increasing in the interval (a, b). The curve is said to be **concave up**.



Examples of concave-up curves.

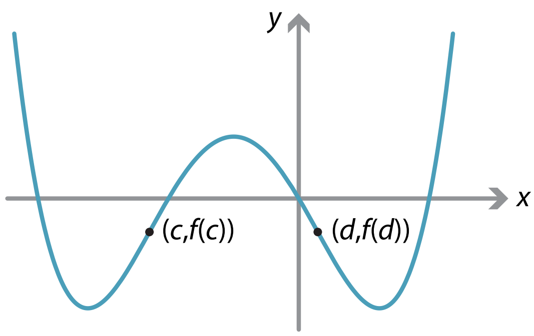
If $f''(x) < 0$, for all $x \in (a,b)$, then the slope of the curve is decreasing in the interval (a,b). The curve is **concave down**.



Example of a concave-down curve.

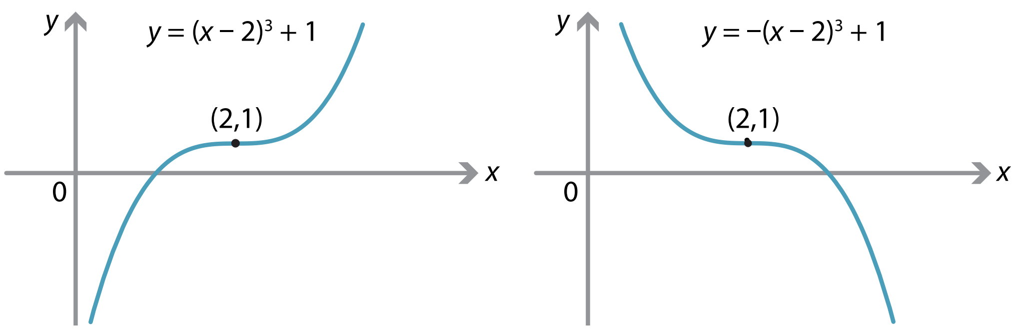
Inflexion points

A point where the curve changes from concave up to concave down, or from concave down to concave up, is called a **point of inflexion**. In the following diagram, there are points of inflexion at $x = c$ and $x = d$.

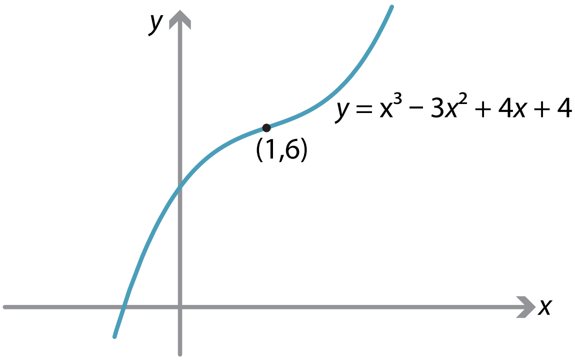


Examples of points of inflexion.

The graphs of $y = (x-2)^3 + 1$ and $y = -(x-2)^3 + 1$ are shown below. The point $(2,1)$ is a point of inflexion for each of these graphs. In fact, the point $(2,1)$ is a stationary point of inflexion for each of these graphs.



The graph of $y = x^3 - 3x^2 + 4x + 4$ is as follows. It has a point of inflexion at $(1, 6)$, but this is not a stationary point. In fact, this function has $\dfrac{dy}{dx} > 0$, for all x.



Note. Clearly, a necessary condition for a twice-differentiable function f to have a point of inflexion at $x = c$ is that $f''(c) = 0$. We will see that this is *not* a sufficient condition, and care must be taken when using it to find an inflexion point. For there to be a point of inflexion, there must be a *change of concavity*.

Example

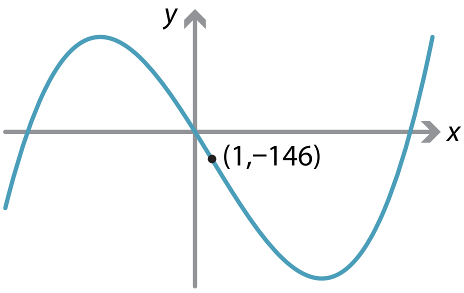
Find the inflexion point of the cubic function $f(x) = x^3 - 3x^2 - 144x$.

Solution

We find the first and second derivatives:

$$f'(x) = 3x^2 - 6x - 144 \qquad \text{and} \qquad f''(x) = 6x - 6.$$

Thus $f''(x) = 0$ implies $x = 1$. For $x < 1$, we have $f''(x) < 0$. For $x > 1$, we have $f''(x) > 0$. The curve changes from *concave down* to *concave up* at $x = 1$. Hence, there is a point of inflexion at $x = 1$.



It is not hard to show that there is a local maximum at $x = -6$ and a local minimum at $x = 8$.

Exercise 3

Find the inflexion points of the function $f(x) = x^4 + 28x^3 + 10x$.

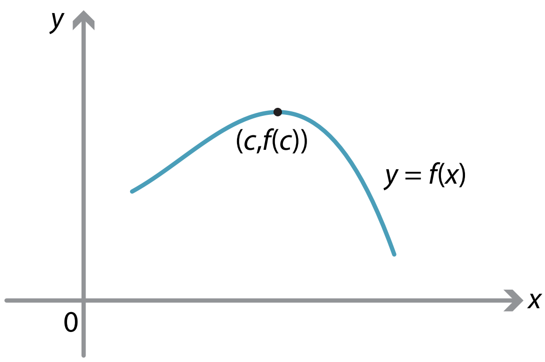
The second derivative test

We have seen how to use the first derivative test to determine whether a stationary point is a local maximum, a local minimum or neither of these. The second derivative provides an alternative test.

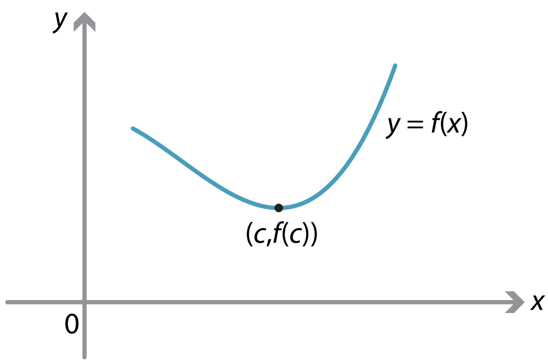
Theorem (Second derivative test)

Suppose f is twice differentiable at a stationary point c.

a *If $f''(c) < 0$, then f has a local maximum at c.*



b *If $f''(c) > 0$, then f has a local minimum at c.*



We will prove the first part of this theorem. The proof of the second part is similar.

Proof

Assume that $f''(c) < 0$. Then $f'(x)$ is strictly decreasing over some interval around c. Hence, there is a positive number δ such that $f'(x)$ is strictly decreasing on the interval $(c-\delta, c+\delta)$.

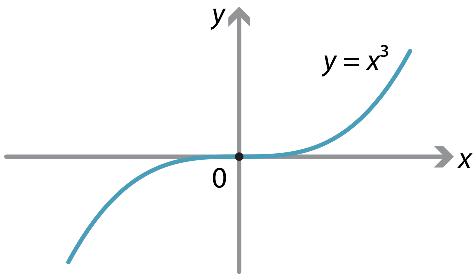
We now use the first derivative test.

- To the left of c: For any $h \in (c-\delta, c)$, we have $f'(h) > f'(c) = 0$.
- To the right of c: For any $k \in (c, c+\delta)$, we have $f'(k) < f'(c) = 0$.

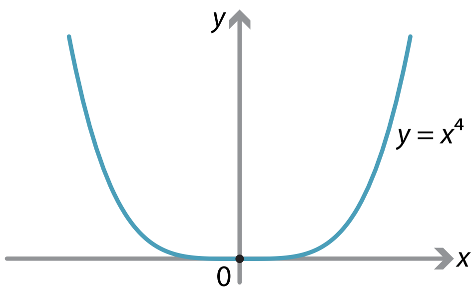
It follows from the first derivative test that c is a local maximum point. □

If c is a stationary point with $f''(c) = 0$, then we cannot use the second derivative test to determine if it is a local maximum, a local minimum or neither of these.

For example, if $f(x) = x^3$, then $f'(0) = 0$ and $f''(0) = 0$. In this case, there is a stationary point of inflexion at 0.



If $f(x) = x^4$, then $f'(0) = 0$ and $f''(0) = 0$. In this case, there is a local minimum at 0.



Thus the second derivative cannot be used as a test when $f'(c) = f''(c) = 0$.

Example

Locate and describe the stationary points of $f(x) = x^4 - 8x^2$.

Solution

The first derivative is $f'(x) = 4x^3 - 16x$, and the second derivative is $f''(x) = 12x^2 - 16$. We find the stationary points by solving $f'(x) = 0$:

$$4x^3 - 16x = 0$$

$$4x(x^2 - 4) = 0.$$

Hence, the stationary points are at $x = 0$, $x = 2$ and $x = -2$. We use the first derivative test by considering a gradient diagram.

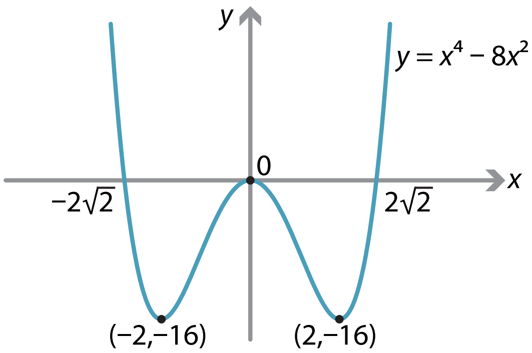
Value of x		-2		0		2	
Sign of $f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Slope of graph $y = f(x)$	╲	—	╱	—	╲	—	╱

- There is a local minimum at $x = -2$.
- There is a local maximum at $x = 0$.
- There is a local minimum at $x = 2$.

We can also check the local minimum and maximum points using the second derivative test, by considering the values of the second derivative:

- $f''(-2) = 32 > 0$, so there is a local minimum at $x = -2$
- $f''(0) = -16 < 0$, so there is a local maximum at $x = 0$
- $f''(2) = 32 > 0$, so there is a local minimum at $x = 2$.

The question has not required us to sketch the graph. But we have enough information to complete a good sketch if we also observe that the x-intercepts are 0, $2\sqrt{2}$ and $-2\sqrt{2}$.



Exercise 4

Locate and describe the stationary points of the graph of $y = f(x)$ if $f'(x) = x^3(x^2 - 5)$.

More graph sketching

Sketching graphs of polynomials

The module *Polynomials* introduces drawing the graphs of polynomials, but does not consider finding the coordinates of the points of inflexion. This is now possible.

By going through the following steps, we can find the important features of the graph of a given polynomial:

1. the x- and y-intercepts
2. stationary points
3. where the graph is increasing and where it is decreasing
4. local maxima and local minima
5. behaviour as x becomes very large positive and very large negative
6. points of inflexion.

Sometimes it is useful to find the coordinates of a few extra points on the graph.

Note. These steps are only a guide. Some flexibility in approach to these questions is desirable. In this module, we adhere to these steps as an aid to readability.

Example

Sketch the graph of $y = x^3 + x^2 - 8x - 12$.

Solution

1. We first find the x-intercepts and y-intercepts.

   When $x = 0$, we have $y = -12$, and so the y-intercept is -12.

   When $y = 0$,

$$\begin{aligned} 0 &= x^3 + x^2 - 8x - 12 \\ &= (x+2)^2(x-3), \end{aligned}$$

   so $x = -2$ or $x = 3$. The x-intercepts are 3 and -2.

2 The derivative is

$$\frac{dy}{dx} = 3x^2 + 2x - 8 = (3x-4)(x+2).$$

So $\frac{dy}{dx} = 0$ implies $x = -2$ or $x = \frac{4}{3}$.

The coordinates of the stationary points are $(-2, 0)$ and $(\frac{4}{3}, -\frac{500}{27})$.

3 The graph of $\frac{dy}{dx}$ against x is a parabola with a positive coefficient of x^2. We can see:

- $\frac{dy}{dx} = 0$ if and only if $x = -2$ or $x = \frac{4}{3}$
- $\frac{dy}{dx} > 0$ if and only if $x < -2$ or $x > \frac{4}{3}$
- $\frac{dy}{dx} < 0$ if and only if $-2 < x < \frac{4}{3}$.

So the graph is increasing on $(-\infty, -2) \cup (\frac{4}{3}, \infty)$ and decreasing on $(-2, \frac{4}{3})$.

Value of x		-2		$\frac{4}{3}$	
Sign of $\frac{dy}{dx}$	$+$	0	$-$	0	$+$
Slope of graph	/	—	\	—	/

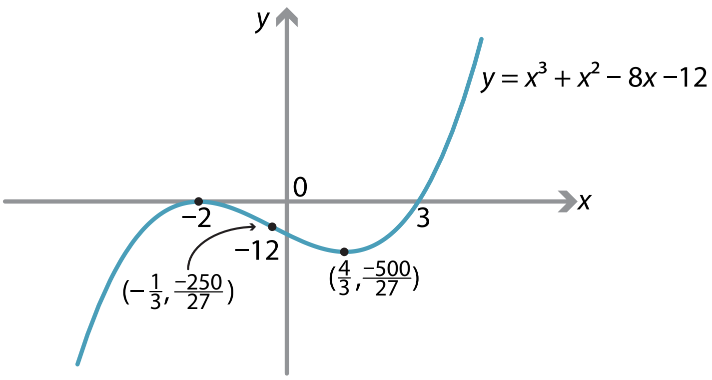
4 From the investigation of the sign of $\frac{dy}{dx}$, we see that there is a local maximum at $x = -2$, and a local minimum at $x = \frac{4}{3}$.

5 As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

6 The second derivative is

$$\frac{d^2y}{dx^2} = 6x + 2.$$

If $x < -\frac{1}{3}$, then $\frac{d^2y}{dx^2} < 0$, and if $x > -\frac{1}{3}$, then $\frac{d^2y}{dx^2} > 0$. There is an inflexion point where $x = -\frac{1}{3}$.



Exercise 5

Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$.

Exercise 6

Sketch the graph of $y = 4x^3 - 18x^2 + 48x - 290$.

Sketching graphs of other functions

The six steps used to help us sketch graphs of polynomials can be used to help sketch other graphs. In the *Links forward* section, we look briefly at graphs where there are local maxima or minima at which the function is not differentiable.

Example

Sketch the graph of $y = \sqrt{x} - x^2$, for $x \geq 0$.

Solution

The first and second derivatives are

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - 2x, \qquad \frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} - 2, \qquad \text{for } x > 0.$$

We follow the six steps for sketching polynomial graphs, which are still useful for this type of graph.

1. We first find the x-intercepts:

$$\begin{aligned} \sqrt{x} - x^2 &= 0 \\ \sqrt{x}(1 - x^{\frac{3}{2}}) &= 0, \end{aligned}$$

   which gives $x = 0$ or $x = 1$.

2. We find the stationary points by solving $\dfrac{dy}{dx} = 0$:

$$\begin{aligned} \frac{1}{2\sqrt{x}} - 2x &= 0 \\ 1 &= 4x^{\frac{3}{2}} \\ x^{\frac{3}{2}} &= \frac{1}{4} \\ x &= \left(\frac{1}{4}\right)^{\frac{2}{3}}. \end{aligned}$$

   So there is a stationary point at $x = \left(\frac{1}{4}\right)^{\frac{2}{3}} = 2^{-\frac{4}{3}}$.

3 If $0 < x < 2^{-\frac{4}{3}}$, then $\dfrac{dy}{dx} > 0$, and if $x > 2^{-\frac{4}{3}}$, then $\dfrac{dy}{dx} < 0$.

We can summarise this in a gradient diagram.

Value of x		$2^{-\frac{4}{3}}$	
Sign of $\frac{dy}{dx}$	$+$	0	$-$
Slope of graph	/	—	\

Hence, there is a local maximum at $x = 2^{-\frac{4}{3}}$. The corresponding y-value is

$$y = \left(2^{-\frac{4}{3}}\right)^{\frac{1}{2}} - \left(2^{-\frac{4}{3}}\right)^2 = 2^{-\frac{2}{3}} - 2^{-\frac{8}{3}}$$

$$= 2^{-\frac{8}{3}}\left(2^2 - 1\right) = 3 \times 2^{-\frac{8}{3}}.$$

4 We can also determine that there is a local maximum at $x = 2^{-\frac{4}{3}}$ by carrying out the second derivative test. When $x = 2^{-\frac{4}{3}}$,

$$\frac{d^2y}{dx^2} = -\frac{1}{4}\left(2^{-\frac{4}{3}}\right)^{-\frac{3}{2}} - 2 = -\frac{1}{4}\left(2^2\right) - 2 = -3.$$

Since $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ at $x = 2^{-\frac{4}{3}}$, the point $x = 2^{-\frac{4}{3}}$ is a local maximum.

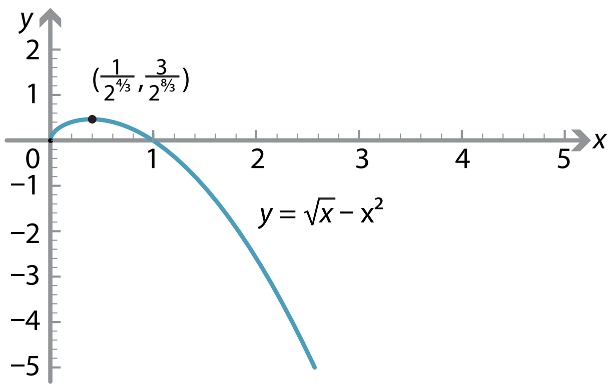
5 As $x \to \infty$, $y \to -\infty$. We do not consider $x \to -\infty$, since the domain of the function is $[0, \infty)$.

6 For all $x > 0$, we have

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} - 2 < 0.$$

Hence, there are no points of inflexion.

We can now sketch the graph.



Example

Sketch the graph of $f(x) = e^x + e^{-2x}$.

Solution

The domain of f is all real numbers. The first two derivatives of f are

$$f'(x) = e^x - 2e^{-2x}$$

$$f''(x) = e^x + 4e^{-2x}.$$

1 For all x, we have $e^x > 0$ and $e^{-2x} > 0$, and therefore $f(x) > 0$. Hence, there are no x-intercepts. We have $f(0) = 2$, and so the y-intercept is 2.

2 We find the stationary points by solving $f'(x) = 0$:

$$\begin{aligned} e^x - 2e^{-2x} &= 0 \\ e^{3x} &= 2 \\ x &= \frac{1}{3}\log_e 2. \end{aligned}$$

So there is a stationary point at $x = \frac{1}{3}\log_e 2$. The value of the function at this point is $f(\frac{1}{3}\log_e 2) = 3 \times 2^{-\frac{2}{3}}$.

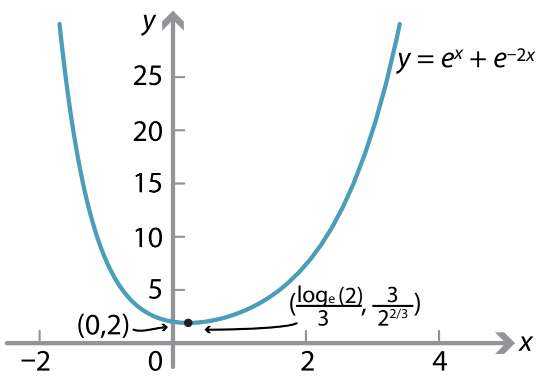
3 If $x < \frac{1}{3}\log_e 2$, the gradient is negative and the graph of $y = f(x)$ is decreasing.

If $x > \frac{1}{3}\log_e 2$, the gradient is positive and the graph of $y = f(x)$ is increasing.

4 We have $f''(x) = e^x + 4e^{-2x} > 0$, for all x. So there is a local minimum at $x = \frac{1}{3}\log_e 2$.

5 As $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to \infty$.

6 Since $f''(x) = e^x + 4e^{-2x} > 0$, for all x, there are no inflexion points.



Example

Sketch the graph of $f(x) = e^{-x} + x$.

Solution

The domain of f is all real numbers. The first two derivatives are $f'(x) = -e^{-x} + 1$ and $f''(x) = e^{-x}$.

1 We have $f(0) = 1$, so the y-intercept is 1. We will see later that there is a local minimum at $x = 0$. Indeed, the minimum value of the function occurs when $x = 0$. Hence, there are no x-intercepts.

2 Solving $f'(x) = 0$ gives

$$-e^{-x} + 1 = 0 \iff e^{-x} = 1 \iff x = 0.$$

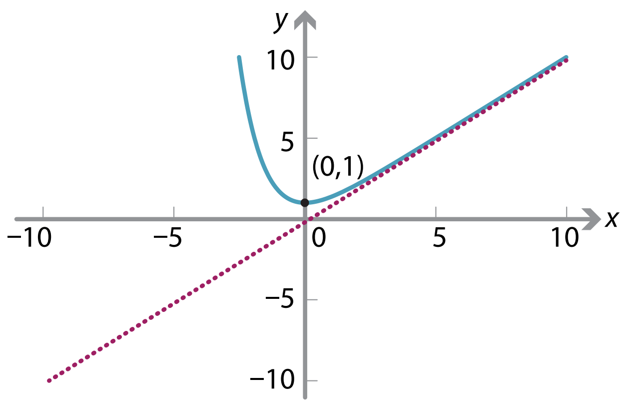
Hence, there is a stationary point at $x = 0$.

3 If $x < 0$, the gradient is negative and the graph of $y = f(x)$ is decreasing.

If $x > 0$, the gradient is positive and the graph of $y = f(x)$ is increasing.

4 We have $f''(x) = e^{-x} > 0$, for all x. Hence, there is a local minimum when $x = 0$.

5 If $x \to -\infty$, then $f(x) \to \infty$. If $x \to \infty$, then the line $y = x$ is an asymptote, since $e^{-x} \to 0$.



Exercise 7

Sketch the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = x^2 \log_e x$.

Maxima and minima at end points

It may be that the maximum or minimum value of a graph occurs at the endpoints of that graph. Here are two examples to illustrate the need to check endpoints when trying to find the maximum and minimum values of a function.

Example

Sketch the graph of the function $f\colon [1,3] \to \mathbb{R}$ given by $f(x) = x^2(x-4)$, and find the maximum and minimum values of the function.

Solution

If $f(x) = 0$, then $x = 0$ or $x = 4$, which are outside the domain. Therefore the graph has no x-intercepts.

We next find the derivative and the second derivative of f:

$$f'(x) = 3x^2 - 8x = x(3x-8)$$

$$f''(x) = 6x - 8.$$

Hence, there is a stationary point at $x = \frac{8}{3}$, and a point of inflexion at $x = \frac{4}{3}$.

To determine if $x = \frac{8}{3}$ is a local maximum or minimum, we use the second derivative test:

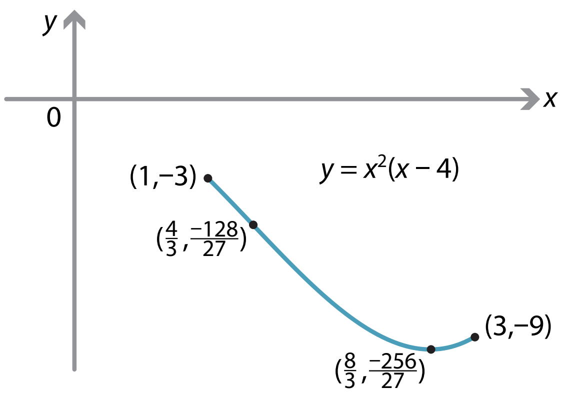
$$f''\Big(\frac{8}{3}\Big) = 8 > 0,$$

and hence there is a local minimum at $x = \frac{8}{3}$.

We now find the values of f at the two endpoints and the local minimum:

$$f\Big(\frac{8}{3}\Big) = -\frac{256}{27}, \qquad f(1) = -3, \qquad f(3) = -9.$$

Hence, the maximum value of f is -3, and it occurs when $x = 1$. The minimum value of f is $-\frac{256}{27}$, and it occurs when $x = \frac{8}{3}$.



Example

Define $f\colon [-7,6] \to \mathbb{R}$ by $f(x) = 3x^4 + 8x^3 - 174x^2 - 360x$. Find the maximum and minimum values of the function.

Solution

The first and second derivatives are

$$\begin{aligned} f'(x) &= 12x^3 + 24x^2 - 348x - 360 \\ &= 12(x^3 + 2x^2 - 29x - 30) = 12(x+1)(x-5)(x+6) \\ f''(x) &= 36x^2 + 48x - 348 = 12(3x^2 + 4x - 29). \end{aligned}$$

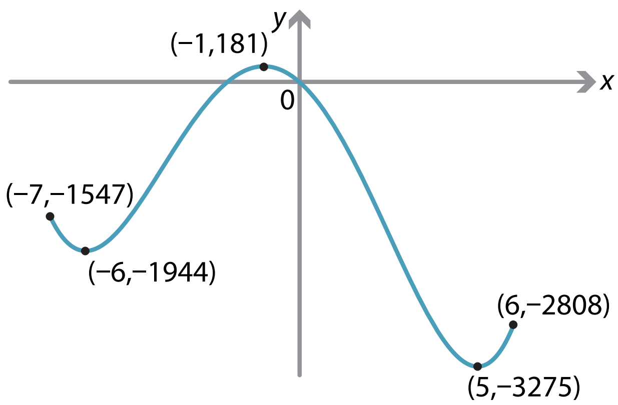
So $f'(x) = 0$ implies $x = 5$ or $x = -1$ or $x = -6$. There are stationary points at these three values. We use the second derivative test:

$$f''(5) = 792 > 0, \qquad f''(-1) = -360 < 0, \qquad f''(-6) = 660 > 0.$$

Hence, there is a local minimum at $x = 5$, a local maximum at $x = -1$, and a local minimum at $x = -6$.

We have the following points on the graph:

- the left endpoint $(-7, -1547)$
- a local minimum $(-6, -1944)$
- a local maximum $(-1, 181)$
- a local minimum $(5, -3275)$
- the right endpoint $(6, -2808)$.



The maximum value of the function f is 181, which occurs at $x = -1$, and the minimum value of f is -3275, which occurs at $x = 5$.

Exercise 8

Sketch the graph of the function $f\colon [-1,2] \to \mathbb{R}$ given by $f(x) = x^2(x+4)$, and find the maximum and minimum values of this function.

Maxima and minima problems

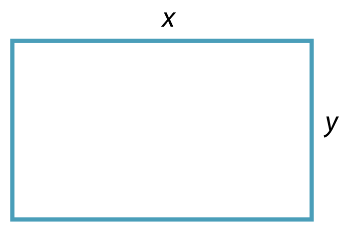
The need to find local maxima and minima arises in many situations. The first example we will look at is very familiar, and can also be solved without using calculus. Examples of solving such problems without the use of calculus can be found in the module *Quadratics*.

Example

Find the dimensions of a rectangle with perimeter 1000 metres so that the area of the rectangle is a maximum.

Solution

Let the length of the rectangle be x m, the width be y m, and the area be A m^2.



The perimeter of the rectangle is 1000 metres. So

$$1000 = 2x + 2y,$$

and hence

$$y = 500 - x.$$

The area is given by $A = xy$. Thus

$$A(x) = x(500 - x) = 500x - x^2. \qquad (1)$$

Because x and y are lengths, we must have $0 \le x \le 500$.

The problem now reduces to finding the value of x in $[0,500]$ for which A is a maximum. Since A is differentiable, the maximum must occur at an endpoint or a stationary point.

From (1), we have

$$\frac{dA}{dx} = 500 - 2x.$$

Setting $\frac{dA}{dx} = 0$ gives $x = 250$.

Hence, the possible values for A to be a maximum are $x = 0$, $x = 250$ and $x = 500$. Since $A(0) = A(500) = 0$, the maximum value of A occurs when $x = 250$.

The rectangle is a square with side lengths 250 metres. The maximum area is 62 500 square metres.

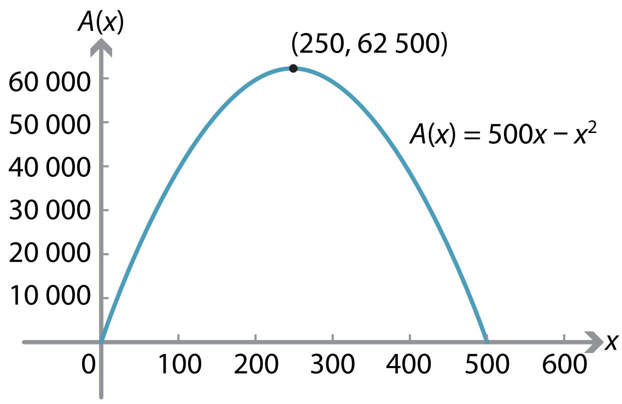
Notes.

1 $\frac{dA}{dx} > 0$, for $0 \le x < 250$, and $\frac{dA}{dx} < 0$, for $250 < x \le 500$. Hence, there is a local maximum at $x = 250$.

2 $\frac{d^2A}{dx^2} = -2 < 0$. This is a second way to see that $x = 250$ is a local maximum.

3 The graph of $A(x) = 500x - x^2$ is a parabola with a negative coefficient of x^2 and a turning point at $x = 250$. This is a third way of establishing the local maximum.

4 It is worth looking at the graph of $A(x)$ against x.



Exercise 9

A farmer has 8 km of fencing wire, and wishes to fence a rectangular piece of land. One boundary of the land is the bank of a straight river. What are the dimensions of the rectangle so that the area is maximised?

The following steps provide a general procedure which you can follow to solve maxima and minima problems.

Steps for solving maxima and minima problems

Step 1. Where possible draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note any restrictions on the values of the variables.

Step 2. Write an expression for the quantity that is going to be maximised or minimised. Eliminate some of the variables. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.

Step 3. If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.

Step 4. Test each point for which $f'(x) = 0$ to determine if it is a local maximum, a local minimum or neither.

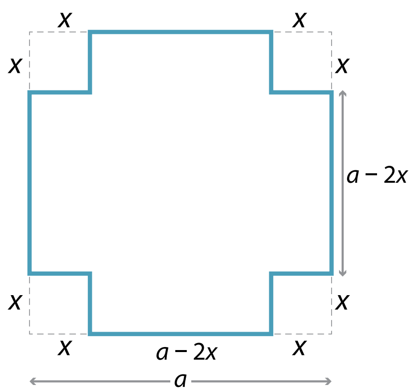
Step 5. If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the end points.

Example

A square sheet of cardboard with each side a centimetres is to be used to make an open-top box by cutting a small square of cardboard from each of the corners and bending up the sides. What is the side length of the small squares if the box is to have as large a volume as possible?

Solution

Step 1.



Let the side length of the small squares be x cm. The side length of the open box is $(a-2x)$ cm, and the height is x cm. Here a is a constant, and x is the variable we will work with. We must have

$$0 \le x \le \frac{a}{2}.$$

Step 2. The volume V cm^3 of the box is given by

$$V(x) = x(a-2x)^2 = 4x^3 - 4ax^2 + a^2x.$$

Step 3. We have

$$\frac{dV}{dx} = 12x^2 - 8ax + a^2 = (2x-a)(6x-a).$$

Thus $\dfrac{dV}{dx} = 0$ implies $x = \frac{a}{2}$ or $x = \frac{a}{6}$.

Step 4. We note that $x = \frac{a}{2}$ is an endpoint and that $V\left(\frac{a}{2}\right) = 0$. We will use the second derivative test for $x = \frac{a}{6}$. We have

$$\frac{d^2V}{dx^2} = 24x - 8a = 8(3x-a).$$

When $x = \frac{a}{6}$, we get

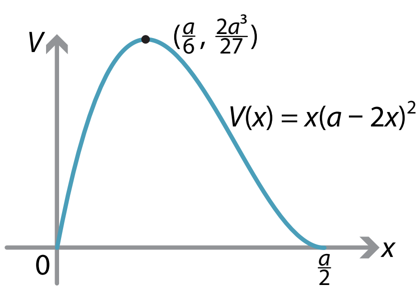
$$\frac{d^2V}{dx^2} = 8\left(3 \times \frac{a}{6} - a\right) = -4a < 0.$$

Hence, $x = \frac{a}{6}$ is a local maximum.

Step 5. The maximum value of the function is at $x = \frac{a}{6}$, as $V(0) = V\left(\frac{a}{2}\right) = 0$. The maximum volume is

$$V\left(\frac{a}{6}\right) = \frac{2a^3}{27}.$$

The following diagram shows the graph of V against x.

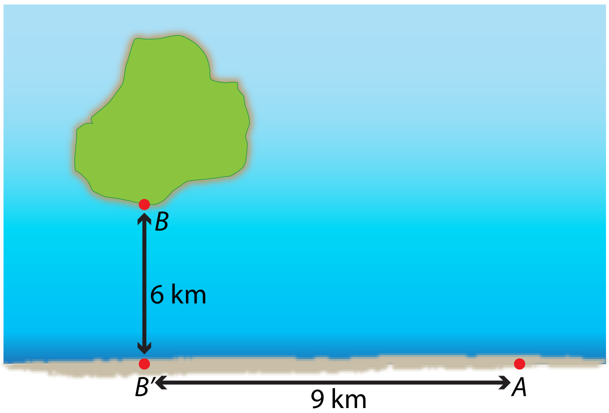


The following example illustrates a number of issues that can occur.

Example

A company wants to run a pipeline from a point A on the shore to a point B on an island which is 6 km from the shore. It costs $\$P$ per kilometre to run the pipeline on shore, and $\$Q$ per kilometre to run it underwater. There is a point B' on the shore so that BB' is at right angles to AB'. The straight shoreline is the line AB'. The distance AB' is 9 km. Find how the pipeline should be laid to minimise the cost if

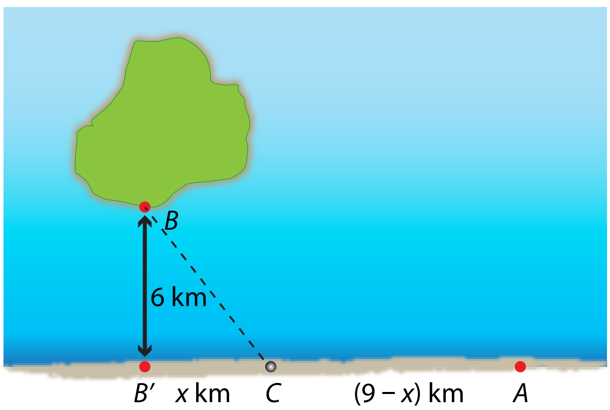
1. $P = 4000$ and $Q = 5000$
2. $P = 5000$ and $Q = 13\,000$
3. $P = 24\,000$ and $Q = 25\,000$.



Solution

We will work through most of the problem without assigning values to P and Q.

Step 1.



Suppose that the pipeline leaves the shore x km from B' at a point C between B' and A. The distance AC is $(9-x)$ km. By Pythagoras' theorem, the distance CB is $\sqrt{36+x^2}$ km. It is important to note that

$$0 \le x \le 9.$$

Step 2. Let $\$T$ be the total cost. Then

$$T(x) = P(9-x) + Q\sqrt{36+x^2}. \qquad (1)$$

Step 3. We have

$$\frac{dT}{dx} = \frac{Qx}{\sqrt{36+x^2}} - P.$$

Hence, solving $\dfrac{dT}{dx} = 0$ gives

$$\begin{aligned}
\frac{Qx}{\sqrt{36+x^2}} - P &= 0\\
Qx &= P\sqrt{36+x^2}\\
Q^2x^2 &= P^2(36+x^2)\\
(Q^2-P^2)x^2 &= 36P^2\\
x &= \sqrt{\frac{36P^2}{Q^2-P^2}} = \frac{6P}{\sqrt{Q^2-P^2}}. \qquad (2)
\end{aligned}$$

Note that we need $Q > P$ for this solution x to exist, and we also need $0 \le x \le 9$.

If $Q \le P$, the pipeline should go directly from A to B, with minimum cost $\$3\sqrt{13}Q$.

Step 4. Using the second derivative test:

$$\frac{d^2T}{dx^2} = \frac{36Q}{(36+x^2)^{\frac{3}{2}}} > 0,$$

for all x. Hence, there is a local minimum at $x = \dfrac{6P}{\sqrt{Q^2-P^2}}$ for the function with rule $T(x)$. Such a local minimum may occur outside the interval $[0,9]$.

Step 5.

- If $x = 0$, then $T = 9P + 6Q$.
- If $x = 9$, then $T = 3\sqrt{13}Q$.

- If $x = \dfrac{6P}{\sqrt{Q^2-P^2}}$, then from (1) we have

$$\begin{aligned} T &= P\Big(9-\frac{6P}{\sqrt{Q^2-P^2}}\Big)+Q\sqrt{36+\frac{36P^2}{Q^2-P^2}} \\ &= 9P-\frac{6P^2}{\sqrt{Q^2-P^2}}+Q\sqrt{\frac{36(Q^2-P^2)+36P^2}{Q^2-P^2}} \\ &= 9P-\frac{6P^2}{\sqrt{Q^2-P^2}}+\frac{6Q^2}{\sqrt{Q^2-P^2}} \\ &= 9P+6\sqrt{Q^2-P^2}. \qquad (3) \end{aligned}$$

The local minimum occurs in the interval $[0,9]$ if and only if

$$\frac{6P}{\sqrt{Q^2-P^2}} \le 9.$$

We now solve this inequality for the ratio $\frac{P}{Q}$, assuming that $Q > P$:

$$\begin{aligned} \frac{6P}{\sqrt{Q^2-P^2}} \le 9 &\iff \frac{36P^2}{Q^2-P^2} \le 81 \\ &\iff 36P^2 \le 81(Q^2-P^2) \\ &\iff 117P^2 \le 81Q^2 \\ &\iff \frac{P^2}{Q^2} \le \frac{81}{117}. \end{aligned}$$

Thus the local minimum occurs in the interval $[0,9]$ if and only if $\frac{P}{Q} \le \frac{3}{\sqrt{13}}$.

We now consider the particular values of P and Q specified in the question.

1 $P = 4000$ and $Q = 5000$.

By equation (1), we have

$$T = 4000(9-x)+5000\sqrt{36+x^2}, \qquad \text{for } 0 \le x \le 9.$$

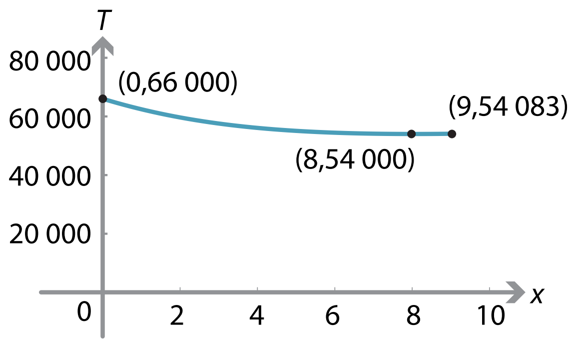
Note that

$$T(0) = 36\,000+30\,000 = 66\,000$$

$$T(9) = 15\,000\sqrt{13} \approx 54\,083.$$

By equation (2), the local minimum point is $x = 8$ and in this case, by equation (3), the minimum cost is

$$T_{\min} = 9 \times 4000 + 6 \times 3000 = \$54\,000.$$



2 $P = 5000$ and $Q = 13\,000$.

By equation (1), we have

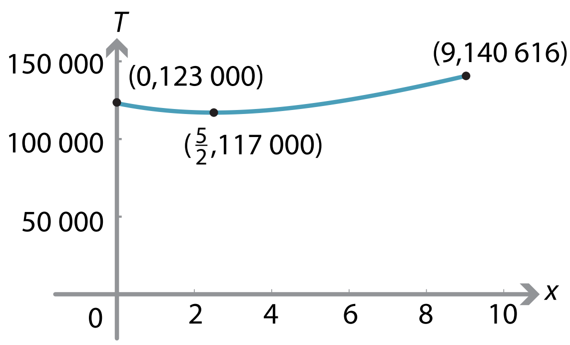
$$T = 5000(9 - x) + 13\,000\sqrt{36 + x^2}, \qquad \text{for } 0 \le x \le 9.$$

We note that

$$T(0) = 123\,000, \qquad T(9) = 39\,000\sqrt{13} \approx 140\,616.$$

By equation (2), the local minimum point is $x = \frac{5}{2}$ and in this case, by equation (3), the minimum cost is

$$T_{\min} = 9 \times 5000 + 6 \times 12\,000 = \$117\,000.$$



3 $P = 24\,000$ and $Q = 25\,000$.

By equation (1), we have

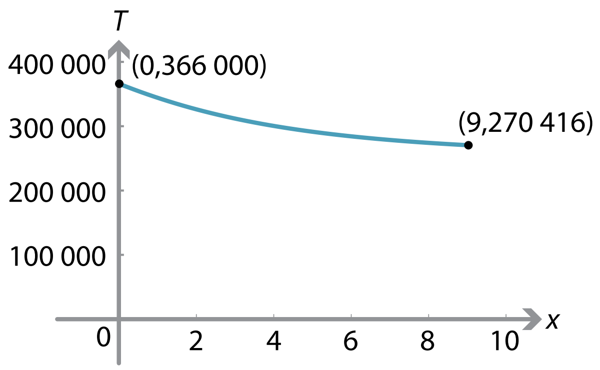
$$T = 24\,000(9 - x) + 25\,000\sqrt{36 + x^2}, \qquad \text{for } 0 \le x \le 9.$$

We note that

$$T(0) = 366\,000, \qquad T(9) = 75\,000\sqrt{13} \approx 270\,416.$$

By equation (2), the local minimum occurs at $x = \dfrac{144}{7}$, which is outside the required domain. In fact, we have $\dfrac{dT}{dx} < 0$, for all $x \in [0,9]$. The minimum cost is

$$T(9) = 75\,000\sqrt{13} \approx \$270\,416.$$



Note. In parts 1 and 2, the minimum occurs at a local minimum. But, in part 3, the minimum occurs at an endpoint.

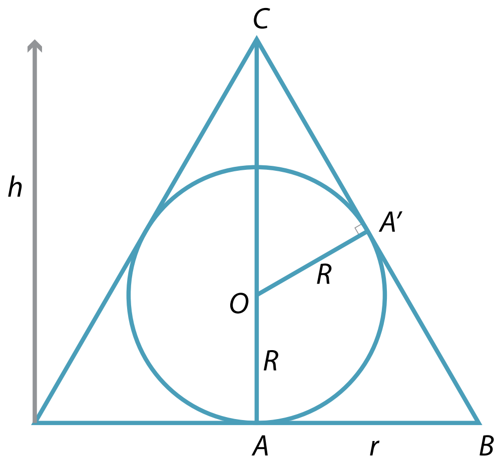
The following example has reasonably demanding algebra and involves some geometry, but the result is surprisingly neat.

Example

A right cone is circumscribed around a given sphere. Find when its volume is a minimum.

Solution

Step 1. The following diagram shows a vertical cross-section of the cone and sphere.



The sphere has radius R, which we treat as a constant.

The cone has radius r and height h. These are variables. From the geometry, we must have $h > 2R > 0$ and $r > R > 0$.

The centre of the sphere is marked by O. The radius OA' is drawn perpendicular to CB.

Step 2. We will find h in terms of r and R.

We begin by noting that $OC = h - R$. By using Pythagoras' theorem in $\triangle OCA'$, we get $CA' = \sqrt{h^2 - 2hR}$. Since $\triangle CA'O$ is similar to $\triangle CAB$ (AAA), we can write

$$\frac{CA'}{CA} = \frac{OA'}{BA}.$$

Hence,

$$\frac{\sqrt{h^2 - 2hR}}{h} = \frac{R}{r}.$$

Solving for h, we obtain

$$\begin{aligned}
\frac{h^2 - 2hR}{h^2} &= \frac{R^2}{r^2} && \text{(square both sides)}\\
r^2(h^2 - 2hR) &= h^2R^2 && \text{(cross-multiply)}\\
r^2h^2 - 2hRr^2 &= h^2R^2\\
r^2h - 2Rr^2 &= hR^2 && \text{(as } h \neq 0\text{)}\\
h(r^2 - R^2) &= 2r^2R\\
h &= \frac{2r^2R}{r^2 - R^2}.
\end{aligned}$$

The volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$. Substituting for h, we obtain

$$V = \frac{2\pi r^4 R}{3(r^2 - R^2)}.$$

We have now expressed the volume in terms of the one variable r.

Step 3. We have

$$\frac{dV}{dr} = \frac{4\pi R r^3(r^2 - 2R^2)}{3(r^2 - R^2)^2}.$$

So $\dfrac{dV}{dr} = 0$ implies that $r^3(r^2 - 2R^2) = 0$, which implies that $r = 0$ or $r = \sqrt{2}R$.

Clearly, $r = \sqrt{2}R$ is the solution we want.

Step 4. Using

$$\frac{dV}{dr} = \frac{4\pi R r^3(r^2 - 2R^2)}{3(r^2 - R^2)^2},$$

we can complete the following gradient diagram.

Value of r		$\sqrt{2}R$	
Sign of $\frac{dV}{dr}$	$-$	0	$+$
Slope of graph	╲	—	╱

Alternatively, we can use the second derivative test. We have

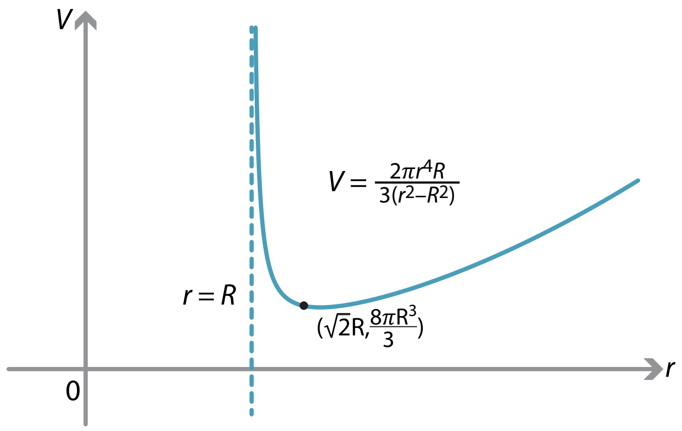
$$\frac{d^2V}{dr^2} = \frac{4\pi R(r^6 - 3r^4R^2 + 6r^2R^4)}{3(r^2 - R^2)^3}.$$

Substituting $r = \sqrt{2}R$ gives

$$\frac{d^2V}{dr^2} = \frac{32\pi R}{3} > 0.$$

Hence, we have a local minimum at $r = \sqrt{2}R$.

The graph of V against r is as follows. There is a vertical asymptote at $r = R$, and the graph approaches a parabola with equation $V = \frac{2\pi R}{3}r^2$ as r becomes very large.



Exercise 10

a Find the maximum area of a rectangle that can be inscribed in the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$. Assume that the sides of the rectangle are parallel to the axes.

b Find the maximum area of a rectangle that can be inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Assume that the sides of the rectangle are parallel to the axes.

Exercise 11

A hollow cone has base radius R and height H. What is the volume of the largest cylinder that can be placed under it?

Related rates of change

Related rates of change are simply an application of the chain rule. In related-rate problems, you find the rate at which some quantity is changing by relating it to other quantities for which the rate of change is known.

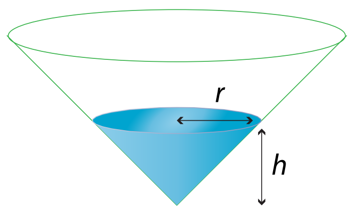
In the following examples, we repeatedly use the result

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}},$$

which is established and discussed in the module *Introduction to differential calculus*.

Example

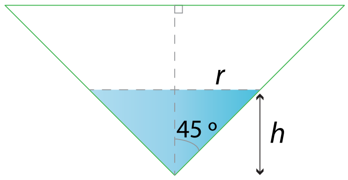
An upturned cone with semivertical angle 45° is being filled with water at a constant rate of 30 cm^3 per second.



When the depth of the water is 60 cm, find the rate at which

1. h, the depth of the water, is increasing
2. r, the radius of the surface of the water, is increasing
3. S, the area of the water surface, is increasing.

Solution



The volume V of the water in the cone is given by

$$V = \frac{1}{3}\pi r^2 h.$$

The cross-section of the cone is a right-angled isosceles triangle, and therefore $r = h$. Hence,

$$V = \frac{1}{3}\pi r^3 = \frac{1}{3}\pi h^3.$$

1 We seek $\frac{dh}{dt}$. Now $V = \frac{1}{3}\pi h^3$, and therefore $\frac{dV}{dh} = \pi h^2$. Hence,

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{1}{\pi h^2} \times 30 \\ &= \frac{30}{\pi h^2}.\end{aligned}$$

When $h = 60$, we have $\frac{dh}{dt} = \frac{1}{120\pi}$ cm/s.

2 We have seen that $r(t) = h(t)$. Thus, when $h = 60$, we have $\frac{dr}{dt} = \frac{1}{120\pi}$ cm/s.

3 We seek $\frac{dS}{dt}$. The area of the water's surface is

$$S = \pi r^2.$$

Therefore

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr} \times \frac{dr}{dt} \\ &= 2\pi r \times \frac{30}{\pi r^2} \\ &= \frac{60}{r}.\end{aligned}$$

When $h = 60$, we have $r = 60$, and so $\frac{dS}{dt} = 1$ cm^2/s.

Example

Variables x and y are related by the equation $y = 2 - \frac{4}{x}$. Given that x and y are functions of t and that $\frac{dx}{dt} = 10$, find $\frac{dy}{dt}$ in terms of x.

Solution

If $y = 2 - \frac{4}{x}$, then $\frac{dy}{dx} = \frac{4}{x^2}$. Therefore

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt}\\ &= \frac{4}{x^2} \times 10\\ &= \frac{40}{x^2}.\end{aligned}$$

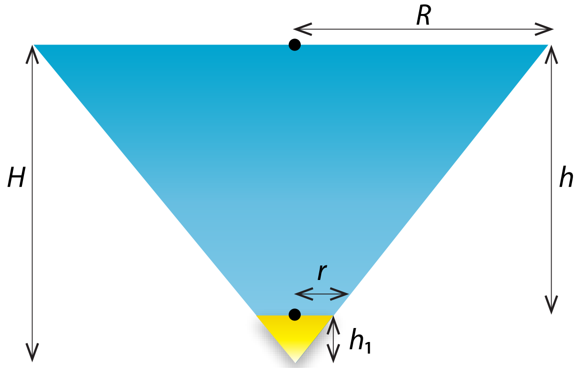
Related rates questions arise in many situations. The following demanding example involving the flow of liquids relates the rate of change of depth to the rate of change of volume.

Example

A vessel of water is in the form of a frustum of a cone with semivertical angle 45°. The bottom circle of the vessel is a hole of radius r cm. Water flows from this hole at a velocity of $c\sqrt{2gh}$ cm/s, where c is a constant and h cm is the height of the surface of the water above the hole.

1 Find the rate in cm^3/s at which water flows from the vessel.

2 Find the rate in cm/s at which h is increasing.



Solution

We first find the volume V of water in the frustum. Consider removing a small cone from the 'tip' of a cone. This forms the hole. Let H be the height of the larger cone (up to the water surface) and let h_1 be the height of the smaller cone (which is removed). Let R be the radius of the larger cone and let r be the radius of the smaller cone.

The height of the water is $h = H - h_1$. Therefore

$$V = \frac{1}{3}\pi(R^2H - r^2h_1) = \frac{1}{3}\pi\big(R^2H - r^2(H-h)\big).$$

Because the semivertical angle is 45°, we have $R = H$ and $R - r = h$. Hence, $R = r + h$.

Substituting for R and H in the equation for V, we have

$$\begin{aligned}
V &= \frac{1}{3}\pi\big(R^2H - r^2(H-h)\big)\\
&= \frac{1}{3}\pi\big(R^3 - r^2(R-h)\big)\\
&= \frac{1}{3}\pi\big((r+h)^3 - r^3\big)\\
&= \frac{1}{3}\pi\big(3r^2h + 3rh^2 + h^3\big).
\end{aligned}$$

1. Let v cm/s be the velocity of the flow of water from the hole at time t. We are given that $v = c\sqrt{2gh}$, where h is the height of the water above the hole at time t. Since

   $$\frac{dV}{dt} = \text{area of the cross-section of the hole} \times \text{velocity of the flow of water from the hole,}$$

   we have

   $$\frac{dV}{dt} = \pi r^2 v = \pi r^2 c\sqrt{2gh}\ \text{cm}^3/\text{s}.$$

2. We found that the volume of water V cm^3 is given by

   $$V = \frac{1}{3}\pi(3r^2h + 3rh^2 + h^3),$$

   and so

   $$\frac{dV}{dh} = \pi(r+h)^2.$$

   Using the chain rule, we have

   $$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{cr^2\sqrt{2gh}}{(r+h)^2}\ \text{cm/s}.$$

Exercise 12

A point P is moving along the curve whose equation is $y = \sqrt{x^3 + 56}$. When P is at $(2, 8)$, y is increasing at the rate of 2 units per second. How fast is x changing?

Exercise 13

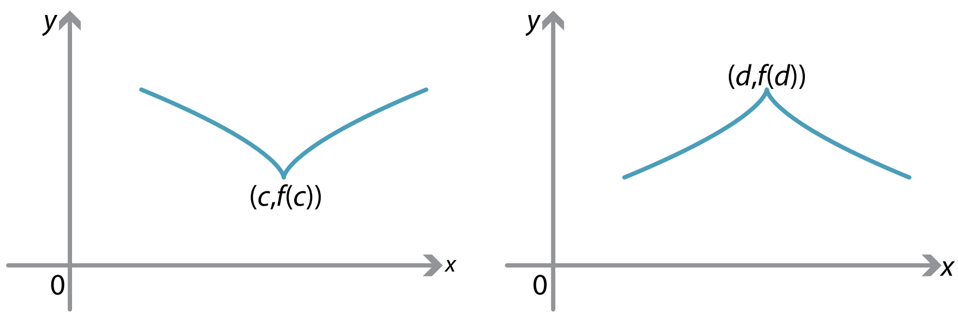
A meteor enters the earth's atmosphere and burns at a rate that at each instant is proportional to its surface area. Assuming that the meteor is always spherical, show that the radius is decreasing at a constant rate.

Links forward

Critical points

In the section *Graph sketching*, we looked at stationary points of a function f, that is, points where $f'(x) = 0$. We gave conditions for a stationary point to be local minimum or a local maximum.

Local maxima and minima also occur in other cases. The point $(c, f(c))$ on the left-hand graph is a local minimum. The point $(d, f(d))$ on the right-hand graph is a local maximum. These are examples of critical points. In each case, the function is not differentiable at that point.



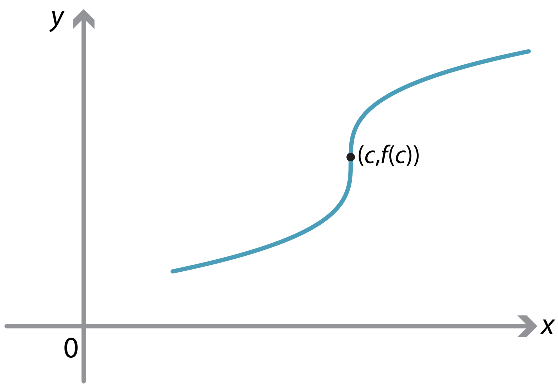
Examples of critical points.

Definition

A **critical point** for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable.

Every stationary point is a critical point.

Not every critical point is a local minimum or maximum point. In the following diagram, the function is not differentiable at the point $(c, f(c))$, since the tangent is vertical. There is a point of inflexion at $(c, f(c))$.



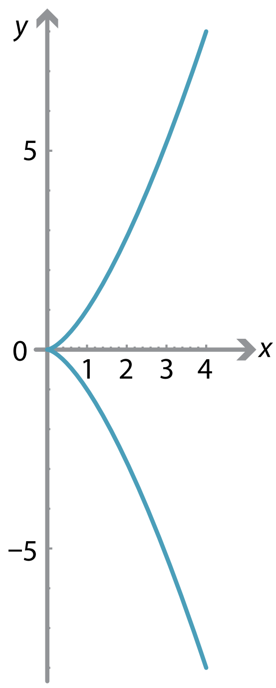
Finding gradients on a parametric curve

We start with a simple example.

Consider the parametric curve defined by

$$x = t^2$$

$$y = t^3.$$



We will find $\frac{dy}{dx}$ in terms of t by using the chain rule and the result

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}.$$

We have $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$. Hence,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx}\\ &= \frac{3t^2}{2t}\\ &= \frac{3t}{2}.\end{aligned}$$

Projectile motion

The motion of a particle projected at an angle α to the horizontal with an initial velocity of u m/s can be described by the parametric equations

$$x = u\cos(\alpha)t$$

$$y = u\sin(\alpha)t - \frac{1}{2}gt^2,$$

where g is the acceleration due to gravity. A method for deriving these equations is given in the module *Motion in a straight line*.

The velocities in the horizontal and vertical directions are

$$\frac{dx}{dt} = u\cos(\alpha) \qquad \text{and} \qquad \frac{dy}{dt} = u\sin(\alpha) - gt.$$

Using the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{u\sin(\alpha) - gt}{u\cos(\alpha)}.$$

This gives the gradient of the path of the projectile in terms of t, for $t \geq 0$. This also allows us to find the angle of inclination of the path of the projectile at any time t. We use the fact that

$$\tan\theta = \frac{dy}{dx},$$

where θ is the angle of inclination of the path to the horizontal. For example, if $\alpha = \frac{\pi}{4}$ and $u = 10$, then

$$\tan\theta = \frac{\frac{10}{\sqrt{2}} - gt}{\frac{10}{\sqrt{2}}} = 1 - \frac{\sqrt{2}gt}{10}.$$

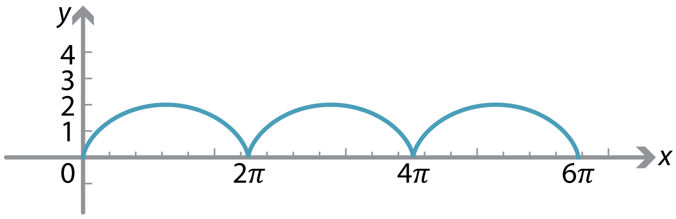
Cycloids

A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line. For a wheel of radius 1, the parametric equations of the cycloid are

$$x = \theta - \sin\theta$$

$$y = 1 - \cos\theta.$$

The graph of y against x for $0 \leq \theta \leq 6\pi$ is as follows.



We can find the gradient at a point on the cycloid by using the chain rule. We have

$$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \text{and} \qquad \frac{dy}{d\theta} = \sin\theta.$$

The chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= \frac{\sin\theta}{1-\cos\theta} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2}.\end{aligned}$$

Also,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) \\ &= \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right) \times \frac{d\theta}{dx} \\ &= -\frac{1}{4\sin^4(\frac{\theta}{2})}.\end{aligned}$$

Notes.

- There are stationary points where $\cot\frac{\theta}{2} = 0$, which is equivalent to $\cos\frac{\theta}{2} = 0$. This occurs when θ is an odd multiple of π, that is, when $\theta \in \{\dots, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, \dots\}$. Note that $\frac{d^2y}{dx^2} < 0$ for these values of θ, and so there are local maxima when θ is an odd multiple of π.
- The function is not differentiable when θ is even multiple of π. That is, these are critical points of the function.
- There are no points of inflexion.
- The x-intercepts occur when $\cos\theta = 1$, that is, when θ is an even multiple of π.

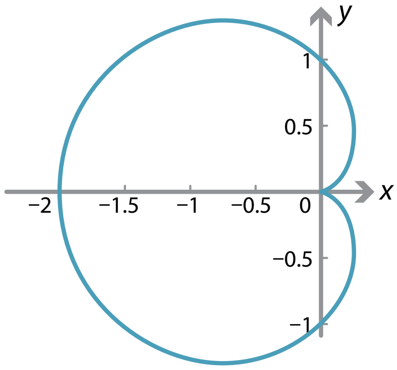
Cardioids

A **cardioid** is the curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius. The cardioid shown in the graph has parametric equations

$$x = (1 - \cos t)\cos t$$

$$y = (1 - \cos t)\sin t.$$

It is plotted for $t \in [0, 2\pi]$.



We can calculate the gradient of the cardioid for a particular value of t as follows:

$$\frac{dx}{dt} = -\sin t + 2\sin t \cos t = \sin 2t - \sin t$$

$$\frac{dy}{dt} = \cos t - \cos^2 t + \sin^2 t = \cos t - \cos 2t,$$

and so

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t}.$$

The gradient is defined for $\sin 2t - \sin t \neq 0$. For $t \in [0, 2\pi]$, this means that gradient is not defined for $t = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$. The corresponding x-values are $0, \frac{1}{4}, -2, \frac{1}{4}, 0$.

There are points of zero gradient when $t = \frac{2\pi}{3}, \frac{4\pi}{3}$, which corresponds to $x = -\frac{3}{4}$.

History and applications

History

The history of calculus is discussed in the modules:

- *Introduction to differential calculus*
- *Motion in a straight line.*

Applications

Application in biology

Example: Chemotherapy

Malignant tumours respond to radiation therapy and chemotherapy. Consider a medical experiment in which mice with tumours are given a chemotherapeutic drug. At the time of the drug being administered, the average tumour size is about 0.5 cm^3. The tumour volume $V(t)$ after t days is modelled by

$$V(t) = 0.005e^{0.24t} + 0.495e^{-0.12t},$$

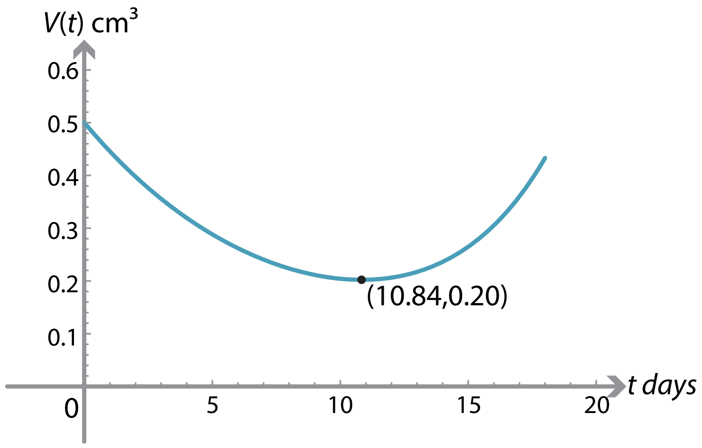
for $0 \le t \le 18$.

Let us find the minimum point. The first and second derivatives are

$$V'(t) = 0.0012e^{0.24t} - 0.0594e^{-0.12t}$$

$$V''(t) = 0.000288e^{0.24t} + 0.007128e^{-0.12t}.$$

Note that $V''(t) > 0$, for all t in the domain. By solving $V'(t) = 0$, we find that the minimum point occurs when $t \approx 10.84$ days. The volume of the tumour is approximately 0.20 cm^3 at this time.



Applications in physics

Calculus is essential in many areas of physics. Some additional applications of calculus to physics are given in the modules *Motion in a straight line* and *The calculus of trigonometric functions*.

Example: Damped simple harmonic motion

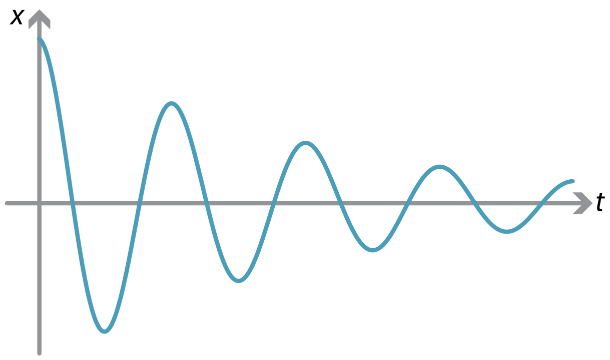
The displacement of a body of mass m undergoing damped harmonic motion is given by the formula

$$x = Ae^{-\frac{bt}{2m}}\cos(\omega t),$$

where

$$\omega = \sqrt{\frac{k}{m} - \Big(\frac{b}{2m}\Big)^2}.$$

Here k and b are positive constants. (Note that damped simple harmonic motion reduces to simple harmonic motion when $b = 0$.)



The motion can be investigated by using calculus to find the turning points.

Since $x = Ae^{-\frac{bt}{2m}}\cos(\omega t)$, we have

$$\begin{aligned}\frac{dx}{dt} &= -\frac{b}{2m}Ae^{-\frac{bt}{2m}}\cos(\omega t) - A\omega e^{-\frac{bt}{2m}}\sin(\omega t)\\ &= -Ae^{-\frac{bt}{2m}}\Big(\frac{b}{2m}\cos(\omega t) + \omega\sin(\omega t)\Big).\end{aligned}$$

The stationary points are found by putting $\dfrac{dx}{dt} = 0$. This implies

$$\tan(\omega t) = -\frac{b}{2m\omega}.$$

Here is example from optics using related rates of change.

Example: Thin lens formula

The thin lens formula in physics is

$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f},$$

where s is the distance of an object from the lens, S is the distance of the image from the lens, and f is the focal length of the lens. Here f is a constant, and s and S are variables.

Suppose the object is moving away from the lens at a rate of 3 cm/s. How fast and in which direction will the image be moving?

We are given $\frac{ds}{dt} = 3$. The question is: What is $\frac{dS}{dt}$?

We know by the chain rule that

$$\frac{dS}{dt} = \frac{dS}{ds} \times \frac{ds}{dt} = \frac{dS}{ds} \times 3.$$

Making S the subject of the thin lens formula, we have $S = \frac{fs}{s-f}$. The derivative of S with respect to s is

$$\frac{dS}{ds} = \frac{-f^2}{(s-f)^2}.$$

Thus

$$\frac{dS}{dt} = \frac{-f^2}{(s-f)^2} \times 3 = \frac{-3f^2}{(s-f)^2}.$$

The image is moving towards the lens at $\frac{3f^2}{(s-f)^2}$ cm/s.

Answers to exercises

Exercise 1

We can form a gradient diagram for this function.

Value of x		1		3	
Sign of $f'(x)$	$-$	0	$-$	0	$+$
Slope of graph $y = f(x)$	╲	—	╲	—	╱

There is a stationary point of inflexion at $x = 1$, and a local minimum at $x = 3$.

Exercise 2

Let $f(x) = x^3 - 5x^2 + 3x + 2$. The derivative is $f'(x) = 3x^2 - 10x + 3 = (3x - 1)(x - 3)$. So the stationary points are $x = \frac{1}{3}$ and $x = 3$.

- When $x < \frac{1}{3}$, $f'(x) > 0$.
- When $\frac{1}{3} < x < 3$, $f'(x) < 0$.
- When $x > 3$, $f'(x) > 0$.

Hence, there is a local maximum at $x = \frac{1}{3}$, and a local minimum at $x = 3$.

Exercise 3

The inflexion points are $x = 0$ and $x = -14$.

Exercise 4

We are given $f'(x) = x^3(x^2 - 5) = x^5 - 5x^3$. So the stationary points are $x = 0$, $x = \sqrt{5}$ and $x = -\sqrt{5}$.

The second derivative is $f''(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3)$.

- $f''(0) = 0$, so we cannot use the second derivative test for $x = 0$. However, note that $f'(-1) = 4$ and $f'(1) = -4$. The gradient changes from positive to negative at $x = 0$. Hence, there is a local maximum at $x = 0$.
- $f''(\sqrt{5}) = 50 > 0$, so there is a local minimum at $x = \sqrt{5}$.
- $f''(-\sqrt{5}) = 50 > 0$, so there is a local minimum at $x = -\sqrt{5}$.

Exercise 5

Let $y = 3x^4 - 44x^3 + 144x^2$. Then the first and second derivatives are

$$\frac{dy}{dx} = 12x^3 - 132x^2 + 288x = 12x(x^2 - 11x + 24) = 12x(x-3)(x-8)$$

$$\frac{d^2y}{dx^2} = 36x^2 - 264x + 288 = 12(3x^2 - 22x + 24) = 12(3x-4)(x-6).$$

1 We first find the x-intercepts:

$$3x^4 - 44x^3 + 144x^2 = 0$$

$$x^2(3x^2 - 44x + 144) = 0,$$

which gives $x = 0$ or $x = \frac{2}{3}(11 - \sqrt{13})$ or $x = \frac{2}{3}(11 + \sqrt{13})$.

2 We have $\dfrac{dy}{dx} = 12x(x-3)(x-8)$. So $\dfrac{dy}{dx} = 0$ implies $x = 0$ or $x = 3$ or $x = 8$.

3

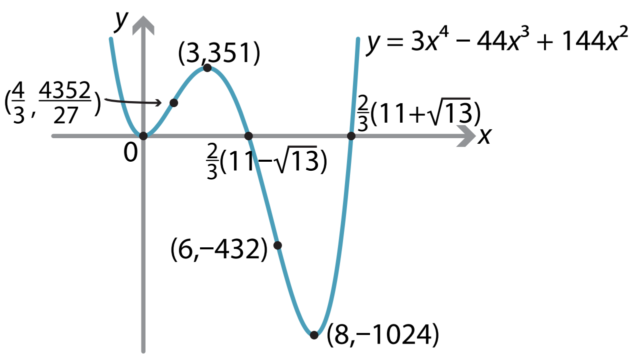
Value of x		0		3		8	
Sign of $\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
Slope of graph	╲	—	╱	—	╲	—	╱

4 We use the second derivative test:

- at $x = 0$, we have $\dfrac{d^2y}{dx^2} = 288 > 0$, so there is a local minimum
- at $x = 3$, we have $\dfrac{d^2y}{dx^2} = -180 < 0$, so there is a local maximum
- at $x = 8$, we have $\dfrac{d^2y}{dx^2} = 480 > 0$, so there is a local minimum.

5 As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to \infty$.

6 $\dfrac{d^2y}{dx^2} = 0$ implies $x = \frac{4}{3}$ or $x = 6$. The second derivative changes from positive to negative at $x = \frac{4}{3}$, and from negative to positive at $x = 6$. So there are points of inflexion at $x = \frac{4}{3}$ and $x = 6$.



Exercise 6

Let $y = 4x^3 - 18x^2 + 48x - 290$. The first and second derivatives are

$$\frac{dy}{dx} = 12x^2 - 36x + 48 = 12(x^2 - 3x + 4)$$

$$\frac{d^2y}{dx^2} = 12(2x - 3).$$

1 The y-intercept is $y = -290$. We now find the x-intercepts:

$$\begin{aligned} 4x^3 - 18x^2 + 48x - 290 &= 0 \\ 2x^3 - 9x^2 + 24x - 145 &= 0 \\ (x-5)(2x^2 + x + 29) &= 0. \end{aligned}$$

The quadratic has no real solutions, and so the only x-intercept is $x = 5$.

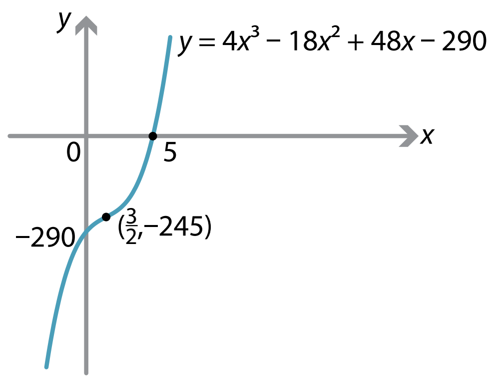
2 $\dfrac{dy}{dx} = 12(x^2 - 3x + 4)$. The discriminant is $12(9 - 16) < 0$. Hence, $\dfrac{dy}{dx} > 0$, for all x, and there are no stationary points.

3 The function is increasing, for all x.

4 There are no local maxima or minima.

5 As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

6 $\dfrac{d^2y}{dx^2} = 12(2x - 3)$. The second derivative changes from negative to positive at $x = \frac{3}{2}$. So there is a point of inflexion at $x = \frac{3}{2}$.



Exercise 7

Define $f\colon (0,\infty) \to \mathbb{R}$ by $f(x) = x^2 \log_e x$. Then

$$f'(x) = 2x\log_e x + x^2 \times \frac{1}{x} = 2x\log_e x + x = x(2\log_e x + 1)$$

$$f''(x) = 2\log_e x + 2 + 1 = 2\log_e x + 3.$$

1 For $x > 0$, we have $f(x) = 0 \iff \log_e x = 0 \iff x = 1$. So the x-intercept is 1.

2 We have $f'(x) = x(2\log_e x + 1)$. So $f'(x) = 0$ implies $x = e^{-\frac{1}{2}}$. There is a stationary point at $x = e^{-\frac{1}{2}}$.

3 We have

$$f'(x) > 0 \iff x(2\log_e x + 1) > 0 \iff 2\log_e x + 1 > 0 \iff x > e^{-\frac{1}{2}},$$

and so $f'(x) < 0 \iff 0 < x < e^{-\frac{1}{2}}$.

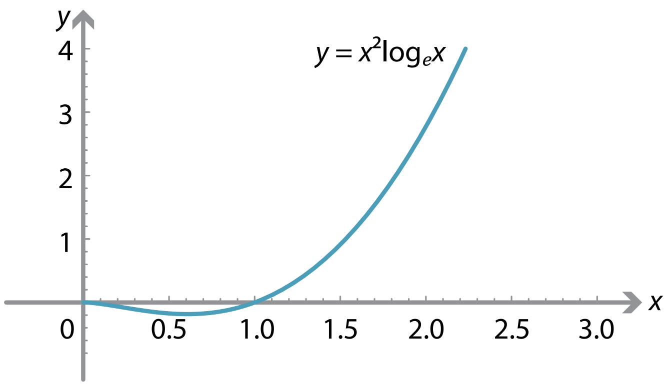
4 $f''(e^{-\frac{1}{2}}) = 2 > 0$. Hence, there is a local minimum at $x = e^{-\frac{1}{2}}$.

5 As $x \to \infty$, $f(x) \to \infty$.

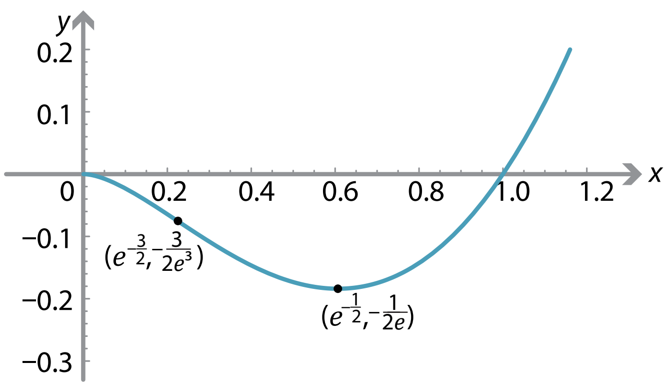
6 We have $f''(x) = 2\log_e x + 3$. So

$$f''(x) > 0 \iff x > e^{-\frac{3}{2}}, \qquad f''(x) < 0 \iff 0 < x < e^{-\frac{3}{2}}.$$

There is a point of inflexion at $x = e^{-\frac{3}{2}}$.



A second graph is drawn here to highlight the important points.



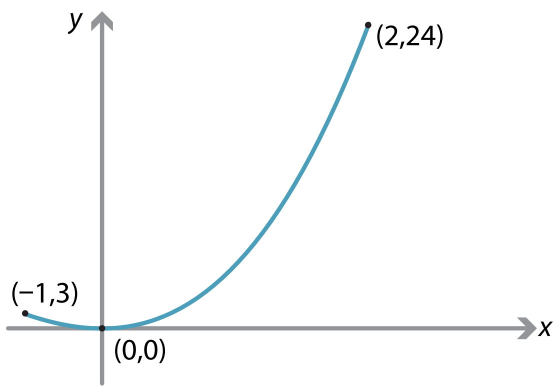
Exercise 8

Define $f\colon [-1,2] \to \mathbb{R}$ by $f(x) = x^2(x+4)$. The first and second derivatives are

$$f'(x) = 3x^2 + 8x, \qquad f''(x) = 6x + 8.$$

So $f'(x) = 0$ when $x = 0$ or $x = -\frac{8}{3}$. The second value is outside the required domain. Since $f''(0) = 8 > 0$, there is a local minimum at $x = 0$, with $f(0) = 0$.

We now find the value of the function at the endpoints: $f(-1) = 3$ and $f(2) = 24$.



The minimum value of the function is 0 and the maximum value is 24.

Exercise 9

Let x km, x km and y km be the lengths of fencing of the three sides of the rectangle to be enclosed. Then $y = 8 - 2x$.

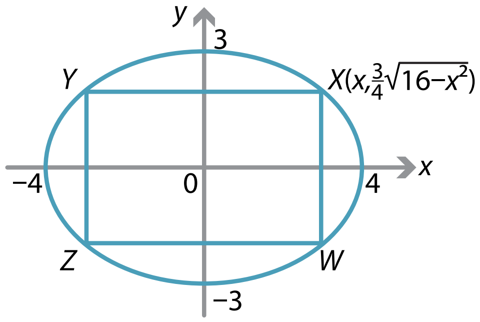
Let $A(x)$ km^2 be the area of the enclosed land. Then $A(x) = x(8-2x) = 8x - 2x^2$. The first two derivatives are

$$A'(x) = 8 - 4x \qquad \text{and} \qquad A''(x) = -4.$$

So $A'(x) = 0$ implies $x = 2$. Since $A''(2) = -4 < 0$, there is a local maximum at $x = 2$. The rectangle of maximum area has dimensions 2 km × 4 km, and the maximum area is 8 km^2.

Exercise 10

a The ellipse has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$.



Assume the top-right corner of the rectangle is at $X(x, \frac{3}{4}\sqrt{16-x^2})$. The area A of the rectangle $XYZW$ is given by

$$A = XY \times XW = 2x \times \frac{3}{2}\sqrt{16-x^2} = 3x\sqrt{16-x^2}.$$

We have

$$\begin{aligned}\frac{dA}{dx} &= 3\Big(\sqrt{16-x^2} - \frac{x^2}{\sqrt{16-x^2}}\Big)\\ &= 3\Big(\frac{16-x^2-x^2}{\sqrt{16-x^2}}\Big)\\ &= 3\Big(\frac{16-2x^2}{\sqrt{16-x^2}}\Big).\end{aligned}$$

So $\frac{dA}{dx}=0$ implies $x=2\sqrt{2}$. Furthermore, we have

$$\frac{d^2A}{dx^2} = \frac{6x(-24+x^2)}{(16-x^2)^{\frac{3}{2}}},$$

and so $\frac{d^2A}{dx^2}<0$ when $x=2\sqrt{2}$. The maximum area is $3\times 2\sqrt{2}\times\sqrt{16-(2\sqrt{2})^2}=24$.

b The ellipse has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a, b>0$. Assume that the top-right corner of the rectangle is at (x, y), where $0\le x\le a$ and $0\le y\le b$. Let A be the area of the rectangle. Then

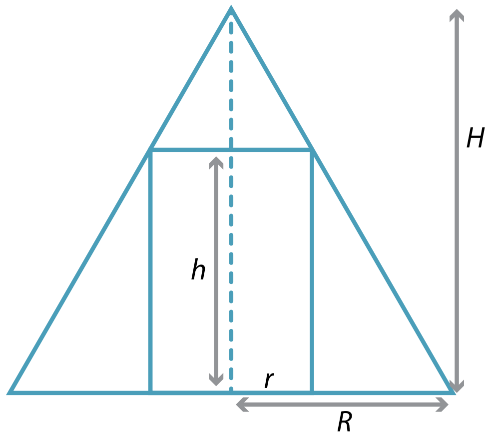
$$A = \frac{4b}{a}x\sqrt{a^2-x^2}$$

$$\frac{dA}{dx} = \frac{4b(a^2-2x^2)}{a\sqrt{a^2-x^2}}.$$

So $\frac{dA}{dx}=0$ implies $x=\frac{a}{\sqrt{2}}$. The maximum area is $2ab$.

Exercise 11

Let the cylinder have radius r and height h. Here $r > 0$ and $h > 0$, and they are variables. Let the cone have radius R and height H. They are constants.



The volume V of the cylinder is given by

$$V = \pi r^2 h.$$

Using similar triangles, we have

$$\frac{H-h}{r} = \frac{H}{R}$$

$$HR - hR = rH$$

$$h = H - \frac{rH}{R}.$$

Substitute for h in the formula for V:

$$V = \pi r^2\Big(H - \frac{rH}{R}\Big) = \frac{\pi H}{R}\big(Rr^2 - r^3\big).$$

Differentiate with respect to r:

$$\frac{dV}{dr} = \frac{\pi H}{R}\big(2Rr - 3r^2\big) = \frac{\pi H r}{R}\big(2R - 3r\big).$$

Check for stationary points: $\dfrac{dV}{dr} = 0$ implies $r = \dfrac{2R}{3}$, since $r \neq 0$. The second derivative is

$$\frac{d^2V}{dr^2} = \frac{\pi H}{R}\big(2R - 6r\big).$$

So, when $r = \dfrac{2R}{3}$, we have $\dfrac{d^2V}{dr^2} = -2\pi H < 0$. A local maximum occurs when $r = \dfrac{2R}{3}$. This gives the maximum volume of the cylinder, which is $V = \dfrac{4\pi R^2 H}{27}$.

Exercise 12

The point P is moving along the curve $y = \sqrt{x^3 + 56}$. We have

$$\frac{dy}{dx} = \frac{3x^2}{2\sqrt{x^3+56}},$$

and so

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt} = \frac{2\sqrt{x^3+56}}{3x^2} \times \frac{dy}{dt}.$$

When $x = 2$, we are given that $\dfrac{dy}{dt} = 2$, and so

$$\frac{dx}{dt} = \frac{2\sqrt{2^3+56}}{3 \times 2^2} \times 2 = \frac{8}{3}.$$

Exercise 13

Assume the meteor is a sphere of radius r. Its surface area is $S = 4\pi r^2$, and its volume is $V = \frac{4}{3}\pi r^3$. The volume is decreasing at a rate proportional to the surface area. That is,

$$\frac{dV}{dt} = -kS = -4\pi k r^2,$$

for some positive constant k. Using the chain rule, we have

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times -4\pi k r^2 = -k.$$

So the radius is decreasing at a constant rate.

0 1 2 3 4 5 6 7 8 9 10 11 12